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AUXILIARY TO GEOMETRY;

WITH AN ACCOUNT OF THE

PLANE SECTIONS OF THE CONE AND CYLINDER,

WHICH CERTAIN GENERAL PROPERTIES OF THE CONIC SECTIONS ARE
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ELEMENTARY COURSE.

In compiling the present Treatise, the leading object has been to furnish the "Library of Useful Knowledge" with a body of geometrical information, in which nothing might be wanting that seemed necessary or desirable, whether to the correct explanation and solid establishing of the science, or to its application in the different branches of natural philosophy. Such an object, it is plain, can never be accomplished by a mere elementary course, which has solely in view the instruction of beginners: it implies many discussions and distinctions, many theorems, scholia, and even whole sections of matter, which it is better that a beginner should pass by, while he confines his attention to the few and simple but important propositions to which perpetual reference is made, and which may be regarded as constituting the high road of Geometry. At the same time, the purposes of instruction have not been lost sight of; and accordingly, while the present work may be considered sufficiently extensive to answer every useful purpose, it will be found also to include an *elementary course of study* complete in itself, by the help of which a person totally unacquainted with the subject may become his own instructor, and advance by easy steps to a competent knowledge of it. With this view, the beginner has only to confine himself to the following portions of the entire work.

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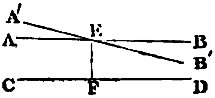
The demonstration of the converse part of Book I. Prop. 14., is attended with a difficulty which is stated at some length in page 11, as we have been anxious that the student should be fully aware of its existence. It will be better, however, in a first perusal, to avoid this difficulty by making, at once, the following assumption: "Through the same point there cannot pass two different straight lines, each of which is parallel to the same straight line."

The converse part of Prop. 14, viz. that "parallel straight lines are at right angles to the same straight line," will then be demonstrated as follows:—

Let A B be parallel to C D, and from any point E of A B let E F be drawn at right angles to C D (12.): E F shall also be at right angles to AB.



For, if A B be not at right angles to E F, through the point E let A′ B′ be drawn at right angles to EF (post. 5.). Then, by the former part of the proposition, because A′ B′ and C D are, each of them at right angles to E F, A′ B′ is parallel to C D. But A B is parallel to C D. Therefore, through the same point E there pass the two straight lines A B and A′ B′, each of which is parallel to C D. But it is assumed that this is impossible. Therefore, the supposition that A B is not at right angles to E F is impossible; that is, A B is at right angles to E F.

It will be found that the Course just laid down, excepting the sixth Book of it only, is not of much greater extent, nor very different in point of matter from that of Euclid, whose "Elements" have at all times been justly esteemed a model not only of easy and progressive instruction in Geometry, but of accuracy and perspicuity in reasoning. A perusal of this work, as translated and edited by Simson, though certainly not essential to an acquaintance with geometry, is strongly recommended to the student.

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GEOMETRY.

BOOK I.

§ 1. *Definitions*—§ 2. *First Theorems*—§ 3. *Parallels*—§ 4. *Parallelograms*—§ 5. *Rectangles under the parts of divided Lines*—§ 6. *Relations of the Sides of Triangles*—§ 7. *Problems.*

SECTION 1. *Definitions.*

GEOMETRY is the science of extension. The subjects which it considers are extent of distance, extent of surface, and extent of capacity or solid content.

The name *Geometry* is derived from two Greek words, signifying *land* and *to measure.* Hence it would appear that the measurement of land was the most important (perhaps the only) use to which this science was, in the first instance, applied. Egypt is described to have been its birth-place, where the annual inundations of the Nile rendered it of peculiar value to the inhabitants as a means of ascertaining their effaced boundaries. From the Egyptians the ancient Greeks derived their acquaintance with it; and, in the hands of this acute people, it was carried, from a state of comparative nothingness, to a degree of perfection which has scarcely been advanced by succeeding ages. If, however, as a science, Geometry has made but little progress, since it was so successfully cultivated by the Greeks, its uses have been both multiplied and extended. In the present day it embraces the measurement equally of the earth and of the heavens: it forms with arithmetic the basis of all accurate conclusions in the mixed sciences: and there is scarcely any mechanical art, our views of which may not be improved by an acquaintance with it.

The truths of Geometry are founded upon definitions, each furnishing at once an exact notion of the thing defined, and the groundwork of all conclusions relating to it. The leading definitions are as follows:—

1. A *solid* is a magnitude having three dimensions—length, breadth, and thickness.

2. A *surface* is the boundary of a solid, having length and breadth only.

3. A *line* is the boundary of a surface, having length only.

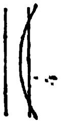
4. A *point* is the extremity of a line, having no dimensions of any kind—neither length, nor breadth, nor thickness.

5. (Euc. i. def. 4.)[*] A *right line,* or *straight line,* is that which lies evenly between its extreme points.

When the word "line" is used by itself in the following pages, a straight line is to be understood.

6. Any line of which no part is a right line is called a *curve.*

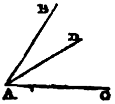
If a curve be cut by a straight line in two points, the curve is said to be *concave* towards that side upon which the straight line lies, and towards the other side, *convex.*



7. (Euc. i. def. 7.) A *plane surface,* or *plane,* is that, in which any two points whatsoever being taken, the straight line between them lies wholly in that surface.

8. A surface, of which no part is plane, is said to be *curved.*

9. If there be two straight lines in the same plane, which meet one another in a point, they are said to form at that point a *plane rectilineal angle.*



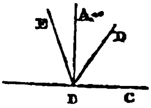
The magnitude of an angle does not depend upon the length of its *legs,* that is, of the straight lines by which it is contained, but upon the opening between them, or the extent to which they are separated the one from the other. Thus, the angle B A C is greater than the angle B A D, by the angle D A C.

If there be only one angle at the point A, it may be denoted by the letter A alone, as "the angle A;" but if there be more angles at the same point, it becomes necessary to indicate the containing sides of each, in order to distinguish it from the others: thus, the angle B A C is distinguished from B A D and D A C. In this case the middle letter, as A, always denotes the angular point.

---

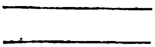
[*] This and the like references are to Simson's Euclid, the Roman numeral indicating the book, and the other the proposition. When the reference is to a definition, as in the present instance, or to an axiom, the same is particularized by the initial syllable Def. or Ax. The most important definitions only, which are taken from Euclid, and stated in nearly the same words, are here referred to.

10. (Euc. i. def. 10.) When a straight line standing upon another straight line makes the adjacent angles equal to one another, each of them is called a *right angle*, and the straight line which stands upon the other is called a *perpendicular* to it.



11. If an angle be not right, it is called *oblique*. An oblique angle is said to be *acute* or *obtuse*, according as it is less or greater than a right angle.

In the adjoined figure, ABC is a right angle, D B C an acute angle, and E B C an obtuse angle.



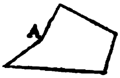
12. (Euc. i. def. 35.) If there be two straight lines in the same plane, which, being produced ever so far both ways, do not meet, these straight lines are called *parallels*.

13. A *plane figure* is any portion of a plane surface which is included by a line or lines.

The whole circuit of any figure, that is, the extent of the line or lines by which it is included, is called its *perimeter*.

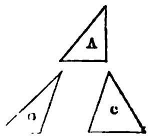
14. A *plane rectilineal figure* is any portion of a plane surface, which is included by right lines. These right lines are called the *sides* of the figure, and it is said to be *trilateral*, or *quadrilateral*, or *multilateral*, according as it has three, or four, or a greater number of sides.

A trilateral figure is more commonly called a *triangle*, and a multilateral figure a *polygon*.



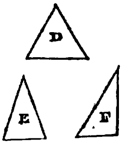
It is further to be understood of rectilineal figures in the present treatise, that the several angles are contained towards the interior of the figure; that is, that they have no such angle as the *re-entering angle* A in the figure which is adjoined. In other words, their perimeters are supposed to be convex externally.

15. A triangle is said to be *right-angled*, when it has a right angle. Of triangles which are not right-angled, and which are therefore said to be *oblique-angled*,—an *obtuse-angled* triangle is that which has an obtuse angle; and an *acute-angled* triangle is that which has three acute angles.



In the adjoined figure, A is a right-angled triangle; B an obtuse-angled triangle; and C an acute-angled triangle.

16. A triangle is also said to be *equilateral*, when its three sides are equal to one another; *isosceles*, when only two of its sides are equal; and *scalene*, when it has three unequal sides. D is an equilateral triangle, E an isosceles triangle, and F a scalene triangle.



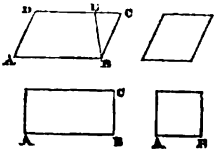
The three sides of any the same triangle are frequently distinguished by giving to one of them the name of *base*, in which case the other two are called *the two sides*, and the angular point opposite to the base is called the *vertex* or summit. In an *isosceles* triangle, considered as such, the *vertex* is the angular point between the two equal sides, and the *base* the side opposite to it.

In a *right-angled* triangle, the side which is opposite to the right angle is called the *hypotenuse*; and of the other two sides, one is frequently termed the *base*, and the other the *perpendicular*.

17. Of quadrilaterals, a *parallelogram* is that which has its opposite sides parallel, as A B C D. A quadrilateral which has only two of its sides parallel is called a *trapezoid*, as A B E D.

A parallelogram, or indeed any quadrilateral figure, is sometimes cited by two letters only placed at opposite angles: as "the parallelogram A C", "the trapezoid A E." This plan is never adopted, however, where confusion might ensue from it: when used, it must always be in such a way as to avoid uncertainty; thus, by "the quadrilateral B D" in the adjoined figure, either ABCD or ABED might be intended, whereas "the quadrilateral A C" is distinct from "the quadrilateral A E."

18. A *rhombus* is a parallelogram which has two adjoining sides equal.



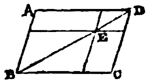
19. A *rectangle* is a parallelogram which has a right angle. A rectangle

is said to be *contained by* any two of its adjoining sides; as A C, which is called the rectangle under A B, B C, or *the rectangle* A B, B C.

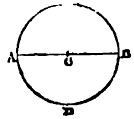
20. A *square* is a rectangle which has two adjoining sides equal. The square described upon any straight line A B, or the square of which A B is a side, is called the square of A B, or A B *square*.

21. The *altitude* of a parallelogram or triangle, is a perpendicular drawn to the base from the side or angle opposite.

22. The *diagonals* of a quadrilateral are the straight lines which join its opposite angles.



23. If through a point, E, in the diagonal of a parallelogram, A B C D, straight lines be drawn parallel to two adjacent sides, the whole parallelogram will be divided into four quadrilaterals; of which two, having the parts of the diagonal for their diagonals, are for that reason said to be *about the diagonal;* and the two others, A E, E C, are called *complements*, because, together with the portions about the diagonal, they complete the whole parallelogram A B C D.



24. A *circle* is a plane figure contained by one line, which is called the circumference, and is such that all straight lines drawn from a certain point within the figure to the circumference are equal to one another. This point is called the *centre* of the circle; and the distance from the centre to the circumference is called the *radius*, or, sometimes, the *semidiameter*, because it is the half of a straight line passing through the centre and terminated both ways by the circumference, which straight line is called a *diameter*.

The point C is the centre of the circle A B D; A B is a diameter; and A C a radius or semidiameter.

The truths and questions of Geometry are, for the sake of perspicuity, stated and considered in small separate discourses called *Propositions;* it being proposed in them either to demonstrate something which is asserted, a proposition of which kind is called a *theorem*, or to show the manner of doing something which is required to be done, a proposition of which kind is called a *problem*.

A proposition has commonly the following parts:—

1°. the *enunciation*, declaring what is to be proved or done;

2°. the *construction*, inserting the lines necessary thereto;

3°. the *demonstration*, or course of reasoning;—And,

4°. the *conclusion*, asserting that the thing required has been proved or done.

A *corollary* to any proposition is a statement of some truth, which is an obvious consequence of the proposition.

A *scholium* is a remark or observation.

The object of a problem, as above stated, is evidently distinct from that of a theorem. If a problem be regarded, however, as demonstrating merely the *existence* of the points and lines required in its enunciation, it becomes, for our purposes, a theorem certifying the existence of such. And hence has arisen the introduction of problems into the theory of Geometry; for, the existence of the lines and points specified in the constructions of some theorems not being altogether self-evident, it became necessary, either to introduce distinct problems for the finding of such, or to point out the certainty of their existence by the way of theorem and corollary, as occasion offered.

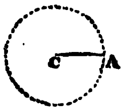
The former plan, exemplified in Euclid's Elements, has been followed by the greater number of geometrical writers; although the problems introduced have not, in all cases, been limited to the very few which are necessary to support the theory. To avoid thus sacrificing unity of purpose, and at the same time not to be wanting to the ends of practical geometry, the problems in the present treatise have been altogether separated from the theorems; and the requisite support has been supplied to the latter, in the second of the two ways above mentioned.

The existence of the following lines, &c. will be taken for granted; and they will, therefore, be referred to by the name of

POSTULATES.*

1. A straight line, which joins or passes through two given points, A, B.

2. A circle, which is described from a given centre, C, with a given radius, C A.



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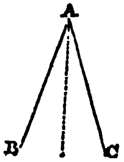
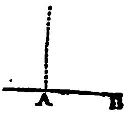
* Things required; from the Latin *postulo*, to require.

3. A point which bisects a given finite straight line, A B, that is, which divides it into two equal parts.

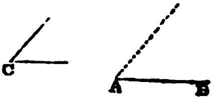


4. A straight line which bisects a given angle, B A C.



5. A perpendicular to a given straight line, erected from a given point in the same.

6. A straight line, which makes with a given straight line, A B, at a given point, A, an angle equal to a given rectilineal angle, C.



The following truths require no steps of reasoning to establish their evidence. It may be said of them, that no demonstration can make them more evident than they are already, without it: they are, therefore, called self-evident truths or *axioms*. They will be found of perpetual recurrence in demonstrating the propositions of the following sections, and are therefore here premised:—

AXIOMS.*

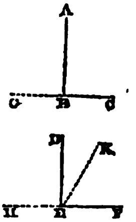
1. Things, which are equal to the same, are equal to one another.
2. If equals be added to equals, the wholes are equal.
3. If equals be taken from equals, the remainders are equal.
4. The doubles of equals are equal.
5. The halves of equals are equal.
6. The greater of two magnitudes, increased or diminished by any magnitude, is greater than the less increased or diminished by the same magnitude.
7. The double of the greater is greater than the double of the less.
8. The half of the greater is greater than the half of the less.
9. If there be two magnitudes, and a third, and if one of them exceed the third by as much as the other falls short of it; the two together shall be double of the third magnitude.
10. Straight lines which pass through the same two points lie in one and the same straight line.
11. Magnitudes, which may be made to coincide with one another, that is, to fill exactly the same space, are equal to one another.

The converse of this last axiom is likewise true of some magnitudes. In what follows, it will be assumed, with regard to straight lines and angles; *i. e.* it will be assumed that if two straight lines are equal, they may be made to coincide with one another, and the same of two angles.

SECTION 2. *First Theorems.*

PROP. 1. (EUC. i. *Ax.* 11.)

All right angles are equal to one another.



Let the angles A B C, D E F be each of them a right angle; the angle A B C shall be equal to the angle D E F.

Produce C B to any point G, and F E to any point H. Then, because A B C is a right angle, it is equal to the adjacent angle A B G (def. 10.); and because DE F is a right angle, it is equal to D E H.

From E draw any straight line E K. Then, because the angle KEH is greater than D E H, and that D E H is equal to D E F, K E H is greater than D E F: but D E F is greater than K E F: much more, then, is K E H greater than KE F.

Now, let the angle A B C be applied to the angle D E F, so that the point B may be upon E, and the straight line B C upon E F; then (ax. 10.) B G will coincide with E H. And, B G coinciding with E H, B A must also coincide with E D; for, should it fall otherwise, as E K, the angle A B G would be greater than A B C, by what has been already demonstrated, whereas, they are equal to one another.

Therefore, B A coincides with E D, and the angle A B C coincides with the angle D E F, and (ax. 11.) is equal to it.

Therefore, all right angles are equal to one another, which was to be demonstrated.

PROP. 2. (EUC. i. 13 & 14.)

The adjacent angles, which one straight line makes with another upon

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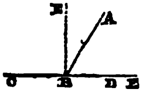
* Authorities, or things having authority; from a Greek word.

one side of it, are either two right angles, or are together equal to two right angles: and, conversely, if the adjacent angles, which one straight line makes with two others at the same point, be together equal to two right angles, these two straight lines shall be in one and the same straight line.



Let the straight line A B make with C D upon one side of it, the adjacent angles A B C, A B D: these are either two right angles, or are together equal to two right angles.

For, if they are equal, then is each of them (def. 10.) a right angle.

But, if not, from the point B draw B E perpendicular to C D (Post. 5.). And because the angle E B D is equal to the two angles, E B A, A B D, to each of these equals add the angle E B C: therefore, (ax. 2.) the two angles E B C, E B D are equal to the three angles E B C, E B A, A B D. And in the same manner it may be shewn, that the two angles A B C, A B D, are equal to the same three angles. Therefore, (ax. 1.) the angles A B C, A B D, are together equal to the angles E B C, E B D, that is, to two right angles.

Next, let the straight line A B make with the two straight lines, B C, B D, at the same point B, the adjacent angles A B C, A B D together equal to two right angles: B C, B D shall be in one and the same straight line.

For, let B F be in the same straight line with B C: then, by the first part of the proposition, because A B makes angles with C F upon one side of it, these angles, viz. A B C, A B F, are together equal to two right angles. But A B C, A B D are also equal to two right angles; therefore, (ax. 1.) A B C, A B D together are equal to A B C, A B F together; and, A B C being taken from each of these equals, the angle A B D is equal to A B F (ax. 3.) Therefore B D coincides with B F; that is, it is in the same straight line with B C.

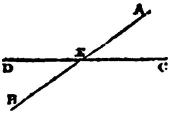
Therefore, &c.*

Cor. 1. If from a point in a straight line there be drawn any number of straight lines upon one side of it, all the angles (made by each with that next to it) shall be together equal to two right angles.

Cor. 2. Any angle of a triangle is less than two right angles.

Prop. 3. (Euc. i. 15.)

If two straight lines cut one another, the vertical or opposite angles shall be equal.



Let the two straight lines A B, C D, cut one another in the point E: the vertical angles A E D, B E C, as also the vertical angles A E C, B E D, shall be equal to one another.

Because the angles A E C, A E D are adjacent angles made by the straight line A E with C D, they are (2.) together equal to two right angles; and for the like reason, the angles A E C, C E B, are together equal to two right angles; therefore, (ax. 1.) the angles A E C, A E D together are equal to the angles A E C, C E B together. From each of these equals take the angle A E C, and the angle A E D is equal to the angle C E B. (ax. 3.) In the same manner it may be shown that the angles A E C, B E D are equal to one another.

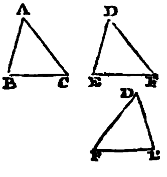
Therefore, &c.

Cor. (Euc. i. 15. *Cor.* 2.) If any number of straight lines pass through the same point, all the angles about that point, (made by each with that next to it,) shall be together equal to four right angles.

Prop. 4. (Euc. i. 4.)

If two triangles have two sides of the one equal to two sides of the other, each to each, and likewise the included angles equal; their other angles shall be equal, each to each, viz. those to which the equal sides are opposite, and the base, or third side, of the one shall be equal to the base, or third side, of the other.

Let A B C, D E F be two triangles, which have the two sides A B, A C, equal to the two sides D E, D F, each to each, viz. A B to D E, and A C to D F, and let them likewise have the angle B A C equal to the angle E D F: their other angles shall be equal, each to each, viz. A B C to D E F, and A C B to D F E, and the base B C shall be equal to the base E F.



For if the triangle A B C be applied to the triangle D E F, so that the point A may be upon D, and the straight line A B upon D E, the straight line A C will coincide with D F, because the angle B A C is equal to E D F. Also the point B will coincide with E, because A B is equal to D E, and the point C with F, because A C is equal to D F; and, because the points B, C, coincide with the points E, F, the straight line B C coincides with the straight line E F (ax. 10.), and (ax. 11.) is equal to it; the angle A B C coincides with D E F, and is equal to it; and the angle A C B with the angle D F E, and is equal to it.

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* Hence the adjacent angle ABD is sometimes said to be *supplementary* to ABC; one angle being called the *supplement* of another, when together with that other it is equal to two right angles.

Therefore, &c.

Cor. The two triangles are equal also as to surface.

Scholium.

It is indifferent which of the two triangles D E F be taken, although in these triangles the side D E lie in opposite directions from D F; viz. to the right of it in the one, and to the left of it in the other.

The same may be observed of the next proposition, and of all cases of *plane* triangles, which are equal in every respect.

Prop. 5. (Euc. i. 26, first part of.)

If two triangles have two angles of the one equal to two angles of the other, each to each, and likewise the interjacent sides equal; their other sides shall be equal, each to each, viz. those to which the equal angles are opposite, and the third angle of the one shall be equal to the third angle of the other.*

Let A B C, D E F (see the last figure) be two triangles which have the two angles A B C, A C B of the one, equal to the two angles D E F, D F E of the other, each to each, and likewise the side B C equal to the side E F: their other sides shall be equal, each to each, and the third angle B A C shall be equal to the third angle E D F.

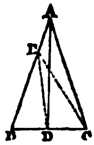
For, if the triangle A B C be applied to the triangle D E F, so that the point B may be upon E, and the straight line B C upon E F, the point C will coincide with the point F, because B C is equal to E F. Also the straight line B A will coincide in direction with E D, because the angle C B A is equal to F E D, and the straight line C A with F D, because the angle B C A is equal to E F D. But, if two straight lines which cut one another, coincide with other two which cut one another, it is manifest that the points of intersection must likewise coincide. Therefore, the point A coincides with D, and the sides A B, A C, coincide with the sides D E, D F, and are equal to them; and the angle B A C coincides with the angle E D F, and is equal to it (ax. 11.).

Therefore, &c.

Cor. The two triangles are equal also as to surface.

Prop. 6. (Euc. i. 5 & 6.)

If two sides of a triangle be equal to one another, the opposite angles shall be likewise equal: and conversely, if two angles of a triangle be equal to one another, the opposite sides shall be likewise equal.



Let A B C be an isosceles triangle, having the side A B equal to the side A C; the angle A C B shall be equal to the angle A B C.

Let the angle B A C be divided into two equal angles by the straight line A D, which meets the base B C in D (Post. 4). Then, because the triangles A D B, A D C have two sides of the one equal to two sides of the other, each to each, and the interjacent angles B A D, C A D equal to one another, their other angles are equal, each to each (4.); therefore the angle A C B is equal to A B C.

Next, let the angle A B C be equal to the angle A C B: the side A C shall be equal to the side A B.

From D, the middle point of B C, erect a perpendicular to B C (Post. 3. and 5.): and, if it do not pass through the vertex A, let this perpendicular, if possible, cut one of the sides as A B in E, and join E C. Then, because the triangles E D B, E D C have two sides of the one equal to two sides of the other, each to each, and the included angles E D B, E D C equal to one another (def. 10.), their other angles are equal, each to each (4.). Therefore the angle ECD is equal to EBD. But E B D or ABD is equal to ACB: therefore the angle ECD is equal to ACB (ax. 1.), the less to the greater, which is impossible. Therefore the perpendicular at D cannot pass otherwise than through the vertex A: and because the triangles A D B, A D C are equal, according to Prop. 4., the side A B is equal to the side A C.

Therefore, &c.

Cor. 1. Every equilateral triangle is also equiangular; and conversely.

Cor. 2. In an isosceles triangle A B C, if the equal sides A B, A C, be produced, the angles upon the other side of the base B C will be equal to one another; for, each of them together with one of the equal angles A B C, A C B, is equal to two right angles (2.).

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* "Interjacent sides," i. e. sides lying between.

Cor. 3. The straight line which bisects the vertical angle of an isosceles triangle, bisects the base at right angles: and conversely, the straight line which bisects the base at right angles, passes through the vertex, and bisects the vertical angle.

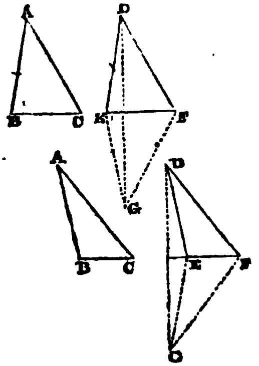
Cor. 4. If there be two isosceles triangles upon the same base (whether they be upon the same side of it or upon different sides), the straight line which joins their vertices or summits, or that straight line produced, shall bisect the base at right angles. For the straight line which bisects the base at right angles passes through the vertex of each.

Prop. 7. (Euc. i. 8.)

If two triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal, the angle contained by the two sides of the one shall be equal to the angle contained by the two sides equal to them of the other.

Let A B C, D E F be two triangles, having the two sides of the one equal to two sides of the other, each to each, and likewise the base B C equal to the base E F: the angle B A C shall be equal to the angle E D F.

At the point E in the straight line E F, make the angle F E G equal to the angle A B C (Post. 6.): take E G equal to B A or E D, and join G F, G D. Then because the triangles A B C, G E F, have two sides of the one equal to two sides of the other, each to each, and the included angles equal to one another, the base G F is equal to A C (4.) that is, to D F. Again, because in the triangle F D G, the side F D is equal to F G, the angle F D G is equal to F G D (6.). For the like reason, the angle E D G is equal to E G D. Therefore the angle E D F, which is the sum or difference of the two E D G, F D G, is equal to the angle E G F, which is the sum or difference of the two E G D, F G D (ax. 2., 3.). But E G F is equal to B A C, because (4.) the triangle G E F is equal to the triangle A B C in every respect: therefore, (ax. 1.) the angle E D F is equal to the angle B A C. When G D coincides with G E, G E D is a straight line, and the angles at G and D are the angles at the base of the isosceles triangle F D G; wherefore the latter is equal to the angle at G, that is, to the angle at A, as before.

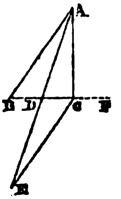


Therefore, &c.

Cor. The two triangles are equal in every respect. (4. & 4 *Cor*.)

Prop. 8. (Euc. i. 17.)

Any two angles of a triangle are together less than two right angles.



Let A B C be any triangle: any two of its angles, ABC and ACB, shall be together less than two right angles.

Bisect B C in D (Post. 3.): join A D, and produce it to E, so that DE may be equal to A D; and join C E.

Then, because the triangles A B D E C D have two sides of the one equal to the two sides of the other, each to each, and the included angles A D B, E D C, equal to one another (3.), the angle E C D is equal to the angle AB D or AB C. (4.) Therefore, the two angles A B C, A C B taken together are equal to the two angles E C D, A C B taken together (ax. 2.), that is, to the angle A C E. But (2. *Cor*. 2.) the angle A C E is less than two right angles. Therefore, the angles A B C, A C B together are less than two right angles.

Therefore, &c.

Cor. 1. (Euc. i. 16.) If one side of a triangle A B C, as B C, be produced to F, the exterior angle A C F shall be greater than either of the interior and opposite angles at A and B; for either of these angles taken with the angle A C B, is less than two right angles, but the angle A C F, taken with the same A C B, is equal to two right angles (2.).

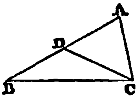
Cor. 2. A triangle cannot have more than one right angle, or more than one obtuse angle.

Prop. 9. (Euc. i. 18. & 19.)

If one side of a triangle be greater

than another, the opposite angle shall likewise be greater than the angle opposite to that other: and conversely.

In the triangle A B C, let the side A B be greater than A C, the angle A C B shall likewise be greater than the angle A B C.



Take A D equal to A C, and join C D. Then, because A D is equal to A C, the angle A C D is equal to the angle A D C (6.). But, because the side B D of the triangle C D B is produced to A, the exterior angle A D C is greater than the interior and opposite angle D B C or A B C. (8. *Cor.* 1.)

Therefore, the angle A C D, and much more A C B, is also greater than A B C.

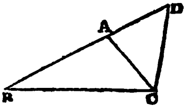
Next, let the angle A C B be greater than the angle A B C; the side A B shall likewise be greater than the side A C. For, A B cannot be equal to A C; because, then, (6.) the angle A C B would be equal to A B C, which is not the case: neither can it be less than A C, because then, by the former part of the proposition, the angle A C B would be less than A B C, which is not the case. Therefore A B cannot but be greater than A C.

Therefore, &c.

Prop. 10. (Euc. i. 20.)

Any two sides of a triangle are together greater than the third side: and any side of a triangle is greater than the difference of the other two.

Let A B C be a triangle: any two of its sides, A B and A C, shall be together greater than the third side B C; and any side A B alone shall be greater than the difference of B C and A C, the other two sides.

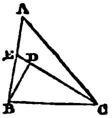


Produce B A to D, so that A D may be equal to A C, and join C D. Then, because A D is equal to A C, the angle A C D is equal to A D C (6.) But the angle B C D is greater than A C D: therefore, the angle B C D is greater also than A D C or B D C. Therefore, (9.) the side B D is, likewise, greater than B C. But, B D is equal to B A and A C together, because A D is equal to A C. Therefore B A and A C together are greater than B C.

And, because B A and A C are together greater than B C, if A C be taken from each, B A alone is greater than the difference of B C and A C.

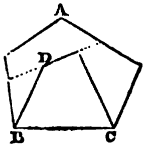
Therefore, &c.

Cor. 1. (Euc. i. 21, part of.) If there be two triangles A B C, D B C, upon the same base B C, and if the vertex of one of them, as D, fall within the other, the two sides of that triangle will be less than the two sides of the other. For, if C D be produced to meet the side A B of the enveloping triangle in E, B D and D C together will be less than B E and E C together, (ax. 6.) because B D is less than B E and E D together: and, for the like reason, B E and E C together are less than B A and A C together: much more, then, are B D and D C together less than B A and A C together.



Cor. 2. Any side of a rectilineal figure is less than the sum of all the other sides.

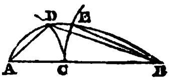
Cor. 3. And, hence, it may easily be demonstrated, that if there be two rectilineal figures A D C, D B C upon the same base B C, one of which wholly envelopes the other, the perimeter of the enveloping figure must be greater than the perimeter of the other.



Scholium.

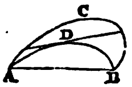
By help of this proposition it may be shown that *a straight line is the shortest distance between two points* A *and* B.

Let A C B be the straight line joining A and B, and A D E B any other line drawn from A to B. In A C B take any point C; and from the centre A with the radius A C describe a circle, cutting A D E B in D; and join A D, D B. Then, because A D and D B are together greater than A B, and that A D is equal to A C, D B is greater than C B (ax. 6.). Therefore, if a circle be described from the centre B with the radius B C, it will cut the straight line D B in some point between D and B; and, consequently, the line A D E B in some point E which is in the part D E B. Join E B. Then, if A D be made to coincide with A C, and B E with B C, it is evident that the parts A D and E B (curvilineal or otherwise) of the whole line A D E B (curvilineal or otherwise) will form a



complete path from A to B, which is shorter than A D E B by the intermediate part D E.

Therefore, there is no path from A to B, the straight line A C B excepted, than which a shorter may not be found between A and B. But, since none of the paths from A to B can be less than of some certain length, there must be some, one or more, shorter than the others. Therefore, the shortest path is the straight line A C B. From this property the straight line which joins two points derives the name of *the distance between* them.

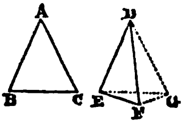


Hence, also, we may infer, that *of any two paths*, A C B, A D B, *leading from* A to B, *and everywhere concave towards the straight line* A B, *that which is enveloped by the other, as* ADB, *is the shortest.* For of all the paths not lying between ADB and the straight line A B, there is none, A D B excepted, than which a shorter may not be found.

And this is the case whether the paths A C B and A D B be both of them curvilineal, or one of them, (ACB or ADB) rectilineal.

Prop. 11. (Euc. i. 24 & 25.)

If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle which is contained by the two sides of the one greater than the angle which is contained by the two sides equal to them of the other, the base of that which has the greater angle shall be greater than the base of the other: and conversely.

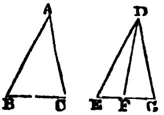


Let A B C, D E F, be two triangles having the two sides AB, A C of the one equal to the two sides D E, D F of the other, each to each, but the angle B A C greater than the angle E D F: the base B C shall be greater than the base E F.

At the point D in the straight line DE, make the angle EDG equal to the angle BAC (Post. 6.); take DG equal to AC, and join E G, G F. Then, because the triangles A B C, D E G have two sides of the one equal to two sides of the other, each to each, and the included angles B A C, E D G equal to one another, the bases B C, E G are equal to one another (4.).

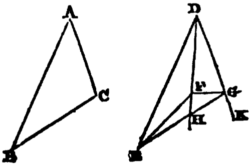
Now, the line D F falls between D E and D G, because the angle E D G is equal to B A C, which is supposed to be greater than E D F. But the point F may fall 1° without the triangle D E G; or 2° upon the base E G; or 3° within the triangle D E G.

In the first case, because D G is equal to A C, that is, to D F, the angle D F G is equal to the angle D G F (6.). But the angle E F G is greater than D F G, and E G F is less than D G F. Therefore much more is the angle E F G greater than the angle EGF. Therefore, also, the side E G, that is B C, is greater than E F (9.).



In the second case, it is at once evident that EG, that is, B C, is greater than E F.

In the third case, let D F and D G be produced to H and K. Then, because D F is equal to D G, the angles H F G, KGF, upon the other side of the base of the isosceles triangle D FG, are equal to one another (6. *Cor.* 2.). But E F G is greater than H F G, and E G F is less than K G F. Therefore much more is the angle E F G greater than the angle E G F, as in the first case, and the side E G, that is B C, is greater than E F (9.).

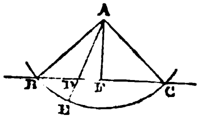


Next, let the base B C be greater than the base E F; the angle B A C shall likewise be greater than the angle EDF. For B A C cannot be equal to E D F, because then (4.) the base BC would be equal to the base EF; neither can it be less than E D F, because then, by the former part of the proposition, the base B C would be less than the base E F. Therefore, the angle B A C cannot but be greater than the angle E D F.

Therefore, &c.

Prop. 12.

A straight line may be drawn perpendicular to a given straight line of indefinite length, from any given point without it; but, from the same point, there cannot be drawn more than one perpendicular to the same straight line.



Let B C be a given straight line of indefinite length, and A any given point without it. A perpendicular may be drawn from the point A to the straight line B C.

In B C take any point D; join A D, and produce it to any point E, (Post. 1.). With the centre A and the radius A E describe a circle cutting B C in the points B and C upon each side of the point D, (Post. 2.). Bisect B C in F, and join AF, AB, AC, (Post. 3.). Then because A B is equal to A C, ABC is an isosceles triangle. Therefore AF, which is drawn from the vertex A to the middle point of the base B C, is perpendicular to the base (6. *Cor.* 3.); that is, a straight line A F may be drawn from the point A perpendicular to the straight line B C.

But, from the same point A there cannot be drawn more than one perpendicular to the same straight line B C. For, if any other straight line AD were perpendicular to B C, the two angles, ADF and AFD, of the triangle A D F, would be together equal to two right angles, which (8.) is impossible.

Therefore, &c.

Cor. 1. If from any point A to a straight line B C, there be drawn a straight line A B which is not at right angles to B C, a second straight line AC may be drawn from A to B C, which shall be equal to AB; for, the perpendicular A F being drawn, and F C being taken equal to FB, it may easily be shown (4.) that A C is equal to A B.

Cor. 2. Of straight lines A B, A D, which are drawn from A to B C upon the same side of the perpendicular A F, that which is nearer to the perpendicular, as A D, is less than the other, which is more remote. For, the angle A B D or ABF being (8. *Cor.* 1.) less than the exterior right angle A F C, and again A F D or A F C less than the exterior angle AD B, much more is the angle A B D less than A D B, and therefore also the side AD less than AB (9.).

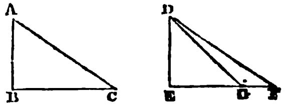
Cor. 3. In the same manner it may be shown that the perpendicular A F is the least of all straight lines which can be drawn from A to B C. For, if A B be any other straight line, the angle A B F being less than the exterior angle A F C, that is than A F B (def. 10.), the side A F is also less than the side A B (9.). For this reason, the perpendicular A F is called also *the distance of the point* A *from the line* B C.

Cor. 4. Hence, if, from the centre A, a circle be described with a radius less than the perpendicular A F, it will not meet the straight line B C; if with a radius equal to A F, it will meet B C in one point only, which is the foot of the perpendicular; and if with a radius greater than A F, it will meet B C in two points, which are at equal distances from the foot of the perpendicular, upon either side of it.

Prop. 13.

If, in two right-angled triangles, the hypotenuse and a side of the one be equal to the hypotenuse and a side of the other; or, if the hypotenuse and an adjacent angle of the one be equal to the hypotenuse and an adjacent angle of the other; the triangles shall be equal to one another in every respect.

Let A B C, D E F, be two triangles, having the angles at B and E right angles: and first, let the hypotenuse A C and side A B of the one be equal to the hypotenuse DF and side DE of the other. The triangles ABC, DEF, shall be equal to one another in every respect.



For, if the side A B be made to coincide with DE, which is equal to it, the right angle A B C will also coincide with the right angle DEF (1. and ax. 11). Therefore, if AC do not coincide with DF, but fall otherwise, as DG, there will be drawn from the point D to the line E F, upon the same side of the perpendicular, two straight lines that are equal to one another, which is impossible (12. *Cor.* 2.). Therefore, A C coincides with D F, and the triangle A B C coincides with the triangle DEF, that is, (ax. 11.) the triangles ABC and DEF are equal in every respect.

Next, let the hypotenuse AC and angle ACB of the one triangle be equal to the hypotenuse D F and angle DFE of the other. In this case, also, the triangles shall be equal in every respect.

For if the hypotenuse A C be made to coincide with D F, which is equal to it, the angle A C B will also coincide with DFE, which is equal to it. Therefore, if A B do not coincide with D E, but fall otherwise, as D G there will be drawn from the point D to the line E F, two perpendiculars, which is impossible (12). Therefore, A B coincides with D E, and the triangle A B C coincides with the triangle DEF, that is, (ax. 11.) the triangles AB C and D E F are equal in every respect.

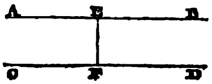
Therefore, &c.

Section 3. *Parallels.*

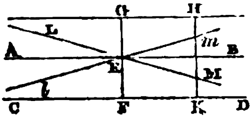
Prop. 14.

Straight lines which are at right angles to the same straight line, are parallel: and, conversely, parallel straight lines are at right angles to the same straight line; that is, if a straight line be drawn through the two at right angles to one of them, it shall be at right angles to the other likewise.

The first part of the proposition is manifest; for if A B and C D be each of them at right angles to the straight line E F, and be not parallel, they must meet one another; in which case, there will be two perpendiculars drawn to the same straight line E F from the same point, viz. the point of concourse. But (12.) this is impossible. Therefore, AB cannot meet CD, though produced ever so far both ways, that is (def. 12.) A B is parallel to C D.



In the next place, let A B be parallel to C D, and from any point E of A B let E F be drawn at right angles to C D: E F shall also be at right angles to A B.



Through E let any straight line L M be drawn which is not at right angles to EF. Produce F E to G, so that E G may be equal to E F, and from G draw G H perpendicular to G F. Through E draw *lm*, making the angle G E *m* equal to the angle F E M. In F D take any point K: make G H equal to F K, and join H K, cutting the lines L M, *l m* in the points M, *m*. Then it may easily be shown (by doubling over the figure, and applying the straight line E F upon E G, so that the point F may coincide with the point G, and therefore the straight line F K with G H, and E M with E *m*,) that E M is equal to E *m*, and M K to *m* H.

Now, *it seems sufficiently evident*, that, the straight lines G H and F K being at right angles to the same straight line, and therefore (by the first part of the proposition) never meeting one another, the distance H K, of any two corresponding points in them, neither increases nor diminishes, but remains always equal to FG; while, on the other hand, the lines EM and E*m* cutting one another in E, the distance M *m*, of any two corresponding points in them, continually increases with the distance from E, and may, by sufficiently producing EM, E*m*, be made greater than any assigned distance, as F G. Therefore, the straight lines E M, E*m* may be produced, so that M*m* may become greater than HK. But, because M K is always equal to *m* H, M *m* cannot become greater than H K, unless the straight line E M cuts the line F D, and E*m* the line G H.

Therefore E M and E *m* may be produced, until they respectively meet the lines F D and G H.

Hence, it appears, that A B, which never meets C D, cannot but be at right angles to E F: for, it has been shown, that any straight line which passes through E, and is not at right angles to E F, may be produced to meet C D.

Therefore, &c.

Cor. 1. Any point E being given, a straight line A B may be drawn through that point, which shall be parallel to a given straight line CD (12. and Post. 5.).

Cor. 2. Through the same given point, there cannot be drawn more than one parallel to the same given straight line.

Cor. 3. If a straight line cut one of two parallels, it may be produced to cut the other likewise.

Scholium.

The second part of this proposition is not supported by that cogency of demonstration which is said, and with truth, to characterize every other part of Geometry. Of the two particulars which have been assumed, one indeed, viz., that which regards the unlimited divergency of cutting lines, seems almost axiomatic or self-evident. The other is not equally so. It may be illustrated by observing that, at equal intervals, upon either side of E F, the distances of corresponding points are equal to one another; and thence arguing, that, from one of these to the other, the distance can neither have been increasing nor diminishing, for that, had either been the case, since the lines are *straight*, the distance of their corresponding points would have *continued* to increase or to diminish. Should this, however, as we can easily imagine, fail to satisfy the student, we must refer him to measurement for such a degree of conviction as it can afford.

In fact, although it may be shown without difficulty, that certain straight lines will never meet one another; the *converse*, viz., that straight lines, which never meet one another, must have certain properties, *has never been strictly demonstrated*. It is agreed by Geometrical writers that some assumption is indispensable.[*]

The following is, perhaps, as simple as any that can be proposed, while it has the advantage also of not being many steps distant from the proposition in question.

"If from two points of one straight line to another, there fall two unequal perpendiculars, the straight lines will meet one another, if produced, upon the side of the lesser perpendicular."

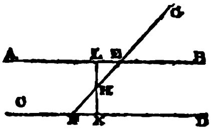
Hence, if two straight lines be parallel, the perpendiculars drawn from the points of the one to the other, must all of them be equal; and hence, if a straight line be drawn at right angles to one of two parallels, it may easily be shown to cut the other at angles, which are equal to one another, that is, at right angles.

It is demonstrated in Prop. 16., that, if two straight lines be parallel, the perpendiculars drawn from the points of the one to the other must all of them be equal: but that demonstration itself rests upon the converse part of Prop. 14., which is here in question. The reader must not imagine, therefore, that the above assumption is at all assisted by that demonstration.

PROP. 15. (EUC. i. 27, 28 and 29.)

Straight lines which make equal angles with the same straight line, towards the same parts, are parallel: and, conversely, if two parallel straight lines be cut by the same straight line, they shall make equal angles with it towards the same parts.

Let the straight lines A B, C D make equal angles B E G, D F G, with the same straight line E F, towards the same parts: AB shall be parallel to CD.



Bisect E F in H (Post. 3.), from H draw H K perpendicular to C D (12.), and produce K H to meet A B in L. Then because the angle H E L is equal to B E G (3.), and that B E G is equal to D F G or H F K, the angle H E L is equal to the angle H F K (ax. 1.). The vertical angles E H L, F H K are likewise equal to one another (3.). Therefore, the triangles H E L and H F K having two angles of the one equal to two angles of the other, each to each, and their sides H E, H F, which lie between the equal angles, also equal to one another, are equal in every respect (5.). Therefore, the angle H L E is equal to the angle H K F, that is, to a right angle. But straight lines which are at right angles to the same straight line K L are parallel (14.). Therefore, A B is parallel to C D.

Next let A B be parallel to C D, and let them be cut by the same straight line G E F; the angles B E G, D F G, which are towards the same parts, shall be equal to one another.

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[*] That of Euclid (the famous twelfth axiom, see 15. *Cor.* 4.) is the converse of our eighth proposition, and asserts that, if the two interior angles made by two straight lines with a third be together less than two right angles, the two straight lines will meet one another, and, with the third, form a triangle, if produced far enough.

Simson's demonstration of this axiom rests upon an assumption, which is scarcely more evident than that of the text, viz., that if the perpendiculars which are drawn from two points of one straight line to another be equal, any other perpendicular, drawn from a point of the first to the other, shall be equal to either of them.

A modern geometer of great celebrity, M. Legendre, has, after more than one alteration, suggested, finally, an experimental proof not very different from that which is here adopted, as best suited to an Elementary Treatise. He is, notwithstanding, of opinion, that the grand truth with which it is so intimately connected, viz., that "the three angles of a triangle are together equal to two right angles," may be referred to the general principle of *homogeneity*. Of this it is unnecessary to say more in this place, than that it teaches us in the present instance, that the angles of a triangle depend, not upon the absolute magnitude of its sides, but upon their relative magnitude; so that a triangle whose sides are 3, 4, and 5 times some given line will have the same angles whether the given line be an inch or a mile; a truth which, indeed, seems to be nearly related to the more simple truth, that an angle is not increased by producing the sides which contain it, and leads directly to the theory of parallel straight lines. But, though the principle be of extensive application, the reasoning by which it is established has been shown to be incomplete, and such as, if great circumspection be not used, may even lead to fallacies.

Bisect E F, and draw the straight line L K, as before. Then, because L K is at right angles to C D, and that A B is parallel to C D, L K is at right angles also to A B (14.). And, because, in the right-angled triangles H E L, H F K, the hypotenuse H E, and adjacent angle E H L of the one, are equal to the hypotenuse H F, and adjacent angle F H K (3.) of the other, they are equal in every respect (13.). Therefore, the angle H E L is equal to the angle H F K, and B E G, which is equal to H E L (3.), is equal to D F G.[*]

Therefore, &c.

Cor. 1. When two straight lines A B, C D, are cut by a third, E F, the angle B E G, is called the *exterior angle*, and the angle D F G, the *interior and opposite angle upon the same side* of the line. Therefore, if one straight line fall upon two other straight lines, so as to make the exterior angle equal to the interior and opposite upon the same side, those two straight lines shall be parallel: and conversely.

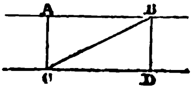
Cor. 2. The angles A E F, E F D, are called *alternate angles*. And A E F is always equal to B E G (3.). Therefore, if one straight line fall upon two other straight lines so as to make the two alternate angles equal to one another, those two straight lines shall be parallel: and conversely.

Cor. 3. The angles B E F, and D F E, are called *interior angles on the same side* of the line. Now, when B E G is equal to D F G, or D F E, the angles B E F and D F E are together equal to two right angles, because B E F and B E G, are together equal to two right angles (2.). Therefore, if one straight line, falling upon two other straight lines, make the two interior angles, on the same side, together equal to two right angles, the two straight lines shall be parallel: and conversely.

Cor. 4. (Euc. i. ax. 12.) If one straight line, falling upon two other straight lines, make the two interior angles, on the same side, together less than two right angles, the two straight lines shall meet upon that side, if produced far enough.

Prop. 16.

Parallel straight lines are every where equidistant; that is, if from any two points of the one, perpendiculars be drawn to the other, those perpendiculars shall be equal to one another.



Let A B, C D be two parallel straight lines; and from the points A, B, of A B, let A C, B D be drawn perpendicular to C D (12.): A C shall be equal to B D.

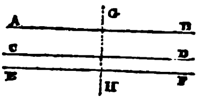
Join B C. Then because A C and B D are perpendicular to the same straight line C D, they are parallel (14.). Therefore, the alternate angles A C B, D B C, are equal to one another (15. *Cor.* 2.). Again, because A B is parallel to C D, the alternate angles A B C, D C B are equal to one another (15. *Cor.* 2.). Therefore the triangles A B C, D C B, having two angles of the one equal to two angles of the other, each to each, and the same line, B C, lying between the equal angles, are equal in every respect (5.). Therefore, A C is equal to B D.

Therefore, &c.

Cor. It appears from the demonstration, that if A C be only parallel to B D, A C and B D will be equal to one another. Therefore, the parts of parallel straight lines, which are intercepted by parallel straight lines, are equal to one another.

Prop. 17. (Euc. i. 30.)

Straight lines, which are parallel to the same straight line, are parallel to one another.



Let the straight lines A B, C D be each of them parallel to E F: A B shall be parallel to C D.

For, if the straight line G H be at right angles to E F, it will be at right angles to A B, because A B is parallel to E F (14.): and, for the like reason, it will be at right angles to C D: therefore, A B and C D being at right angles to the same straight line, G H, are parallel to one another (14.).

Therefore, &c.

Cor. Hence it appears that the quadrilaterals into which the parallelogram A B C D is divided in def. 23., by lines drawn parallel to two adjacent sides are likewise themselves parallelograms (def. 17.)

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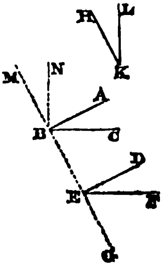
[*] A shorter demonstration may be had, by considering that if the angles B E G, D F G, be equal to one another, the two interior angles upon each side of E F will be together equal to two right angles; and, therefore, cannot be two angles of a triangle (8.), that is A B, C D cannot meet upon either side of E F; and hence the converse, because (14. *Cor.* 2.) only one parallel can be drawn through the same point to the same straight line. That which is given in the text, however, seems preferable, as pointing out the connexion of the proposition with Prop. 14., which immediately precedes it.

PROP. 18.

If, of two angles in the same plane, the sides of the one be parallel to the sides of the other, or perpendicular to the sides of the other, in the same order, the two angles shall be equal.



Let AB C, D E F be two angles in the same plane, having the sides AB, B C of the one parallel to the sides D E, E F of the other, each to each; and let the sides A B, D E, lie in the same direction from the sides B C, E F: the angle A B C shall be equal to the angle D E F.

Join B E, and produce it to G. Then, because A B, D E are parallel, and G B falls upon them, the exterior angle D E G is equal (15. *Cor.* 1.) to the interior and opposite angle A B E: and, for the like reason, F E G is equal to C B E: therefore, if these two equals be taken respectively from the former two, (ax. 3.) the remaining angle A B C will be equal to D E F.

Secondly, let H K L be an angle, the sides of which are perpendicular to those of the angle A B C, each to each, viz., H K to A B, and K L to B C: the angle H K L shall be equal to A B C. Draw B M perpendicular to BA (Post. 5.), and therefore (14.) parallel to K H, and B N perpendicular to B C, and therefore parallel to K L. Then, by the first part of the proposition, the angle M B N is equal to H K L. But, because the right angle M B A is equal to the right angle N B C (1.), and the part N A B common to both, the remaining angle M B N is equal to A B C (ax. 3.). Therefore, (ax. 1.) A B C is equal to H K L.

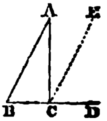
Therefore, &c.

Cor. The demonstration of the second case, viz. that in which the sides of the one angle are perpendicular to the sides of the other, requires only that the angle M B A be equal to N B C.

Therefore, if, of two angles, the sides of the one make equal angles with the sides of the other, respectively, in the same order, and towards the same parts, the two angles shall be equal.

PROP. 19. (EUC. i. 32.)

If one side of a triangle be produced, the exterior angle shall be equal to the two interior and opposite angles; and the three angles of every triangle are together equal to two right angles.



Let the side B C of the triangle A B C be produced to D: the exterior angle A C D shall be equal to the two interior and opposite angles at A and B; and the three angles of the triangle A B C shall be together equal to two right angles.

Draw CE parallel to BA (14. *Cor.* 1.). Then, because AC meets these parallels, the alternate angles A C E and A are equal; and because B D falls upon the same parallels, the angles E C D and B are equal (15. *Cor.* 1. and 2.). Therefore, the whole angle A C D, which is made up of the two angles A C E, E C D together, is equal to the angles at A and B together (ax. 2.).

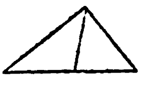
To each of these equals add the angle ACB: therefore, (ax. 2.) the three angles of the triangle A B C are together equal to the angles A C D, A C B, that is, to two right angles (2.).

Therefore, &c.

Cor. 1. If two triangles have two angles of the one equal to two angles of the other, their third angles will likewise be equal to one another.

Cor. 2. (Euc. i. 26, second part of.) Hence, if two triangles have two angles of the one equal to two angles of the other, each to each, and one side equal to one side, the equal sides being opposite to equal angles in each, the two triangles shall be equal in every respect (5.).

Cor. 3. In a right-angled triangle, the right angle is equal to the sum of the other two angles: and, conversely, if one angle of a triangle be equal to the sum of the other two, that angle shall be a right angle.

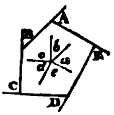


Cor. 4. In a right-angled triangle, the straight line, which is drawn from the right angle to the middle of the opposite side, is equal to half that side: and, conversely, if this be the case in any triangle, the angle from which the straight line is drawn shall be a right angle.

For, if the right angle be divided into parts equal respectively to the two acute angles, the triangle will be divided (6.) into two isosceles triangles. And, in the converse, the triangle being made up of two isosceles triangles, one of its angles is equal to the other two.

Prop. 20. (Euc. i. 32. *Corr.*1. and 2.)

All the exterior angles of any rectilineal figure are together equal to four right angles: and all the interior angles, together with four right angles, are equal to twice as many right angles as the figure has sides.



For, if from any point in the same plane, straight lines be drawn, one after the other, parallel to the sides of the figure, the angles contained by these straight lines about that point, will be equal to the exterior angles of the figure (18.), each to each, because their sides are parallel to the sides of the figure. Thus, the angles *a, b, c, d, e,* are respectively equal to the exterior angles *A, B, C, D, E.* But the former angles are together (3. *Cor.*) equal to four right angles; therefore, all the exterior angles of the figure are together equal to four right angles (ax. 1.).

Again, since every interior angle, together with its adjacent exterior angle, is equal to two right angles (2.); all the interior angles, together with all the exterior angles, are equal to twice as many right angles as the figure has angles. But all the exterior angles are, by the former part of the proposition, equal to four right angles; and the figure has as many angles as sides: therefore all the interior angles together with four right angles are equal to twice as many right angles as the figure has sides.

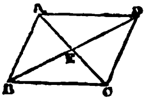
Therefore, &c.

Cor. The four angles of a quadrilateral are together equal to four right angles.

Section 4. *Parallelograms.*

Prop. 21. (Euc. i. 33.)

The straight lines which join the extremities of two equal and parallel straight lines towards the same parts, are likewise themselves equal and parallel.



Let the straight lines A D, B C be equal and parallel, and let them be joined towards the same parts by A B, C D: A B shall be both equal to C D, and parallel to it.

Join B D. Then, because in the triangles A D B, C B D, the two sides A D, D B are equal to the two C B, B D, each to each, and the angles A D B, C B D equal to one another, for they are alternate angles (15.), the side A B is *equal* to C D, and the angle A B D to C D B (4.); and, hence, because A B D, C D B are alternate angles (15.), A B is also *parallel* to C D.

Therefore, &c.

Prop. 22. (Euc. i. 34, first part of.)

The opposite sides and angles of a parallelogram are equal, and its diagonals bisect one another: and, conversely, if, in any quadrilateral figure, the opposite sides be equal; or if the opposite angles be equal; or if the diagonals bisect one another; that quadrilateral shall be a parallelogram.

Let A B C D be a parallelogram (see the last figure), and let its diagonals A C, B D cut one another in the point E: the sides A D, B C, as also, A B, C D, shall be equal to one another; the angles A and C, as also B and D shall be equal; and the diagonals A C, B D shall be bisected in E.

For, in the first place, that the opposite sides, as A D and B C, are equal, is evident, because they are parts of parallels intercepted by parallels (16. *Cor.*). Also, the opposite angles are equal, as at D and B; for the angle at D is equal to the vertical angle formed by C D, AD produced (3.), and the latter to the angle B (18.).

Lastly, with regard *to the bisection of the diagonals:* because A D is parallel to B C, the two triangles E A D, E C B have the two angles E A D, E D A of the one equal to the two angles E C B, E B C of the other, each to each (15.); and it has been shown, that the interjacent sides A D, B C are equal to one another; therefore, (5.) E A is equal to EC, and E D to E B, that is, A C, B D are bisected in E.

Next, let the opposite sides of the quadrilateral A B C D be equal to one another: it shall be a parallelogram. For, in the triangles A B D, C D B, the three sides of the one are equal to the three sides of the other, each to each; therefore, the angle A B D is equal to C D B (7.), and (15.) A B is parallel to C D. And, for the like reason, A D is parallel to B C.

Or, let the opposite angles be equal: then, because the angles at A and B together are equal to the angles at C and D together, and that the four angles of the quadrilateral (20. *Cor.*) are equal to four right angles, the angles at A and B are together equal to two right angles, and

(15. *Cor.* 3.) AD is parallel to BC. And, for the like reason, AB is parallel to CD.

Or, let the diagonals AC, BD, bisect one another in E: then, because the triangles EAD, ECB, have two sides of the one equal to two sides of the other, each to each, and the included angles AED, CEB, (3.) equal to one another, the angle EAD is (4.) equal to ECB, and, therefore, (15.) AD is parallel to BC. And, for the like reason, AB is parallel to CD.

Therefore, in each of the three cases, the figure is a parallelogram.

Therefore, &c.

Cor. 1. (Euc. i. 34. second part of.) A parallelogram is bisected by each of its diagonals; for the triangles into which it is divided are equal to one another.

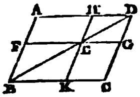
Cor. 2. The diagonals of a rhombus bisect one another at right angles. For the triangles into which it is divided by either of its diagonals are isosceles triangles, of which that diagonal is the base (6. *Cor.* 4.)

Cor. 3. (Euc. i. 46. *Cor.*) If one angle of a parallelogram be a right angle, all its angles will be right angles.

Cor. 4. By help of this Proposition more complete notions may be acquired of the rhombus, rectangle, and square: for, hence it appears, that a rhombus has *all* its sides equal to one another; that a rectangle has *all* its angles right angles; and that a square has *all* its sides equal, and *all* its angles right angles.

PROP. 23. (Euc. i. 43.)

The complements of the parallelograms, which are about the diagonals of any parallelogram, are equal to one another.



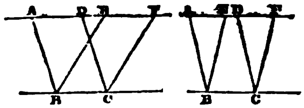
Let ABCD be a parallelogram, and through any point E in the diagonal BD let there be drawn the straight lines FG, HK parallel to the sides BC, DC respectively: the complement AE shall be equal to EC.

Because ABCD is a parallelogram, of which BD is a diagonal, the triangle ABD is (22. *Cor.* 1.) equal to CDB. In like manner, because FK, and HG (17. *Cor.*) are parallelograms, the triangles FBE, HED are equal to the triangles KEB, GDE: therefore, taking these equals from the former, there remains (ax. 3.) the complement AE equal to EC.

Therefore, &c.

PROP. 24. (Euc. i. 35.)

Parallelograms upon the same base and between the same parallels, ar equal to one another.



Let the parallelograms ABCD, EBCF, be upon the same base BC and between the same parallels AF, BC; the parallelogram ABCD shall be equal to the parallelogram EBCF.

Because AD and EF are each of them (22.) equal to BC, they are (ax. 1.) equal to one another. Therefore, the whole or the remainder AE is equal to the whole or the remainder DF (ax. 2. or 3.). Therefore, the two triangles EAB, FDC, having two sides of the one equal to two sides of the other, each to each, and the included angles EAB, FDC equal, are equal to one another (4. *Cor.*). Therefore, taking each of these equals from the whole figure ABCF, there remains (ax. 3.) the parallelogram EBCF equal to the parallelogram ABCD.

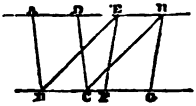
If the points D, E coincide, AE is the same with AD, and DF the same with EF; therefore, AE and DF, being each of them equal to BC, are equal to one another; and, hence, the triangle EAB is equal to the triangle FDC, and the parallelogram EBCF to the parallelogram ABCD, as before.

Therefore, &c.

Cor. Every parallelogram is equal to a rectangle of the same base and altitude.

PROP. 25. (Euc. i. 36.)

Parallelograms upon equal bases, and between the same parallels, are equal to one another.



Let the parallelograms ABCD, EFGH, be upon equal bases BC, FG, and between the same parallels AH, BG. The parallelogram ABCD shall be equal to the parallelogram EFGH.

Join EB, HC. Then because EH (22.) and BC are each of them equal to FG, they are equal to one another; and they are likewise parallel; therefore, EB and HC are also equal and parallel

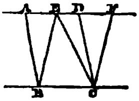
(21.), and E B C H is a parallelogram. But the parallelogram E B C H is equal to A B C D, because it is upon the same base BC, and between the same parallels (24.): and for the like reason E|B C H is equal to E F G H. Therefore (ax. 1.) A B C D is equal to E F G H.

Therefore, &c.

Cor. The squares of equal straight lines are equal to one another: and conversely.

Prop. 26. (Euc. i. 41.)

If a parallelogram and a triangle be upon the same base and between the same parallels, the parallelogram shall be double of the triangle.



Let the parallelogram A B C D and the triangle E B C be upon the same base B C, and between the same parallels A D, B C. The parallelogram A B C D shall be double of the triangle E B C.

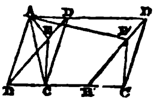
Complete the parallelogram E B C F.* (14. *Cor.* 1.) Then the parallelogram E B C F is double of the triangle EBC, because it is bisected by the diagonal E C (22. Cor.); and A B C D is equal to E B C F, because it is upon the same base, and between the same parallels (24.). Therefore, the parallelogram A B C D is also double of the triangle E B C.

Therefore, &c.

Cor. Every triangle is equal to the half of a rectangle of the same base and altitude.

Prop. 27. (Euc. i. 37, 38, 39, & 40.)

Triangles upon the same base, or upon equal bases, and between the same parallels, are equal to one another: and conversely, equal triangles, upon the same base, or upon equal bases in the same straight line, and towards the same parts, are between the same parallels.



The first part of the proposition is manifest; for the triangles are the halves of parallelograms (26.) upon the same base, or upon equal bases, and between the same parallels; and because these parallelograms are equal to one another (24. or 25.), the triangles, which are their halves, are also equal (ax. 5.).

In the second place, therefore, let the triangles A B C, D B C, standing upon the same base B C, or upon equal bases B C, B C, in the same straight line, and towards the same parts, be equal to one another; and let A D be joined: A D shall be parallel to B C.

For, if not, let A E (14. *Cor.* 1.) be parallel to B C; and let it meet D B in E. Join E C. Then, by the first part of the proposition, because A B C, E B C, are upon the same base, or upon equal bases, and between the same parallels, the triangle E B C is equal to A B C, that is, to D B C; the less to the greater, which is impossible. Therefore A E is not parallel to B C; and in the same manner it may be shown that no other straight line which passes through A, except A D only, can be parallel to B C; that is, (14. *Cor.* 1.) A D is parallel to B C.

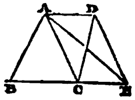
Therefore, &c.

Cor. 1. It is evident that the second (or converse) part of the proposition applies equally to parallelograms, as to triangles.

Cor. 2. If a quadrilateral be bisected by each of its diagonals, it must be a parallelogram: for the two triangles A B C, D B C (see the figure of Prop. 21.) which stand upon any one of its sides BC for a base, and which have their vertices in the side opposite, being equal, each of them, to half the quadrilateral, are equal to one another; and therefore A D is parallel to B C; and, for a like reason, A B is parallel to D C.

Prop. 28.

A trapezoid is equal to the half of a rectangle having the same altitude, and a base equal to the sum of its parallel sides.



Let A B C D be a trapezoid, having the side A D parallel to the side B C. The trapezoid ABCD shall be equal to the half of a rectangle having the same altitude, and a base equal to the sum of A D, B C.

Join A C; through D draw D E parallel to A C, and let it meet the base B C produced in E, and join A E.

Then, because D E is parallel to A C, the triangle A C D is (27.) equal to A C E. Therefore, the triangles A B C, A C D, together, are equal to the triangles A B C, A C E, together (ax. 2.); or, the trapezoid A B C D is equal to the triangle A B E. But the base of the triangle A B E is equal to the sum of A D, B C, because (22.) C E is equal to A D; and (26. *Cor.*) every triangle is equal to the half of a rectangle of the same base and altitude. Therefore, the trapezoid ABCD is equal to the half of a rectangle of the same altitude, and upon a base which is equal to the sum of A D, B C.

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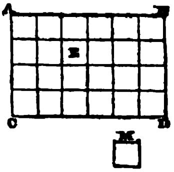
* To complete the parallelogram, in this case, it is only requisite that C F should be drawn through the point C parallel to B E, to meet A D produced in F. The word *complete*, indeed, almost explains itself: in future constructions it will be introduced without further notice.

Therefore, &c.

Prop. 29.

If the adjoining sides of a rectangle contain, each of them, the same straight line, a certain number of times exactly, the rectangle shall contain the square of that straight line, as often as is denoted by the product of the two numbers, which denote how often the line itself is contained in the two sides.



Let A B D C be a rectangle, and let its adjacent sides A B, A C, contain each of them the straight line M a certain number of times exactly, viz., A B 6 times, and A C 4 times: the rectangle ABDC shall contain the square of M, 6 × 4, or 24 times.

Divide A B, A C, each of them, into parts equal to M; and, through the division-points of each, draw straight lines parallel to the other, thereby dividing the rectangle into six upright rows of four parallelograms each, that is, upon the whole, into twenty-four parallelograms. Now these parallelograms are all of them rectangular, because their containing sides are parallel to AB, AC, the sides of the right angle A (18.). They are also equilateral: for any one of them, as E, has its upright sides each of them (22,) equal to a division of A C, that is, to M; and its other two sides each of them equal to a division of A B, that is, to M. Therefore, they are squares (def. 20.), equal, each of them, to the square of M. And they are twenty-four in number. Therefore, the square of M is contained twenty-four times in the rectangle A B D C.

The same may be said, if, instead of 6 and 4, any other two numbers be taken.

Therefore, &c.

Cor. 1. In like manner, it may be shown, that, if there be two straight lines, one of which is contained an exact number of times in one side of a rectangle, and the other an exact number of times in the side adjoining to it; the rectangle under those two straight lines shall be contained as often in the given rectangle, as is denoted by the product of the two numbers which denote how often the lines themselves are contained in the two sides

Cor. 2. The square of twice M is equal to 4 times M square, because it is a rectangle, in which each of the sides contains M twice. In like manner, the square of 3 times M is equal to 9 times M square—of 4 times M to 16 times M square—of 5 times M to 25 times M square, &c.

Cor. 3. The square of 5, or 25, is equal to the sum of 16 and 9. Consequently the square of 5 times M is equal to the square of 4 times M, together with the square of 3 times M.

Scholium.

From the theorems of this Section rules are easily deduced for the mensuration of rectilineal figures. For every rectilineal figure may be divided into triangles; and every triangle, being equal (26. Cor.) to half the rectangle under its base and altitude, contains half as many *square units* as is denoted by the product of the numbers which express how often the corresponding *linear unit* is contained in its base and in its altitude. Let this linear unit be, for example, a foot; and let it be required to find how many square feet there are in a triangle whose altitude is 10 feet, and its base 9 feet. The rectangle, of which these are the sides, contains 10 × 9, or 90 square feet (by Prop. 29.); and, therefore, the triangle contains 45 square feet.

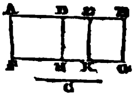
Hence, a rectangle is sometimes said to be equal to the *product* of its base and altitude, a triangle to half the *product* of its base and altitude, and the like; expressions which must be understood as above, the words *rectangle,* &c. *base,* &c. being briefly put for *the number of square units in the rectangle, &c. the number of linear units in the base, &c.* By the *length* of a line is commonly understood the number of linear units which it contains; and the term *superficial area,* or *area,* is similarly applied to denote the number of square units in a surface. With regard to the measuring unit, a less or a greater is convenient, according to the subject of measurement: a glazier measuring his glass by square inches, a carpenter his planks by square feet, a proprietor his land by acres, and a geographer the extent of countries by square miles.

SECTION 5. *Rectangles under the parts of divided lines.*

PROP. 30. (EUC. ii. 1.)

If there be two straight lines, one of which is divided into any number of parts, the rectangle contained by the two lines shall be equal to the sum of the rectangles contained by the undivided line, and the several parts of the divided line.

Let A B and C be two straight lines, of which A B is divided into the parts A D, D E, E B: the rectangle under C and A B shall be equal to the sum of the rectangles under C and A D, C and D E, C and E B.



Draw the straight line A F at right angles to A B, and equal to C (post. 5.): complete the rectangle A G, and through the points D and E draw the straight lines D H and E K parallel to A F (14. *Cor.*). Then, because D H and E K are each of them (22.) equal to A F, that is, to C, the rectangles A H, D K, E G are equal to the rectangles under C and A D, C and D E, C and E B. But these rectangles make up the whole rectangle A G, which is equal to the rectangle under C and A B. Therefore, the rectangle under C and A B is equal to the rectangles under C and A D, C and D E, C and E B.

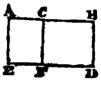
Therefore, &c.

Cor. (Euc. ii. 2.) If a straight line be divided into any two parts, the rectangles contained by the whole line and each of the parts, shall be together equal to the square of the whole line.

PROP. 31. (EUC. ii. 3.)

If a straight line be divided into any two parts, the rectangle contained by the whole line and one of the parts, shall be equal to the rectangle contained by the two parts, together with the square of the aforesaid part.

Let the straight line A B be divided into any two parts A C, C B: the rectangle under A B, B C shall be equal to the rectangle under A C, C B, together with the square of B C.



Draw the straight line B D at right angles to A B (post. 5.), and equal to B C: complete the rectangle A B D E and through C draw C F parallel to B D.

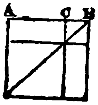
Then, the figure A F is equal to the rectangle A C, C B, because (22.) C F is equal to B D, that is to C B; and the figure C D is equal to the square of C B; and the figure A D to the rectangle A B, B C. But A D is equal to A F, together with C D. Therefore, the rectangle A B, B C, is equal to the rectangle A C, C B, together with the square of B C.*

Therefore, &c.

PROP. 32. (EUC. ii. 4.)

The square of the sum of two lines is greater than the sum of their squares, by twice their rectangle.

Let the straight line A B be the sum of the two straight lines A C, C B: the square of A B shall be greater than the squares of A C, C B, by twice the rectangle A C, C B.



Because the straight line A B is divided into two parts in the point C, (30. Cor.) the square of A B is equal to the sum of the rectangles under A B, A C, and A B, B C. But the rectangle under A B, A C (31.) is equal to the rectangle under A C, C B together with the square of A C; and, in like manner, the rectangle under A B, B C is equal to the rectangle under A C, C B together with the square of C B. Therefore, the square of A B is equal to twice the rectangle A C, C B, together with the squares of A C, C B; or, which is the same thing, the square of A B is greater than the squares of A C, C B, by twice the rectangle A C, C B.

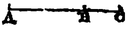
Therefore, &c.

The figure shews in what manner the square of AB may be divided into two squares equal to those of A C, C B, and two rectangles, each equal to the rectangle A C, C B.

PROP. 33. (EUC. ii. 7.)

The square of the difference of two lines is less than the sum of their squares by twice their rectangle.

Let the straight line, A B, be the difference of the two straight lines, A C, C B: the square of A B shall be less than the squares of A C, C B, by twice the rectangle A C, C B.



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* This proposition, being an obvious consequence of the preceding, might have been added to it as a second corollary: it is of so great importance, however, that it seemed preferable to force it upon the attention of the student, by placing it among the propositions.

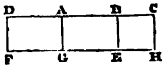
Because A C is the sum of A B and B C (32.), the square of A C is equal to the squares of A B, B C, together with twice the rectangle, A B, B C: therefore the square of A B is less than the square of A C by the square of B C, together with twice the rectangle A B, B C. Therefore, the same square of A B is less than the squares of A C, B C, by twice the square of B C, together with twice the rectangle A B, B C. But twice the square of B C, together with twice the rectangle A B, B C, is (32.) equal to twice the rectangle A C, C B. Therefore, the square of A B is less than the squares of A C, B C, by twice the rectangle A C, C B.

Therefore, &c.

Cor. (Euc. ii. 8.) The square of the sum of two lines is greater than the square of their difference by four times their rectangle: for the former square is greater than the sum of their squares by twice their rectangle (32.), and the latter square is less than the sum of their squares by twice their rectangle.

Prop. 34. (Euc. ii. 5.)

The difference of the squares of two lines is equal to the rectangle under their sum and difference.



Let A B, A C, be any two straight lines, and let B A be produced to D, so that A D may be equal to A B, and therefore C D equal to the sum, and C B to the difference of A B, A C: the difference of the squares of A B, A C, shall be equal to the rectangle under C D, C B.

Draw the straight line C H perpendicular to C D (post. 5.), and equal to C B; complete the rectangle C H F D; and, through the points A and B, draw A G and B E parallel to C H.

Then, because B E is (22.) equal to C H, that is, to B C, the rectangle A E is equal to the rectangle A B, B C; and, because A D is equal to A B, the rectangle D G is equal (25.) to the rectangle A E; and C E is the square of C B: therefore, the whole rectangle C F is equal to the square of B C, together with twice the rectangle A B, B C. But, because the square of A C is equal (32.) to the squares of A B, B C, together with twice the rectangle A B, B C; if the square of A B be taken from each side, (ax. 3.) the difference of the squares of A C, A B, is equal to the square of B C, together with twice the rectangle A B, B C. Therefore (ax. 1.) the difference of the squares of A C, A B, is equal to the rectangle C F, that is, to the rectangle under C D, C B.

Therefore, &c.

Prop. 35. (Euc. ii. 9. and 10.)

The squares of the sum, and of the difference of two lines, are together double of the squares of the two lines.

For (32.) the square of the sum of two lines is greater than the sum of their squares, by twice their rectangle, and the square of their difference is (33.) as much less than the sum of their squares. Therefore (ax. 9.), the square of the sum, together with the square of the difference, is equal to twice the sum of their squares.

Therefore, &c.

Scholium.

The theorems of this section admit of being enunciated more briefly and perspicuously by the use of certain conventional signs =, +, −, ×, ( ), &c. borrowed from Algebra.

That A is equal to B, is thus denoted: A = B, which is read "A is equal to B."

The sum of A and B thus: A + B, which is read "A plus B."

The excess of A above B : A − B...... "A minus B."

Twice A, four times A, &c.........2 A, 4 A, &c.

The rectangle under A and B....A × B, or A B........" A, B."

The square of A........A A, or A^2........ "A square;" and,

$(A + B - C)$ or $\overline{A + B - C}$ signifies that A + B − C (that is, the excess of the sum of A and B above C) is to be taken as a single quantity.

The theorems of this section may, therefore, be more briefly expressed as follows:—

Prop. 30. $A\times(B+C+D) = AB + AC + AD$.

Prop. 31. $(A+B)\times B = AB + B^2$.

Prop. 32. $(A+B)^2 = A^2 + B^2 + 2AB$.

Prop. 33. $(A-B)^2 = A^2 + B^2 - 2AB$.

Prop. 33. Cor. $(A+B)^2 - (A-B)^2 = 4AB$.

Prop. 34. $(A+B)\times(A-B) = A^2 - B^2$.

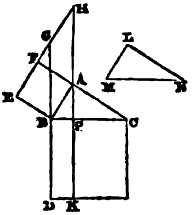
Prop. 35. $(A+B)^2 + (A-B)^2 = 2(A^2 + B^2)$.

In this borrowing of its notation, may be seen the first glimmerings of the application of *Algebra* to Geometry. It requires but a very slight acquaintance with the former science, to perceive at once, that, when the lines contain each of them some common part M a certain number of times exactly, these theorems are but so many examples of its rules of addition, subtraction, and multiplication. The perspicuity alone, however, which is displayed in the above expressions, will enable the uninitiated reader to form some notion of the advantages resulting from a more intimate union of the two sciences.

SECTION 6. *Relations of the sides of Triangles.*

PROP. 36. (EUC. i. 47. and 48.)

In every right-angled triangle, the square of the hypotenuse, or side opposite to the right angle, is equal to the sum of the squares of the sides which contain that angle: and conversely, if the square of one side of a triangle be equal to the sum of the squares of the other two sides, the angle contained by those two sides shall be a right angle.



Let A B C be a right-angled triangle, having the right angle, B A C: the square of B C shall be equal to the sum of the squares of B A, A C.

Upon B C, B A, describe the squares C D, A E; produce D B to meet E F, or E F produced, in G; and, through A, draw H K parallel to B D.

Then, because the sides E B, B G, of the angle E B G, are perpendicular respectively to the sides A B, B C, of the angle A B C, these two angles (18.) are equal to one another; and the right angle B E G is equal to B A C; therefore the two triangles E B G, A B C, having two angles of the one equal to two angles of the other, each to each, and the interjacent sides, E B, A B, equal to one another, are equal in every respect, and B G is equal to B C or B D (5.). And, because the parallelogram A G, and the square A E, are upon the same base, and between the same parallels (24.), A G is equal to A E. Again, because the parallelogram A G, and the rectangle K B, are upon equal bases, B G, B D, and between the same parallels (25.), A G is equal to K B. Therefore, also, A E, that is, the square of A B, is equal to the rectangle B K (ax. 1.). And, in like manner, it may be shewn that the square of [illegible] is equal to the A c rectangle K C. Therefore, the squares of A B, A C, together, are equal to the rectangles B K, K C together, that is, to the square of B C.

Next, let the square of B C be equal to the sum of the squares of B A, A C: the angle B A C shall be a right angle. For, let L M N be another triangle having its sides, L M, L N, equal to the sides A B, A C, respectively, and the angle at L a right angle. Then, by the former part of the proposition, the square of M N is equal to the squares of L M, L N, that is, to the squares of A B, A C, or to the square of B C, and M N is equal to B C (25. *Cor.*). Therefore, in the triangles A B C, L M N, the three sides of the one are equal to the three sides of the other, each to each; and, consequently, the angle B A C is (7.) equal to the angle M L N, that is, to a right angle.[*]

Therefore, &c.

Cor. 1. In a right-angled triangle, the square of either of the two sides is equal to the difference of the squares of the hypotenuse and the other side.

Cor. 2. It appears, from the demonstration, that if a perpendicular be drawn from the right angle to the hypotenuse, the square of either side is equal to the rectangle under the hypotenuse and segment adjacent to that side. And conversely, if this be the case, the angle at A must be a right angle. For, if the rectangle B C, B P, be equal to the square of B A; then, taking the square of B P from each (31. and 36.) the rectangle B P, P C, will be equal to the square of A P; and therefore, adding the square of P C to each, the rectangle B C, C P, will be equal to the square of A C: therefore the two rectangles, B C, B P, and B C, C P, together, that is (30. Cor.) the square of B C, will be equal to the squares of B A, A C, and the angle B A C will be a right angle (36.).

Cor. 3. And hence it follows that, if B A C be a right angle, the square of the perpendicular on the hypotenuse

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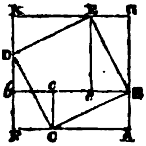
[*] It is evident that the sides of the squares upon A B, A C, which are opposite to A B, A C, would meet, if produced, in the point H. The latter part of the demonstration is, accordingly, equally applicable to shew that *if upon the two sides* A B, A C, *of any triangle* A B C *(right-angled or otherwise) any two parallelograms be described whose sides opposite to* A B, A C, *meet in a point* H, *and if, upon the base* B C *of the triangle, a parallelogram be likewise described, having its sides adjoining to the base equal and parallel to* A H, *the parallelogram upon the base shall be equal to the sum of the parallelograms upon the two sides.*

will be equal to the rectangle under the segments of the hypotenuse; and that, conversely, if this be the case, the angle B A C will be a right angle. (See the proof of *Cor.* 2.)

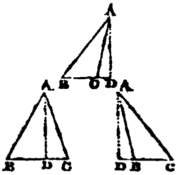
Scholium.

Among the different proofs which have been invented of this celebrated theorem, there is one of no little elegance, which has the advantage of pointing out in what manner the squares of the two sides may be dissected, so as to form, by juxta-position of their parts, the square of the hypotenuse. It will be readily apprehended from the following outline. B A C is a triangle, having the angle at A a right angle: upon the hypotenuse B C is described the square B C D E, and through the points D, E, straight lines are drawn parallel to B A, C A, to meet one another in K, and A C, A B produced, in the points F, G. Then it may be shewn that the four right-angled triangles, upon the sides of the square B D, are equal to one another; and, therefore together equal to twice the rectangle B A, A C: also, that the figure A F K G is a square, and equal to the square of B A+A C: and hence it easily follows that the square of B C is equal to the squares of B A, A C (32. and ax. 3.). To shew the dissection of the squares, E *e* and C *c* are drawn parallel to A B, to meet B *b*, which is drawn parallel to A C, in the points *c*, *e*: then C *b* is equal to the square of A B, and E *b* to the square of A C; and the former is divided into two, and the latter into three parts, which may be placed (as indicated by the divisions of B D,) so as to fill up the space B C D E, which is the square of B C.



PROP. 37. (EUC. ii. 12. and 13.)

In every triangle, the square of the side which is opposite to any given angle, is greater or less than the squares of the sides containing that angle, by twice the rectangle, contained by either of these sides, and that part of it, which is intercepted between the perpendicular let fall upon it from the opposite angle, and the given angle: greater, when the given angle is greater than a right angle, and less, when it is less.



Let A B C be any triangle, and C one of its angles; and from the angle A to the opposite side B C, or B C produced, let there be drawn the perpendicular A D: the square of A B shall be greater or less than the squares of A C, C B, by twice the rectangle B C, C D; greater, if C be greater than a right angle, and less if it be less.

When C is greater than a right angle, the opposite side, A B, and the perpendicular, A D, must lie upon different sides of A C; since, otherwise, in the triangle A C D, one of the angles would be a right angle, and another greater than a right angle, which is impossible (8.). For the like reason, when C is less than a right angle, the opposite side, A B, and the perpendicular, A D, must lie upon the same side of A C.

Therefore, according as C is greater or less than a right angle, the line B D will be the sum, or the difference of B C, CD and (32. and 33.) the square of B D will be greater or less than the squares of B C, C D by twice the rectangle B C, C D. Add to each the square of A D: therefore, the squares of B D, A D will be greater or less than the squares of B C, C D, A D, by twice the rectangle B C, C D. But the square of A B is equal (36.) to the squares of B D, A D, and the square of A C is equal to the squares of C D, A D. Therefore, the square of A B will be greater or less than the squares of B C, A C by twice the rectangle B C, C D.

Therefore, &c.

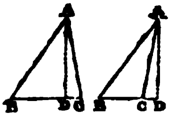
Cor. Any angle of a triangle is greater or less than a right angle, according as the square of the side opposite to it is greater or less than the squares of the sides by which it is contained.

Scholium.

When the point D coincides with C, there is no rectangle B C, C D, the angle C is a right angle, and we come to the conclusion of Prop. 36.: when D coincides with B, the rectangle BC, CD becomes the square of B C, the angle B is a right angle, and we come to the conclusion of Prop. 36. Cor. 1.

PROP. 38.

In every triangle, if a perpendicular be drawn from the vertex to the base, *or to the base produced; the difference of the squares of the sides shall be equal to the difference of the squares of the segments of the base, or of the base produced.*



Let A B C be a triangle, and from the vertex A to the base B C, or B C produced, let there be drawn the perpendicular A D: the difference of the squares of AB, A C, shall be equal to the difference of the squares of B D, C D.

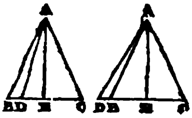
For the square of B D is as much greater than the square C D, as the squares of B D, A D together are greater than the squares of C D, A D together, that is (36.), as the square of A B is greater than the square of A C.

Therefore, &c.

We may remark that, if the triangle be isosceles, the segments of the base will be equal; and, that, in other cases, the greater segment of the base is always adjoining to the greater side. (12. *Cor.* 1. and 2.)

Prop. 39.

In an isosceles triangle, if a straight line be drawn from the vertex to any point in the base, or in the base produced, the square of this straight line shall be less or greater than the square of either of the two sides, by the rectangle under the segments of the base, or of the base produced.



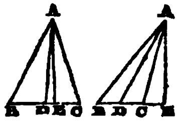
Let A B C be an isosceles triangle, having the side AB equal to the side A C, and from the vertex A to any point D in the base, or in the base produced, let there be drawn the line A D: the square of A D shall be less or greater than the square of A C, by the rectangle B D, D C.

Bisect BC in E, (post. 3.) and join AE. Then, DC is equal to the sum of DE and E C; and because B E is equal to E C, B D is equal to the difference of D E and E C. Therefore the rectangle B D, D C is (34.) equal to the difference of the squares of D E, E C. But the line AE is perpendicular to BC, because the base BC of the isosceles triangle is bisected in E (6. Cor. 3.). Therefore, (38.) the difference of the squares of DE, EC, is equal to the difference of the squares of AD, AC. Therefore (ax. 1.) the rectangle BD, DC, is equal to the difference of the squares of AD, AC; that is, the square of AD is less or greater than the square of A C, by the rectangle B D, D C.

Therefore, &c.

Prop. 40.

In every triangle the squares of the two sides are together double of the squares of half the base, and of the straight line, which is drawn from the vertex to the bisection of the base.



Let A B C be any triangle, and from the vertex A to D, the middle point of the base, let there be drawn the line A D. The squares of A B, A C, shall be together double of the squares of B D, D A.

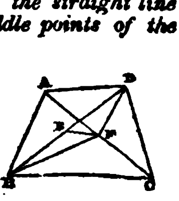
From the point A to B C, or B C produced, draw the perpendicular A E (12.). Then BE is equal to the sum of BD, DE; and, because B D is equal to D C, E C is equal to the difference of B D, D E. Therefore, (35.) the squares of B E, E C are together double of the squares of BD, DE. And the square of EA, taken twice, is double of the square of E A. Therefore the squares of B E, E A, C E, E A, are together double of the squares of B D, D E, E A; that is (36.), the squares of B A, A C are together double of the squares of B D, D A.

When A C B is a right angle, the perpendicular A E coincides with the side A C. Therefore B D is equal to D E, and the square of B E is double of the squares of BD, DE (29. *Cor.* 2.); also the square of BE (36. *Cor.* 1.) is equal to the difference of the squares of B A, A E, that is, of B A, A C; and hence the squares of B A, A E, that is, of B A, A C, are together double of the squares of B D, D A, as before.

Therefore, &c.

Prop. 41.

The squares of the four sides of a quadrilateral are together greater than the squares of its diagonals, by four times the square of the straight line which joins the middle points of the diagonals.



Let A B C D be a quadrilateral, and let its diagonals A C, B D be bisected in the points F, E: the squares of A B, B C, C D, D A, are together greater than the squares of A C, B D, by four times the square of E F.

Join B F, F D. Then, because the base A C of the triangle B A C is bisected in F, the squares of A B, B C are (40.) equal to twice the squares of B F, A F: and for the like reason the squares of C D, D A are equal to twice the squares of D F, A F: therefore, the squares of A B, B C, C D, D A are together equal to twice the squares of B F, D F, together with four times the square of A F. But, because the base BD of the triangle FBD is bisected in E, the squares of BF, DF, taken twice, are (40.) equal to four times the squares of B E, E F: therefore the squares of A B, B C, C D, D A are together equal to four times the squares of A F, B E, together with four times the square of EF; that is, to the squares of A C, B D, (39. *Cor.* 2.) together with four times the square of E F.

Therefore, &c.

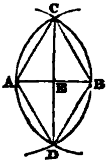
Cor. In a parallelogram, and in a parallelogram only, the points E and F coincide, because (22.) the diagonals bisect one another: therefore, in a parallelogram, and in a parallelogram only, the squares of the diagonals are together equal to the squares of the four sides.

SECTION 7.—*Problems.*

Upon a piece of paper for a plane, with a pen,* a ruler, and a pair of compasses, it is evident, that, first, a straight line may be drawn from any one point to any other point; 2ndly, a terminated straight line may be produced to any length in a straight line; 3dly, from the greater of two straight lines, a part may be cut off equal to the less; and 4thly, a circle may be described from any centre, and with any distance from that centre. Operations so simple require no further notice.

PROP. 42. Prob. 1. (EUC. I. 1.)

To describe an equilateral triangle upon a given finite straight line A B.



From the points A, B, as centres, with the common radius A B, describe two circles, intersecting one another in C, and join C A, C B.

Then, because C A, C B are each of them equal to A B (def. 24.), they are (ax. 1.) equal to one another, and the triangle C A B is equilateral. Therefore, an equilateral triangle has been described upon the given finite straight line AB, which was required to be done.

PROP. 43. Prob. 2. (EUC. I. 10.)

To bisect a given finite straight line A B.

Upon either side of A B (42.) describe an equilateral triangle: join the vertices or summits C,D, and let CD cut AB in E.

Then, because the triangles C A D, C B D have the three sides of the one equal to the three sides of the other, each to each, (7.) they are equal in every respect, and the angle A C D is equal to B C D. Therefore, the triangles A C E, B C E, having two sides of the one equal to two sides of the other, each to each, and the included angles equal to one another, are equal in every respect (4), and A E is equal to E B.

Therefore, &c.

Cor. By the same construction, a straight line may be drawn, which shall bisect any given straight line at right angles.

N.B. It is evident, from 6. *Cor.* 4., that the same end will be obtained by joining the point of intersection C of any two equal circles described from the centres A, B, and cutting one another above A B, with the point of intersection D of any other two equal circles described from the same centres, and cutting one another either above or below A B: for C, D will be the vertices of isosceles triangles upon the same base. This observation may be of use when one of the triangles is already described, as in the first method of Prob. 4.

PROP. 44. Prob. 3. (EUC. I. 11.)

To draw a straight line at right angles to a given straight line A B, *from a given point* C *in the same.*

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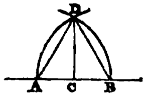
* Although the paper be not an exact plane, nor the pen such as may serve to draw an exact line, these defects admit of being removed to any required degree, and do not, in the least, affect the accuracy of our conclusions, with regard to exact lines and planes.

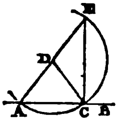
An edge which is a right line, or nearly so, (because it is the common section of two planes, see Book IV.) may be obtained by doubling over a piece of paper upon itself, and a right angle (def. 10.) by doubling over this edge upon itself. These are both useful upon occasion, especially the right angle, which is so frequently required in geometrical constructions that a case of instruments is commonly provided with one. Among practical mechanics it is known under the name of *the square*. Parallel lines also occur so frequently, that it is convenient to have a ruler expressly for drawing them, called a *parallel ruler*. This may be made of two or three different forms, the best of which is that of a flat ruler running upon two equal rollers.

First method. In C A take any point A, and make C B equal to C A: upon A B describe (42.) the equilateral triangle D A B, and join D C.

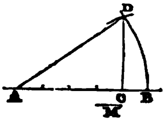


Then, because the triangles D C A, D C B have the three sides of the one equal to the three sides of the other, each to each, (7.) the angle D C A is equal to D C B; and they are adjacent angles; therefore each of them is a right angle (def. 10.), and D C is at right angles to A B.

Second method. Take any point D which is not in A E, and from the centre D, with the radius D C describe the circle A C E, cutting the line *AB* a second time in the point *A*: join AD, and produce it to meet the circumference in E, and join E C. Then because in the triangle E C A, the line D C drawn from the angle C to the middle of the opposite side equal to half that side, the angle C is (19. *Cor.* 4.) a right angle, and C E is at right angles to A B. If the points A, C, coincide, D C is (12. *Cor.* 3.) at right angles to A B.



Third method. Take any straight line M, and make C A equal to four times M: from the centre C, with a radius equal to three times M, describe a circle: from the centre A, with a radius equal to five times M, describe a second circle, cutting the former in D, and join D C, D A. Then, because the square of five times M is (29. *Cor.* 3.) equal to the square of four times M, together with the square of three times M, the square of A D is equal to the squares of A C, C D; and, therefore, (36.) the angle A C D is a right angle, or D C is at right angles to A D.



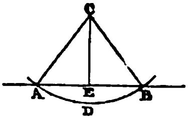
Therefore, &c.

The last two methods are particularly convenient, when the given point C is near the edge of the paper.

PROP. 45. Prob. 4. (EUC. i. 12.)

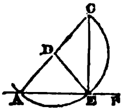
To draw a straight line at right angles to a given straight line A B, *from a given point* C *without it.*

First method. Take any point D upon the other side of A B, and join C D: from the centre C, with the radius C D, describe a circle cutting A B in the points A, B: bisect (43.) AB in E, and join C E.

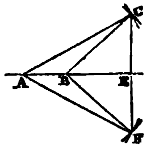


Then, because the triangles C E A, C E B have the three sides of the one equal to the three sides of the other, each to each, (7.) the angle C E A is equal to C E B, and (def. 10.) C E is at right angles to A B.

Second method. Draw from C to A B any straight line C A: bisect C A (43.) in D: from the centre D, with the radius D A or D C, describe a circle, cutting A B in a second point E, and join C E. Then, for the same reason, as in the second method of the preceding problem, C E is at right angles to A B. Also, if the points A, E coincide, C A is at right angles to A B.



Third method. In A B take any two points A, B, and from A, B, as centres, with the radii A C, B C, describe circles cutting one another a second time in F: join CF, and let C F cut A B in E. Then, because A C F, B C F, are isosceles triangles upon the same base CF, the line AB which joins their summits, bisects the base at right angles (6. *Cor.* 4.); that is, C E is at right angles to A B.



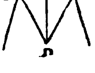
Therefore, &c.

When C A is equal to C B, (which may happen, when the points A, B are on different sides of E,) the last of these three methods is, in practice, nearly the same as the first. The last two methods are applicable to the case, in which the point C falls near the edge of the paper.

PROP. 46. Prob. 5. (EUC. i. 9.)

To bisect a given rectilineal angle B A C.

In A B take any point B: make A C equal to A B: join B C: upon B C describe (42.) the equilateral triangle B D C, and join A D. Then, because the triangles A B D, A C D have the three sides of the one equal to the three sides of the other, each to each, (7.) the angle B A D is equal to the angle C A D.



Therefore, &c.

Cor. By repeating this process with the halves, quarters, &c. of the given angle, it may be divided into four, eight, &c., equal parts.

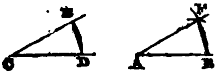
Scholium.

There is no construction in *Plane* Geometry, *i. e.* no construction practicable with the assistance of the right line and the circle only, by which a given angle may be divided into three, five, &c. equal parts. It is true that, by the description of other lines which are plane curves, (as the hyperbola, conchoïd, and cissoïd,) any given angle may be trisected, or divided into three equal parts; but such constructions, not being effected by the two species of lines which are the subject of these Elements, are for certain reasons (to be found in the application of Algebra to Geometry) said to be out of the range of *Plane* Geometry. There are, however, some particular cases in which, by means of the circle, an angle may be divided into three, five, and other numbers of equal parts. Such is the division of a right angle into three equal parts. In this case we are already aware, that, since an equilateral triangle has its three angles equal to one another, and the three taken together equal to two right angles, each of them is two thirds of a right angle. The problem, therefore, will be solved by describing an equilateral triangle upon either of the legs, taken of any length at pleasure. In like manner, by means of an isosceles triangle, each of whose base-angles is double of the vertical angle, (see book iii.) and is, therefore, four-fifths of a right angle, *Plane* Geometry enables us to divide a right angle into five equal parts. But these are only particular cases, and indicate no general process.

PROP. 47. Prob. 6. (EUC. i. 23.)

At a given point A, *in a given straight line* A B, *to make an angle equal to the given rectilineal angle* C.

From the centre C, with any radius, describe a circle, cutting the sides of the given angle in the points D E, and

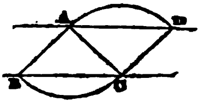


join D E: from the centre A, with a radius equal to C E or C D, describe a circle, cutting A B in B, and from the centre B with a radius equal to D E, describe a circle cutting the last circle in F; and join F A, F B. Then, because the triangles E C D, F A B have the three sides of the one equal to the three sides of the other, each to each, (7.) they are equal in every respect, and the angle at A is equal to the angle at C.

Therefore, &c.

PROP. 48. Prob. 7. (EUC. I. 31.)

Through a given point A, *to draw a straight line parallel to a given straight line* B C.



From A to B C draw any straight line A C, and at the point A make the angle C A D equal to A C B (as directed in Prop. 47., or as in the adjoined figure): then, because these angles are alternate angles, A D is (15. *Cor.* 2.) parallel to B C.

Therefore, &c.

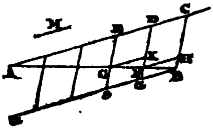
N. B.—In the figure, equal circles are, in the first place, described from the centres A, C, the former cutting B C a second time in B; and the point D is then determined by describing a circle from the centre A with the radius B C.

This construction is equally short, in practice, with that of Prop. 47.; and when it is required to obtain the point D at as great a distance as possible from A, may be preferred, the line A C being drawn more obliquely for that purpose.

PROP. 49. Prob. 8. (EUC. vi. 9).

To divide a given straight line, A B, *into any number of equal parts.*

Let it be required to divide A B into five equal parts. Draw A C, making any angle with A B, and taking any distance M, set it off upon A C five times from A to C: join C B, and through the points of division in A C, viz. D, E, &c. draw (48.) parallels to C B, cutting A B in the points F, G, &c. A B shall be divided at the latter points into five equal parts.



Through F and G draw (48.) F H and G K, each of them parallel to A C. Then, because F C is a parallelogram, F H is equal to D C (22.); and for the like reason G K is equal to E D: but E D, D C, are equal to one another; therefore (ax. 1.), F H is equal to G K. But F H, G K, are sides of the triangles F H B, G K F, which have their angles equal to one another, each to each, because their sides are parallel (18.): therefore (5.), their other sides are also equal to one another, and F B is equal to G F. In like manner it may be shown, that each of the other five parts into which AB is divided, is equal to FB or GF.

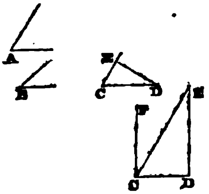
And it is evident that, by a similar process, A B may be divided into any other number of equal parts.

Therefore, &c.

It is convenient to draw the parallels D F, E G, &c. by making the angle A B *a*, equal to B A C (47.), taking B *d*, *d e*, &c. each equal to M, and joining corresponding points D *d*, E *e*, &c. For the straight lines which join the extremities of equal and parallel straight lines, are also themselves equal and parallel (21.). A similar process may be recommended, whenever a number of parallels are to be drawn without the assistance of a parallel ruler.

PROP. 50. Prob. 9. (Euc. i. 22.)

To construct a triangle from three parts given, of which one, at least, is a side.

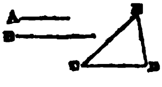


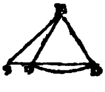
Case 1. Let the given parts be two angles and a side; viz. the angles A B, and the side C D.

If the given side is to be *interjacent* or to lie between the given angles, make (47.) at the points C, D, the angles D C E, C D E, equal to A, B, respectively: and the triangle E C D will evidently be the triangle required.

But if C D is to be *opposite* to one of the given angles, as B, make at the point C, the angles D C E, E C F, equal to A, B, respectively, and through D, draw D E parallel to C F (48.): then the angle C E D is (15. Cor. 2.) equal to the alternate angle E C F, that is, to B, and E C D is the triangle required.

Case 2. Let the given parts be two sides and an angle; viz. the sides A, B, and the angle C.

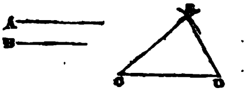




If the given angle is to be *included*, take C D, C E, upon its sides, equal to A, B, respectively, and join D E; the triangle C D E will evidently be the triangle required.

But if C is to be *opposite* to one of the given sides as A, take C E equal to B as before: from the centre E, with a radius equal to A, describe a circle cutting C D in the points D, D; and join E D, ED. Then, if the points D, D fall both on the same side of C with the angle, each of the triangles E C D will satisfy the given conditions; but if they fall upon different sides, only one of them, as E C D, will satisfy those conditions, and therefore that one will be the triangle required.

If the points D, D, being upon the same side with the angle, coincide with one another, which will happen when E D C is a right angle (12. *Cor.* 3.), there is only one triangle E D C, which is the triangle required.



Case 3. Let the given parts be three sides, viz. A, B, and C D. From the centres C, D, with radii equal to A, B, respectively, describe circles intersecting in E, and join E C, E D: then E C D is evidently the triangle required.

Therefore, &c.

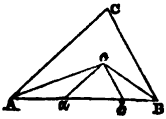
Scholium.

In each of these three cases we have supposed the *data*, or given parts, to be such that the problem is possible: and the same will be supposed in all future problems. Many of them, however, will be found to be of that description, that, if the relations of the data be not confined within certain *limits*, the required solutions will be impossible. In problems of a more complex nature than the present, the determination of these limits is facilitated by the consideration of *loci*, to be noticed hereafter (in Book III. Sect. 6), and which may here be explained as circumscribing the range, if it be limited, within which every particular *datum* confines the solution of the problem; for it is obvious that, if a second datum cannot be satisfied within that range, the two will be inconsistent, and the solution impossible.

In the mean time, it may be observed, the limits of the data are in many cases readily suggested by the known properties of the figure, or by the necessary construction of the problem. It is evident, for instance, that in every case of the problem before us, there are certain limits, beyond which the solution will be impossible. This will happen in the first case if the two given angles should be together equal to or greater than two right angles, for there can be no triangle having two of its angles together equal to or greater than two right angles (8.). The problem will be impossible in the second case, if when the side A is to be opposite to the given angle C, it should be less than the perpendicular let fall from E upon C D (12. Cor. 4.); or again, if the given angle being right, or obtuse, the side A, which is to be opposite to it be less than the side B, for no triangle can have the less side opposite to the greater angle (9.). And in the third case, the problem will be impossible, if every two of the given sides be not greater than the third side (10.).

Prop. 51. Prob. 10.

Given the three angles of a triangle (together equal to two right angles), and the perimeter A B, *to construct the triangle.*

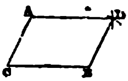


At the points A, B, make (47.) the angles BAC, ABC, equal to two of the given angles, each to each; bisect (46.) the angles at A, B, by the straight lines A *c*, B *c*, meeting in *c*, and through *c* draw (48.) *ca*, *cb*, parallel to C A, C B, respectively: the triangle ABC shall have its angles equal to the three given angles, and its perimeter equal to the given perimeter A B. The first is evident, for its sides are parallel to the sides of the triangle C A B, which has its angles at A and B, and therefore at C, equal to the given angles (18.). Again, because *c a* is parallel to C A, the angle *ac*A is equal (15.) to the alternate angle C A *c*, that is to the angle *a* A *c*, and therefore (6.) *ac* is equal to *a*A; and in the same manner it may be shown that *bc* is equal to *b* B; therefore, the three sides of the triangle *a b c* are together equal to A B, that is, to the given perimeter.

Therefore, &c.

Prop. 52. Prob. 11.

Given two sides and the included angle of a parallelogram, to construct the parallelogram.



Let A C, B C, be the two given sides, and C the given angle. From the centres A, B, with radii equal to B C, A C, respectively, describe circles intersecting in D, and join D A, D B.

Then because the quadrilateral A D has its opposite sides equal to one another, it is a parallelogram (22.); and it has two sides and the included angle the same with those given; therefore, it is the parallelogram required.

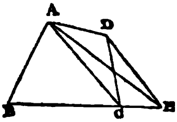
Therefore, &c.

N. B. The same end will be obtained by drawing parallels to C B, C A through the points A, B.

It may be observed that if A C be equal to B C, the parallelogram is a rhombus; if A C be at right angles to B C, it is a rectangle; and if A C be both equal to B C, and at right angles to it, it is a square. This problem, therefore, includes the following as particular cases; 1st, to describe a rhombus with a given side and angle; 2d, to describe a rectangle with two given sides; and 3d, to describe a square upon a given finite straight line.

Prop. 53. Prob. 12.

To describe a triangle which shall be equal to a given quadrilateral A B C D, *and shall have a side and angle adjacent to it, the same with a given side* A B *and adjacent angle* B *of the quadrilateral*



Join A C: through D (48.) draw D E parallel to A C, to meet B C produced in E, and join A E.

Then, because D E is parallel to A C, the triangle A C E is equal to A C D (27.): therefore A B C, A C E together are equal (ax. 2.) to A B C, A C D together: that is, the triangle A B E is equal to the quadrilateral A B C D; and it has the same side A B, and adjacent angle B with the quadrilateral.

Therefore, &c.

Prop. 54. Prob. 13.

To describe a triangle which shall be equal to a given rectilineal figure A B C D E F, *and shall have a side and adjacent angle the same with a given side* A B *and adjacent angle* B *of the figure.*

Join A C, A D, A E: through F draw F G parallel to A E (48.), to meet D E produced in G: through G draw G H parallel to A D, to meet C D produced in H; through H draw H K parallel to H C, to meet B C produced in K, and join A K, A H, A G.*

Then because the triangle A D G is

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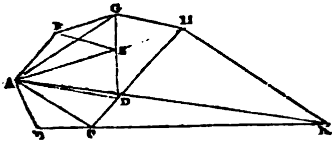
* A H is not joined in the figure, to avoid confusion: for the same reason, A M and A L are not joined in the corresponding figure of Prop. 55.

equal to the quadrilateral A D E F (53.), and that the triangle A D H is equal to A D G (27.), the triangle A D H is equal to the quadrilateral A D E F. To each of these equals add the quadrilateral A B C D; therefore the quadrilateral A B C H is equal to the whole figure A B C D E F. But again (53.) the triangle A B K is equal to the quadrilateral A B C H · therefore the triangle A B K, is equal to the figure A B C D E F, and it has the same side A B, and adjacent angle B with the figure.



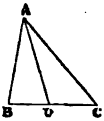
Therefore, &c.

PROP. 55. Prob. 14.

To bisect a given triangle or rectilineal figure by a straight line drawn from a given angle.

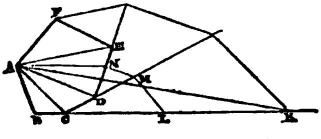
First, let it be required to bisect the triangle A B C by a straight line to be drawn from the angle A.

Bisect (43.) B C in D, and join A D. Then because B D is equal to D C, the triangle A B D is (27.) equal to A D C, and the triangle A B C is bisected by the straight line A D.



Next, let it be required to bisect the rectilineal figure A B C D E F by a straight line, to be drawn from the angle A.

With the same side A B and angle B, *describe* (54.) the triangle A B K, equal to the given rectilineal figure: bisect (43.) B K in L: through L draw (48.) L M parallel to A C, to meet C D produced in M: through M draw M N parallel to A D, to meet D E in N, and join A N, A M, A L. Then, it may be shown, as in the last proposition, that the triangle A B L is equal to the figure A B C D N; but A B L is equal to half the triangle A B K, that is to half of the given figure; therefore the figure A B C D N, is also equal to half of the given figure, and the latter is bisected by the straight line A N.



Therefore, &c.

Cor. 1. By a similar construction, any part required, for example a fifth, may be cut off from a given triangle or rectilineal figure, by a straight line drawn from one of its angles. For in the case of the triangle, if the base B C be divided into five equal parts in the points D, &c. the triangles A B D, &c. will be equal to one another (27.); and therefore any one of them, as A B D, will be equal to one fifth of the given triangle. And hence the passage is easy to the division of the rectilineal figure; for it is evident (53.) that, whatever be the point L taken in A K, the figure A B C D N, constructed as above, will be equal to the triangle A B L, and will therefore be a fifth part of the given figure, if A B L be a fifth of the triangle A B K.

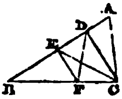
It is manifest in this case, that, if the given fractional part be such, that B L is less than B C, the problem will be solved at once by joining A L.

Cor. 2. And hence it appears in what manner a triangle or rectilineal figure may be divided into any number of equal parts by straight lines drawn from one of its angles.

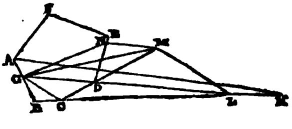
PROP. 56. Prob. 15.

To bisect a given triangle or rectilineal figure by a straight line drawn from a given point in one of its sides.

First, let it be required to bisect the triangle A B C, by a straight line to be drawn from the point D in the side A B. Join D C: bisect (43.) A B in E: through E draw (48.) E F, parallel to DC, and join DF, EC. Then, because D C is parallel to E F, the triangle D E F is (27.) equal to C E F; therefore D E F, B E F together, are equal to C E F, B E F together, that is, the triangle D B F, is equal to the triangle C E B, or (27.) to half of the triangle A B C. Therefore, A B C is bisected by the straight line D F.



Next, let it be required to bisect the figure A B C D E F, by a straight line, to be drawn from the point G in the side A B



Describe the triangle A B K (54.) equal to the given figure, and, as already shewn, bisect it by the straight line G L, drawn from the point G: join G C, G D, &c.; through L draw (48.) L M, parallel to G C, to meet C D produced in M: through M draw M N parallel to G D to meet D E in N, and join G N, G M, G L.* Then, as in the last proposition, it may be shown that the figure G B C D N is equal to the triangle G B L, that is to half of the given figure; therefore the latter is bisected by the straight line G N.

Therefore, &c.

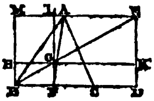
Cor. 1. By a similar construction, any part required may be cut off from a given triangle or rectilineal figure, by a straight line drawn from a given point in one of its sides. For we have but to make C B E the same part of C B A in the first case, and G B L the same part of A B K in the case of the rectilineal figure, and proceed as before.

If the fractional part be such that B L is less than B C, the problem will be solved at once by joining G L.

Cor. 2. And hence a given triangle or rectilineal figure may be divided into any number of equal parts, by straight lines drawn from a given point in one of its sides.

Prop. 57. Prob. 16. (Euc. i. 44.)

Upon a given base B D *to describe a rectangle which shall be equal to a given triangle* A B C.



From the point D draw D E, at right angles to B D (44.): through A draw A E parallel to B D, to meet D E in E (48.): join E B: bisect B C in F (43.): through F draw F G, parallel to D E, to meet B E in G: through G draw H K parallel to B D, and complete the rectangle D H: D H shall be the rectangle required.

Complete the rectangles D L, D M. Then, because G M, G D are complements of the rectangles F H, K L, which are about the diagonal of the rectangle D M, G D is (23.) equal to G M. Therefore, adding F H to each, the whole H D is equal to the whole M F, that is, (26.) to twice the triangle A B F, or to the triangle A B C; and H D is described upon the given base B D.

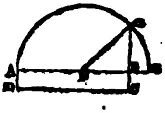
Therefore, &c.

Cor. 1. Hence, upon a given base, a rectangle may be described which shall be equal to a given rectilineal figure: for a triangle (54.) may be described equal to the figure, and a rectangle equal to the triangle.

Cor. 2. It is evident that, with a like construction, a parallelogram may be described upon a given base B D which shall be equal to a given triangle A B C, and shall have one of its angles equal to a given angle B D E.

Prop. 58. Prob. 17. (Euc. ii. 14).

To describe a square which shall be equal to a given rectangle A B C D.



Produce A B to E, so that B E may be equal to B C: bisect A E in F (43.): from the centre F, with the radius F A or F E, describe a circle, and produce C B to meet the circumference in G: the square of B G shall be equal to the rectangle A B C D.

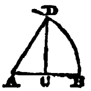
Join F G, then, because F A is equal to F E, A B is equal to the sum, and B E to the difference, of F E, F B: therefore (34.) the rectangle A B, B E, that is, the rectangle A B C D, is equal to the difference of the squares of F E, F B, that is to the difference of the squares of F G, F B, or (36. *Cor.* 1.) to the square of B G.

Therefore, &c

Cor. Hence a square may be described which shall be equal to a given rectilineal figure (57. *Cor.* 1.).

Prop. 59. *Prob.* 18,

To describe a square which shall be equal to the difference of two given squares, viz. the squares of A B, A C



From the point C draw (44.) C D perpendicular to A B, and from the centre A with the radius A B, describe the circle B D cutting C D in D: the square of C D shall be equal to the difference of the squares of A B, A C. For (36. *Cor.* 1.) the square of C D is the difference of the squares of A D, A C, of which A D is equal to A B.

Therefore, &c.

Cor. Hence a square may be described, which shall be equal to the difference of two given rectilineal figures (58. *Cor.*).

Prop. 60. Prob. 19.

To describe a square which shall be

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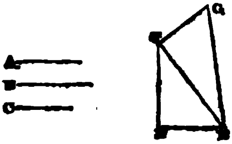
* G D and G L do not necessarily coincide, as in the figure.

equal to the sum of two, three, or any number of given squares, viz. the squares of A, B, C, *&c.*

Take D E equal to A; from the point D, draw D F (44.) at right angles to D E, and equal to B: join E F: from F, draw F G at right angles to E F, and equal to C; and so proceed for the rest of the given squares. Join E G. Then, because E D F is a right angled triangle, the square of E F is equal to the squares of E D, D F (36.), that is, to the squares of A, B: again, because E F G is a right angled triangle, the square of E G is equal to the squares of E F, F G, that is, to the squares of A, B, C; and so on.



Therefore, &c.

Cor. Hence a square may be described, which shall be equal to the sum of two, three, or any number of given rectilineal figures (58 *Cor.*).

BOOK II.

§ 1. *Ratios of Commensurable Magnitudes*—§ 2. *Proportion of Commensurable Magnitudes*—§ 3. *General Theory of Proportion*—§ 4. *Proportion of the sides of Triangles*—§ 5. *Proportion of the surfaces of Rectilineal Figures*—§ 6. *Properties of Lines divided Harmonically*—§ 7. *Problems.*

SECTION 1. *Ratios of Commensurable Magnitudes.*

In the language of Mathematics, the Latin word *ratio* has been adopted to express what is more generally understood by the term proportion: thus, instead of "the proportion which" one thing bears to another, we say "*the ratio* which" one bears to the other, meaning its comparative magnitude—instead of saying that A is to B "in the proportion of 5 to 6," we say "in the ratio of 5 to 6.

The word *proportion* has on the other hand been appropriated to express *the equality of ratios*, as hereafter defined; or, as it may be here less minutely explained, the case in which one magnitude is as many times greater or less than another, as a third magnitude is greater or less than a fourth.

The ratio of one magnitude to another is independent of the *kind* of magnitudes compared; for it is obvious that one may contain the other, or the sixth, or twelfth, or hundredth part of the other the same number of times, whether they be lines, or surfaces, or solids, or again, weights, or parts of duration.

It is required only for the comparison we speak of, that the magnitudes be of the same kind, containing the same magnitude, each of them, a certain number of times, or a certain number of times nearly. Upon these numbers, and upon these only, the ratio depends.

Hence it appears that this theory pertains in truth to Arithmetic. The use of Proportion is, however, so indispensable in Geometry, that it has been usual, either to introduce its theorems into the body of the science, after the example of Euclid, or to premise a few of the most important of them as a manual for reference. In the present, and two following sections, the subject will be discussed at a length commensurate with its importance.

Def. 1. When one magnitude is compared with another of the same kind, the first is called the *antecedent*, and the second the *consequent*.

2. One magnitude is said to be a *multiple* of another, when it contains that other a certain number of times exactly: and the other magnitude, which is contained in the first a certain number of times exactly, is said to be a *submultiple*, or *measure*, or *part* of the first.

Hence, also, one magnitude is said to *measure* another when it is contained in the other a certain number of times exactly.

3. Two magnitudes are said to be *equimultiples* of two others, when they contain those others the same number of times exactly: and the other magnitudes which are contained in the first the same number of times exactly, are said to be *like parts* of the two first.

Thus, 7 A, 7 B, are equimultiples of A, B; and A, B, are like parts of 7 A, 7 B.

4. Two magnitudes are said to be *commensurable* with one another, when *a common measure* of the two may be found, *i. e.* a magnitude which is contained in each of them a certain number of times exactly.

In like manner, any number of magnitudes are said to be *commensurable*, when there is some magnitude which is contained in each of them a certain number of times exactly.

Magnitudes which have no common measure, are said to be *incommensurable.*

Magnitudes A, B, which have one common measure M, have also many others, indeed, an unlimited number of common measures, for (as will be shown in Prop. 1.) every magnitude which is contained an exact number of times in M, is contained also an exact number of times in A and B; and whether M be divided into two, or three, or any other number of equal parts, one of these parts will be contained an exact number of times in M

Among the common measures of the same two magnitudes, there is, however, always one which is greater than any of the others, and which (as will be shown in Prop. 6.) is measured by every other. This *greatest common measure* is always to be understood when "the common measure" is spoken of without further specification.

5. The *numerical ratio* of one magnitude to another with which it is commensurable, is a certain number, whole or fractional, which expresses *how many*, and *what* parts of the second are contained in the first: for example, if the common measure of A and B be contained in A five times, and in B six times, or, which is the same thing, if A contain $\frac{5}{6}$ths of B, then A is said to have to B the numerical ratio "5 to 6" which is thus written 5 : 6, or, in the fractional form, $\frac{5}{6}$.

In fact, the particular ratio of two given magnitudes, whether commensurable or otherwise, can be conceived only by means of the numbers which denote how often the same magnitude is contained, or nearly contained, in each: without these, no idea can be formed of their relative magnitude; they constitute its measure, true or approximate.*

To these numbers, therefore, when speaking of commensurable magnitudes, the term "ratio" alone, *i. e.* without the addition of "numerical," will be found commonly applied in what follows.

6. *The terms of the ratio* of two magnitudes, are the numbers which denote how often a common measure of the two is contained in each of them. They are distinguished by the names of *antecedent* and *consequent*, according to the corresponding magnitudes. In the foregoing example 5 and 6 are the terms of the ratio of A to B, 5, the antecedent, and 6, the consequent.

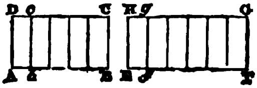
The terms of the same ratio of A to B, will be different according to the common measure by which they are determined; the *lowest terms* being in all cases those which are determined by the greatest common measure. It must be observed, however, that no other terms can express the same ratio, but such as are either the lowest, or equimultiples of the lowest terms; for the magnitudes compared, can have no common measure, which is not either the greatest, or contained a certain number of times in the greatest common measure. On the other hand, any terms whatever which are equimultiples of the lowest terms will express the same ratio: thus, if A contain $\frac{3}{5}$ths of B, it will contain also $\frac{6}{10}$ths of B, $\frac{9}{15}$ths, and, generally, $\frac{3\times n}{5\times n}$ths of B, where any number whatever may be substituted for *n*.

The ratio of B to A has the same terms with the ratio of A to B, but in an inverse order: thus, if 5 : 6 be the numerical ratio of A to B, 6 : 5 will be that of B to A.

The ratio of B to A is accordingly said to be the *inverse* or *reciprocal* of the ratio of A to B.

If the terms of the ratio of A to B be equal, it is evident that the magnitudes A, B, must likewise be equal. In this case the ratio is said to be a *ratio of equality.*

Def. [7.] If there be two magnitudes of the same kind, and other two, and if the first contain a measure of the second, as often as the third contains a like measure of the fourth; or, which is the same thing, if the ratios of the first to the second, and of the third to the fourth be expressed by the same terms; the first is said to have to the second *the same ratio* which the third has to the fourth; and the four magnitudes are called *proportionals.*



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* The numerical ratio is accordingly designated, by some writers, "The measure of the ratio" of one magnitude to another. This term has, however, been applied in a different sense, to which deference is more particularly due, as it has given rise to the word "logarithm," of which it is the literal interpretation.

For example, let A B C D, and E F G H be two rectangles, having equal altitudes, and let their bases A B, E F, contain the same straight line M 5 and 6 times respectively: divide A B, E F, into the parts A *b*, &c. E *f*, &c. each equal to M, and through the points of division draw right lines *b c*, &c. *f g*, &c. parallel to A D, and EH, thereby dividing the rectangles A B C D, E F G H, into 5 and 6 smaller rectangles respectively, A *b c* D, &c. E *f g* H, &c. all equal to one another (I. 25.). Then, 5 : 6 is the ratio of the rectangle A B C D to the rectangle E F G H, because A B C D and E F G H contain the same rectangle 5 and 6 times respectively; and the same 5 : 6 is also the ratio of the base A B to the base E F, because A B and E F contain the same straight line 5 and 6 times respectively. Therefore, the ratios of the first to the second, and of the third to the fourth, are expressed by the same terms, and the two rectangles, and their two bases, are proportionals.

In the preceding, and in every other instance of commensurable proportionals, the first two, and the second two, have a *common numerical ratio:* and, in every case, if two magnitudes, and other two, have a common numerical ratio, the four magnitudes are, according to this definition, proportionals.

It is evident from the observations on def. 6, that if four magnitudes be proportionals, any other terms expressing the ratio of the first to the second, must likewise express the ratio of the third to the fourth. For the terms which determine the proportion, are either the lowest terms, or equimultiples of them (see *Prop.* 6. *Cor.* 1.): and in either case, the lowest terms which express the ratio of the first to the second, and of the third to the fourth, must be the same; therefore, because any other terms expressing the ratio of the first to the second must be equimultiples of the lowest terms, that is, of the lowest terms of the ratio of the third to the fourth, such terms express also the ratio of the third to the fourth (see observations on Def. 6.)

The same will be demonstrated more at large in Prop. [9].

Def. [8]. If there be two magnitudes of the same kind, and other two, the first is said to have to the second *a greater ratio* than the third has to the fourth, when the first contains some measure of the second a greater number of times than the third contains a like measure of the fourth: also, when this is the case, the third is said to have to the fourth *a less ratio* than the first has to the second.

This definition can, in no case, apply to the same magnitudes which come under def. [7], *i. e.* one magnitude cannot have to another the *same ratio* as a third to a fourth by def. [7], and at the same time a *greater* or a *less ratio* than the third has to the fourth, by this definition. (See *Prop.* 9. *Cor.* 1 and 2.)

Much less can one magnitude have to another a *greater* ratio than a third has to a fourth, and at the same time a *less* ratio than the third has to the fourth, by this definition.

9. When four magnitudes A, B, C, D, are proportionals, they are said to constitute *a proportion*, which is thus written,

A : B : : C : D.

i. e. "A is to B as C is to D."

Of a proportion, the first and last terms are called the *extremes*, and the second and third the *means;* thus A, D are the extremes, and B, C the means, of the proportion A : B : : C : D. The terms A and C are said to be *homologous*, as also B and D; the former being antecedents, and the latter consequents, in the proportion.

It is indifferent in the statement of a proportion which of the ratios precedes the other, for it is evident, that if A has to B the same ratio as C has to D, C has to D the same ratio as A to B. Hence, A : B : : C : D, and C : D : : A : B, signify the same proportion—the only difference being, that the extremes of one expression are the means of the other.

Since magnitudes cannot be compared, except they be of the same kind, it is manifest that the first and second terms of a proportion must be of the same kind, as also the third and fourth; yet the first and third may be of different kinds: *e. g.* 10lbs : 6lbs : : 15 ft. : 9ft. is a true proportion; for the ratio of the first to the second, as well as of the third to the fourth, is 5 : 3, and yet 10lbs. and 15ft. are not magnitudes of the same kind.

10. *Three magnitudes of the same kind* are said to be *proportionals* or *in continued proportion*, when the first has to the second the same ratio which the second has to the third. Magnitudes A, B, C, which are in continued proportion, may be written thus, A : B : C, *i. e.* "A *is to* B, *as* B *to* C." In this case B is called a *mean proportional* or *geometrical mean* between A and C, and C *a third proportional* to A and B.

11. *Any number of magnitudes of the same kind* are said to be *in continued proportion*, when the first is to the second, as the second to the third, as the third to the fourth, and so on. Magnitudes A, B, C, D, &c. which are in continued proportion, may be thus written, A : B : C : D : &c.

The magnitudes of such a series are said to be in *geometrical progression* · and B, C, are called *two geometrical means* between A and B ; again, B, C, D, *three geometrical means* between A and E ; and so on.

Also, in this case, the first A is said to have to the third C, *the duplicate ratio* of that which it has to the second B—to the fourth D, the *triplicate ratio* of that which it has to B, and so on: and reciprocally, A is said to have to B the *subduplicate ratio* of that which it has to C, the *subtriplicate ratio* of that which it has to D, and so on.

12. If there be any number of magnitudes of the same kind, A, B, C, D, the first A is said to have to the last D a ratio which is *compounded of the ratios* of A to B, B to C, and C to D.

Also, if K and L, M and N, P and Q, be any other magnitudes, and if the ratios of K to L, of M to N, and of P to Q, be the same respectively with the ratios of A to B, of B to C, and of C to D, A is said to have to D a ratio which is compounded of the ratios of K to L, M to N, and P to Q.

Axioms. (Euc. v. Ax. 1, 2, 3, 4.)

1. Equimultiples of the same or of equal magnitudes are equal to one another.

2. Those magnitudes of which the same, or equal magnitudes are equimultiples, are equal to one another.

3. A multiple of a greater magnitude is greater than the same multiple of a less.

4. That magnitude of which a multiple is greater than the same multiple of another, is greater than that other magnitude.

Prop. 1.

If one magnitude be a multiple of another, and if this be likewise a multiple of a third magnitude, the first shall be a multiple of the third.

Let A contain B three times, and let B contain C four times; then because B contains C four times, three times B or A must contain C thrice as often, or twelve times, *i. e.* a certain number of times exactly; therefore A is a multiple of C. The same may be said, if, instead of three and four, any other numbers whatever be taken, *i. e.* if A be *any* multiple of B, and B *any* multiple of C.

Therefore, &c.

Cor. 1. If one magnitude measure another, it will measure any multiple of that other.

Cor. 2. Hence, the ratio of A to B being expressed by certain given terms, as 5 : 6, the same ratio may be expressed by any terms which are equimultiples of the given terms, as $10 \times 5 : 10 \times 6$. For, if M be the common measure which is contained in A five times and in B six times, any measure of M, as the tenth part, will also measure A and B (*Cor.* 1.) and will be contained in A 10×5 times, and in B 10×6 times.

Cor. 3. Hence, also, reversely, the ratio of A to B being expressed by any terms, as $10 \times 5 : 10 \times 6$, which have a common factor, the same ratio may be expressed by any terms, as 5 : 6, which are like parts of the given terms.

For, if M be the common measure which is contained in A 10×5 times, and in B 10×6 times, it is evident that 10 M will be contained in A 5 times, and in B 6 times.

Prop. 2. (Euc. v. 3.)

If two magnitudes be equimultiples of two others, and if these be likewise equimultiples of two third magnitudes, the two first shall be equimultiples of the two third.

Let A and A′ contain B and B′ respectively three times, and let B and B′ contain C and C′ respectively four times; then, as in the demonstration of the preceding proposition, A and A′ being equal to three times B and three times B′ respectively, contain C and C′ respectively 3×4 times, *i. e.* the same number of times exactly; therefore A and A′ are equimultiples of C and C′.

The same may be said, if, instead of 3 and 4, any other numbers be taken, *i. e.* if A, A′ be *any* equimultiples of B, B′, and B, B′ *any* equimultiples of C, C′.

Therefore, &c.

Cor. If two magnitudes A, A′ be equimultiples of two others B, B′, and likewise of two third magnitudes C, C′, and if one of the second, B, be a multiple of the corresponding one, C, of the third, the other second, B′, shall be the same multiple of the other third, C′.

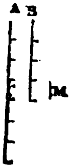
Prop. 3.

A common measure M of any two

magnitudes A *and* B, *measures also their sum* A + B, *and their difference* $A \sim B$.

For, if M be contained in A any number of times, as 7, and in B any number of times, as 4; it will evidently be contained in the sum of A and B, 7 + 4 or 11 times, and in their difference 7 − 4 or 3 times, and therefore will measure their sum and difference.

The same may be said, if, instead of 7 and 4, any other numbers be taken.[*]

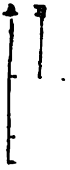


Therefore, &c.

Prop. 4.

If there be two magnitudes A, B, *and if one of them be contained in the other a certain number of times with a remainder; any common measure of the two magnitudes shall measure the remainder, and any common measure of the remainder and the lesser magnitude, shall measure the greater also.*

Let B be contained in A twice with a remainder R, and let M be any common measure of A and B; then, since M is contained a certain number of times in B, it is also contained a certain number of times in twice B (1.)[†] and measures twice B: but it also measures A; therefore (3.) it measures the difference of A and twice B, that is R.



Next, let N be any common measure of R and B: then, as before, N measures twice B; therefore (3.) it measures the sum of R and twice B, that is, A.

And the reasoning, in either case, is independent of the particular numbers assumed.

Therefore, &c.

Cor. The greatest common measure of the remainder and lesser magnitude is also the greatest common measure of the two magnitudes. For, since every common measure of A and B is also a common measure of B and R; the greatest common measure of A and B will be found among the common measures of B and R; and it has been shown that every one of the latter measures both A and B; therefore the greatest among them is the greatest common measure of A and B.

Prop. 5.

By repeating the process indicated in the last proposition, with the remainder and the lesser magnitude, and again with the new remainder (if there be one) and the preceding, and so on, the greatest common measure of two given commensurable magnitudes A, B *may be found.*

Let B, for instance, be contained in A twice (as in the last proposition), with a remainder R; let R be contained in B three times, with a second remainder R_2; let R_2 be contained in R four times, with a third remainder R_3, and let R_3 be contained in R_2 five times exactly. Then, because (by 4. *Cor.*) the greatest common measure of A and B is the greatest common measure of B and R, that is (by the same *Cor.*) of R and R_2, that is, again, of R_2 and R_3, and because R_3, being contained in itself once and a certain number of times in R_2, is the greatest common measure of R_2 and R_3, it is likewise the greatest common measure of A and B.

The same may be said, if instead of 2, 3, 4, 5, any other numbers, supposed to arise from a similar examination of any two given commensurable magnitudes, be taken. At every step of the process, the remainder, as R, is diminished by the following remainder R_2 or by as many times R_2 as are contained in it, that is, in either case, by a magnitude greater than the supposed greatest common measure, to procure the new remainder R_3. In all cases therefore, after a number of steps, which is less than the number of times the lesser magnitude contains the supposed greatest common measure, a remainder will be found which is equal to the greatest common measure.

Therefore, &c.

Cor. 1. If the process admit of being continued through an unlimited number of steps, without arriving at a remainder which measures the next preceding, the magnitudes which are subjected to it, have no common measure, *i. e.* they are incommensurable.

Cor. 2. By proceeding in a similar manner, the greatest common measure of three magnitudes A, B and C may be found; for if M be taken, the greatest common measure of A and B, and M, the greatest common measure of M and C, then because the greatest common measure of A, B and C is to be found among the common measures of M and C, and because every one of the latter, being a measure of M, measures A and B (1.), the greatest among them, that is, M_2, is the greatest common measure of A, B and C.

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[*] The example of Euclid has been followed in annexing straight lines to illustrate this and many subsequent propositions, which are, however, not the less to be understood as applicable to and demonstrated of magnitudes generally, as is evident from the language of the enunciation and demonstration.

[†] The reference is here to the first proposition of the present Book; and generally, in such references as have no Roman numeral to indicate the Book, the current Book is always to be understood.

It is obvious, that in the same manner the Rule may be extended to any number of magnitudes.

Cor. 3. By help of this proposition the lowest terms of the ratio of two given commensurable magnitudes may be determined: for the lowest terms of their ratio are the numbers which denote how often their greatest common measure is contained in each (see def. 6.).

Scholium.

It may be observed that the foregoing process includes the arithmetical rule for finding the greatest common factor of two numbers: which is to *divide the greater number by the lesser, and find the remainder; the lesser by the remainder, and find the second remainder, if there be one; the preceding remainder by this, and find the third remainder; and so on, until a remainder be found which is contained an exact number of times in the next preceding; this last remainder will be the greatest common factor required.*

Thus, if the numbers be 628 and 272, the successive remainders will be 84, 20, and 4, of which 4 is contained in 20 an exact number of times: therefore, 4 is the greatest common factor of the numbers 628 and 272.

If there be found no remainder which is exactly contained in the preceding, until the course of the Rule produces a remainder 1, the numbers have no common factor but 1, and are said to be *prime to one another.*

The greatest common factor of two numbers being thus found, if the numbers be divided by it, the quotients will, manifestly, be prime to one another. With regard to other common factors of the same two numbers, *every other common factor must be contained an exact number of times in the greatest:* for it is contained an exact number of times in each of the remainders of the Rule, the last of which is the greatest common factor.

Hence it follows, that, any two numbers being given, *there is no other common factor but the greatest, by which if the numbers be divided, the quotients will be prime to one another;* for, after division by any other common factor, a common factor must still remain, viz. the number which denotes how often the first is contained in the greatest common factor.

Thus, the greatest common factor of 204 and 240 is 12, as found by the Rule; therefore those numbers have no common factor which is not a factor of 12; and if they be divided by any factor of 12, as 6, the quotients 34 and 40 have still a common factor 2, which is the number of times the factor 6 is contained in 12.

It is impossible to have a clear and correct apprehension of the subject before us, without a reference, not merely to numbers, but also to the properties just mentioned (see Arithmetic, art. 54, 55, 56, 57, 58, 63.).

Prop. 6.

If a magnitude measure each of two others, it shall either be the greatest common measure of the two, or it shall be contained an exact number of times in the greatest common measure.

For, in the process of Prop. 5, it was seen that every common measure of the two magnitudes A and B measures also the successive remainders, the last of which is the greatest common measure of A and B.

Therefore, &c.

Cor. 1. The lowest terms of the ratio of two magnitudes being determined by the greatest common measure of the two, and any other common measure being contained an exact number of times in the greatest, any other terms expressing the same ratio must be equimultiples of the lowest terms.

This corollary has been cited by anticipation in the observations upon def. 6.

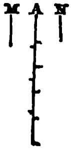
Cor. 2. The numerical ratio of two magnitudes being given, if not already in its lowest terms, may be reduced to them by dividing the terms by their greatest common factor: for the lowest terms, being determined by the greatest common measure of the two magnitudes, must be prime to one another, and there is no other common factor but the greatest, by which if two numbers be divided, the quotients will be prime to one another. (See Prop. 5. *Scholium.*)

For example, the terms of the ratio 628 : 272 have 4 for their greatest common factor: therefore, dividing them by 4, the quotients 157 and 68 are the lowest terms in which the ratio can be expressed.

Cor. 3. If this Rule be applied to two numerical ratios which are differently expressed, it will show whether they are different ratios or only different forms of the same ratio: viz. according as the lowest terms of the two are different or the same.

Prop. 7.

If a magnitude A *be measured by each of two others* M *and* N, *a common measure of these two may be found; and, conversely, if two magnitudes be commensurable, a magnitude may be found which is measured by both.*



Let A contain M 4 times, and N 5 times: then, if A be divided into 4 × 5, or 20 equal parts, M will contain 5 and N 4 of those parts; and, therefore, one of those parts being contained an exact number of times in M, and an exact number of times in N, is a common measure of M and N.

Next, let M contain a magnitude *a* 5 times, and let N contain the same *a* 4 times: then, if there be taken a magnitude A, which contains *a* 5 × 4, or 20 times, A will contain M 4 times, and N 5 times, and, therefore, will be measured by each of the magnitudes M and N.

And the demonstration will be the same whatever numbers are taken.

Therefore, &c.

Prop. 8.

If there be three magnitudes A, B, C, *the first of which is commensurable with the second, and the second with the third, the first shall be commensurable with the third.*

Let M be a common measure of A and B, and N a common measure of B and C. Then, (7.) because the two, M, N, are both of them measures of the same magnitude B, a common measure *m* of these may be found. And, because *m* measures M, it measures also A, which is a multiple of M (1. *Cor.* 1.); and in the same manner, because it measures N, it measures C, which is a multiple of N. Therefore *m* is a common measure of A and C, that is, A and C are commensurable.

Cor. 1. If there be any number of magnitudes, the first of which is commensurable with the second, the second with the third, the third with the fourth, and so on to the last, all the magnitudes shall be commensurable.

Cor. 2. If two magnitudes be commensurable with one another, and if one of them be incommensurable with a third magnitude, the other shall likewise be incommensurable with the third.

Section 2.—*Proportion of Commensurable Magnitudes.*

In the foregoing Section it has been seen how the greatest common measure, and hence the lowest terms of the ratio, of two magnitudes may be determined (5. and 5. *Cor.* 3.). It has been seen, also, that every other common measure of the same two magnitudes is contained in the greatest a certain number of times exactly, and hence that no other terms but such as are equimultiples of the lowest can express the same ratio (6. and 6. *Cor.* 1.); which last simple relation affords an easy Rule for reducing any given terms to the lowest terms, and hence determining whether two given numerical ratios which are differently expressed be the same or not (6. *Cor.* 2 and 3).

We now proceed to the theorems of proportion, the end of which is for the most part to show, that if two magnitudes have the same ratio with other two, magnitudes which are related after a certain manner to the two former, will have the same ratio to one another with magnitudes which are similarly related to the two latter.

In the demonstration of these theorems, an illustration by particular numbers will be preferred, for the sake of the less practised reader, to the use of general symbols for numbers. In following this plan hitherto, we have had occasion to observe at the conclusion of each demonstration, that the steps are not upon this account the less general, but apply equally to any other numbers which may be substituted in place of the particular numbers assumed. The same observation will be found equally true, and is accordingly here premised with regard to particular numbers, wherever they are introduced in the following demonstrations. The demonstrations are exactly similar, whatever numbers be substituted in the place of them; and, accordingly, the general propositions are not less evident than if the reasoning had been conducted by general symbols.

The present Section treats only of commensurable proportionals as described in def. [7]. The theorems, however, (that is, all of them, the first excepted which is expressly enunciated of commensurables) are likewise true with regard to the more general description of proportionals to be considered in Section 3. Hence the brackets which the reader will have already noticed in "def. [7]" and "def. [8]," and in which the numbers 9, 10, &c. of the following propositions are inclosed. Hence, too, the absence of all reference to Euclid, whose theorems on the subject of Proportion are stated not of commensurable magnitudes only, but of such as, whether commensurable or otherwise, come under a more general definition of proportionals. In the next Section, these propositions, (prop. [9] excepted) will be repeated with reference to a similar general definition, the brackets will be removed from the numbers 9, 10, &c., and the references to Euclid will be annexed as usual.

Prop. [9].

If there be four magnitudes A, B, C, D, *which are commensurable proportionals, the second and fourth, and any like parts of the second and fourth, shall be contained in the first and third the same number of times exactly, or the same number of times with corresponding remainders less than the parts.*

Let the common ratio of A to B and C to D be any whatever, for example, 84 : 5; and first, let the second and fourth, viz. B and D, be taken: B and D shall be contained in A and C respectively, the same number of times exactly, or the same number of times with corresponding less remainders.

Let M and N be the common measures of A and B and of C and D, by which the ratio 84 : 5 is determined; and which are therefore contained in B and D respectively 5 times, and in A and C respectively 84 times. Then, because M and N are contained in A and C respectively 84 times, and that B and D contain M and N respectively 5 times; B and D are contained in A and C respectively, as often as the number 5 is contained in 84, that is, the same number of times exactly, if 5 be exactly contained in 84, or the same number of times with corresponding remainders, if 5 be contained in 84 with a remainder.

Next, let any like parts, for example the 6ths of B and D, be taken: these shall be contained in A and B the same number of times, either exactly, or with corresponding less remainders.

Let M and N be the common measures of A and B and of C and D, as before, by which the ratio 84 : 5 is determined; and which are therefore contained in B and D respectively 5 times, and in A and C respectively 84 times. And let M be divided into 6 parts each equal to *m*, and N also into 6 parts each equal to *n*. Then it is evident that *m* and *n* must be contained in A and C respectively 84 × 6 times, and in B and D respectively 5 × 6 times (1.). And, because the sixth-parts in question, *i. e.* the sixth-parts of B and D, are contained in B and D respectively 6 times, they must contain *m* and *n* respectively 5 times. Therefore, the parts in question are contained in A and C respectively as often as the number 5 is contained in 84 × 6, that is, the same number of times exactly, if 5 be exactly contained in 84 × 6, or the same number of times with corresponding remainders, if 5 be contained in 84 × 6 with a remainder.

Therefore, &c.

Cor. 1. Hence it appears that, if two magnitudes and other two have a common numerical ratio, any other terms expressing the ratio of the first to the second must also express the ratio of the third to the fourth; as was observed in the remarks upon def. [7].

Cor. 2. Hence it appears, also, that A cannot be said to have to B the same ratio which C has to D, according to def. [7], and at the same time a greater or a less ratio than C has to D, according to def. [8]; as was observed at def. [8].

Cor. 3. And much less can A be said to have to B a greater ratio than C has to D, according to def. [8], and at the same time a less ratio than C has to D, according to the same definition.*

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* As the General Theory of Proportion in Section 3. contains the remaining propositions of this Section with reference to the new definitions 7 and 8 there given, the reader may, if he pleases, pass on to that Section, or rather, to the concluding Scholium of the present one; by which he will omit nothing that will be cited in the future pages of this treatise. On the other hand, it is recommended to beginners, and such as are not curious about the general and complete theory of proportion, to peruse with care the remainder of the present Section, with the opening paragraphs of Section 3., and then pass on to Section 4.; which they may do with the assurance of no difficulty being presented to them upon that account in the remainder of the treatise.

PROP. [10].

Equal magnitudes have the same ratio to the same magnitude: and the same has the same ratio to equal magnitudes.

For if any part, as a 5th, of C be contained in A any number of times, as 4, and if B be equal to A, the same part of C will evidently be contained in B the same number of times, 4 ; and therefore, 4 : 5 will be the common ratio of A to C, and of B to C.

Again, if any part of A, as a 4th, be contained in C any number of times, as 5, and if B be equal to A, the like part of B (ax. 2.) will evidently be contained in C the same number of times, 5 ; and, therefore, 5 : 4 will be the common ratio of C to A, and of C to B.

Therefore, &c.

Cor. If a ratio which is compounded of two ratios be a ratio of equality, one of these must be the reciprocal of the other. For if there be three magnitudes of the same kind A, B, and C, and if A be equal to C, the ratio of B to C must be the same with the ratio of B to A: in other words, if the ratio of A to C, which is compounded of the ratios of A to B and of B to C, be a ratio of equality, the ratio of A to B must be the reciprocal of the ratio of B to C, (def. 6. and 12.).

PROP. [11].

Of two unequal magnitudes the greater has a greater ratio to the same magnitude; and the same magnitude has a greater ratio to the lesser of the two.

For, the magnitudes A, B, and C, being supposed to be commensurable, if A be greater than B, it must contain the common measure of A, B, and C, that is, a measure of C, a greater number of times than B contains the same measure of C, and therefore (def. [8]) A has to C a greater ratio than B has to C.

Again, if A be greater than B, and if C be any magnitude commensurable with each of them, like measures of A and B may be found, which are each of them contained a certain number of times in C. For, since A, B, and C (8.) contain the same magnitude M, each of them a certain number of times, let them be equal to 7 M, 5 M, and 12 M, respectively. Then, if *a* and *b* be taken, the 35th parts (7 × 5) of A and B, *a* will be a 5th part of M, and therefore will be contained an exact number of times in C, and *b* a 7th part of M, and therefore also contained an exact number of times in C (1. *Cor.* 1.). But any measure of A is greater than the like measure of B (ax. 4.); therefore, of the like measures in question, viz. *a* and *b*, C will contain that of A a less number of times than it contains that of B. Therefore (def. [8]) C has to A a less ratio than it has to B, or a greater ratio to B than it has to A.

Therefore, &c.

Cor. 1. Magnitudes which have the same ratio to the same magnitudes are equal to one another: as likewise those to which the same magnitude has the same ratio ([9] *Cor.* 2.)

Cor. 2. A ratio which is compounded of two ratios, one of which is the reciprocal of the other, is a ratio of equality. For, if there be three magnitudes of the same kind, A, B, and C, and if the ratio of B to C be the same with the ratio of B to A, A must be equal to C: in other words, the ratio of A to C, which is compounded of the ratios of A to B and of B to C, one of which is the reciprocal of the other, is a ratio of equality, (def. 6. and 12.).

Cor. 3. If one of two magnitudes have a greater ratio to the same magnitude than the other has, the first must be greater than the other: and if the same magnitude have a greater ratio to one of two magnitudes than it has to the other, the first must be less than the other ([9] *Cor.* 2, 3.).

PROP. [12].

Magnitudes A, B *and* C, D, *which have the same ratio with the same magnitudes* P, Q, *have the same ratio with one another.*

For, since A, B have the same ratio with P, Q, some part of B is contained in A as often as a like part of Q is contained in P; therefore, if any other part of Q be contained a certain number of times in P, a like part of B will be contained as often in A ([9]).

But, because C, D have the same ratio with P, Q, some part of D is contained in C as often as a like part of Q is contained in P. Therefore, whatever part this be, which is taken of D, and contained in C as often as the like part of Q in P, the like part of B is contained as often in A; that is, A : B :: C : D.

Therefore, &c.

Cor. 1. If A have to B the same ratio as C to D, and C to D a greater or a less ratio than E to F, A shall have to B a greater or a less ratio than E to F. For, from what we have seen ([9]) it follows, that whatever part of D it be that is contained in C a greater or less number of times than the like part of F is contained in E, the like part of B must be contained in A the same greater or less number of times.

Cor. 2. And in the same manner it may be shown, that if A have to B a greater or a less ratio than C to D, and C to D the same ratio as E to F, A shall have to B a greater or a less ratio than E has to F.

Prop. [13].

If four magnitudes be proportionals, and if the first be any multiple or part of the second, the third shall be the same multiple or part of the fourth: and conversely, if one magnitude be the same multiple, or part, of another, that a third magnitude is of a fourth, the four magnitudes shall be proportionals.

For, if A, B, C, D be proportionals, and if the second, or any part of the second, be contained a certain number of times in the first, the fourth, or a like part of the fourth, will be contained the same number of times in the third ([9]). Therefore, if the second be contained in the first seven times, the fourth will also be contained in the third seven times; or again, if a seventh part of the second be contained in the first once, a seventh part of the fourth will also be contained in the third once.

Again, if A contain B seven times, and C also contain D seven times, A, B and C, D have a common ratio 7 : 1; and, in like manner, if A be contained in B seven times, and C in D also seven times, A, B, and C, D, have a common ratio 1 : 7. Therefore, in either case, A, B, C, D are proportionals.

Therefore, &c.

Prop. [14].

If four magnitudes A, B, C, D, *be proportionals, and if the first be greater than the second, the third shall be greater than the fourth, if equal, equal, and if less, less.*

For if A be greater than B, any common measure M of A and B will be contained a greater number of times in A than it is in B, and therefore ([9]) the like measure, N of D, will be contained in C a greater number of times than it is in D, that is, C will be greater than D.

And in like manner it may be shown, that if A be equal to B, C will be equal to D, and if less, less.

Therefore, &c.

Prop. [15].

If four magnitudes A, B, C, D, *be proportionals, they shall also be proportionals when taken* inversely; *that is,* invertendo[*] B : A :: D : C.

For if 7 : 5 be the common ratio of A to B and of C to D, there will be common measures M, N, the first of A and B, and the other of C and D, which are contained in B and D respectively five times, and in A and C respectively seven times; and therefore 5 : 7 will be a common ratio of B to A and of D to C.

This may be stated as follows—"the reciprocals of equal ratios are equal to one another."

Therefore, &c.

Prop. [16].

If four magnitudes, A, B, C, D *be proportionals, and if there be taken any equimultiples of the first and third, and also any equimultiples of the second and fourth; these equimultiples shall likewise be proportionals.*

Let 4 A, 4 C be any equimultiples of A, C, and 6 B, 6 D, any equimultiples of B, D: and let 7 : 5 be the common ratio of A to B, and of C to D, as in the last proposition. Then, if the measures M and N by which it is determined be taken, because M and N are contained in A and C respectively 7 times, they are contained in 4 A and 4 C respectively 4 × 7 times; and in like manner it may be shown that they are contained in 6 B and 6 D respectively 6 × 5 times. Therefore 4 × 7 : 6 × 5 is at once the ratio of 4 A to 6 B, and of 4 C to 6 D.

Therefore, &c.

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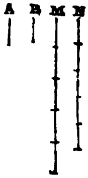
[*] The Latin words "invertendo," "alternando," "dividendo," "convertendo," "componendo," "ex æquali in proportione directâ," "miscendo," and "ex æquali in proportione perturbatâ," ([15], [19], [20], [20] *Cor.* 1, [21], [24], [24], *Cor.* 2. and [26]) carry with them, particularly to such as are strangers to the language, an air of mystery we should rather have dispensed with. They are in such constant use, however, that we cannot well do without them. They mean no more than "by inverting," "by alternating," "by separating," "by exchanging," "by combining," "by reason of equal intervals in direct proportion," "by mixing," and "by reason of equal intervals in cross proportion:" and they serve as so many titles to their respective theorems, which are those most frequently cited out of the whole theory of proportion.

Cor. If A, B, C, D be proportionals, and if any like parts of the first and third be taken, and also any like parts of the second and fourth, these like parts will likewise be proportionals.

Prop. [17].

Magnitudes have the same ratio to one another, which their equimultiples have.



Let M, N be equimultiples of A, B: that is, let M, N contain A, B the same number of times respectively, as 5: then it is evident that if any measure K of B be contained in A a certain number of times, the like measure 5 K of 5 B, or N will be contained in 5 A, or M, the same number of times. Therefore A : B :: M : N.

Therefore, &c.

Cor. 1. Magnitudes have the same ratio to one another, which their like parts have.

Cor. 2. If A, B, C, D be proportionals, and if equimultiples be taken of the first and second, and also equimultiples of the third and fourth, these equimultiples will likewise be proportionals.

The same, it is evident, may be stated with regard to any like parts taken of the first and second, and also of the third and fourth.

Prop. [18].

If four magnitudes of the same kind be proportionals, and if the first be greater than the third, the second shall be greater than the fourth; if equal, equal; and if less, less.

Let A, B, C, D be proportionals, and let M and N be the common measures of A and B and of C and D, by which their common ratio is determined; that is, let M and N be like parts of B and D, which are contained the same number of times in A and C respectively, (def. [7]). Then, because M and N are like parts of A and C, it is evident that if A be greater than C, M must be greater than N; and therefore also B, which is a multiple of M, greater than D, which is the same multiple of N (ax. 3.).

In the same manner it may be shown, that if A be equal to C, B will be equal to D; and if less, less.

Therefore, &c.

Cor. Hence also, if four magnitudes of the same kind be proportionals, and if the second be greater than the fourth, the first will be greater than the third; if equal, equal; and if less, less.

Prop. [19].

If four magnitudes A, B, C, D *of the same kind be proportionals, they shall also be proportionals when taken* alternately; *that is,* alternando A : C :: B : D.

For, if M and N be the common measures of A and B and of C and D, by which their common ratio is determined; then, because A and C are equimultiples of M and N ([17]), A : C :: M : N,—and, for the like reason, B : D :: M : N. Therefore ([12].) A : C :: B : D.

Therefore, &c.

Prop. [20].

If four magnitudes A, B, C, D *be proportionals, they shall also be proportionals when taken* dividedly; *that is, the difference of the first and second shall be to the second as the difference of the third and fourth to the fourth;—or* dividendo, A∼B : B :: C∼D : D.*

For, M and N being taken as in the preceding propositions, if they be contained in A and C 7 times, and in B and D 4 times, they will be contained in A∼B and C∼D, 7∼4 or 3 times; and therefore 3 : 4 will be the ratio of A∼B to B, and also of C∼D to D. Therefore A∼B : B :: C∼D : D.

And the same may be said, when A and C are less than B and D respectively.

Therefore, &c.

Cor. 1. If four magnitudes A, B, C, D be proportionals, they shall also be proportionals *by conversion;* that is, the first shall be to the difference of the first and second, as the third to the difference of the third and fourth; or *convertendo* A : A∼B :: C : C∼D.

For *invertendo*, B : A :: D : C, *dividendo*, A∼B : A :: C∼D : C, and *invertendo* A : A∼B :: C : C∼D.

Cor. 2. If four magnitudes A, B, C, D of the same kind be proportionals, the greatest and least of them together shall be greater than the other two together.

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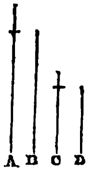
* The sign ∼ placed between two letters denotes the difference of the magnitudes which are represented by them, without supposing the first to be the greater, as is the case when we write A−B by itself.



For, if one of the extremes, as A, be the greatest, then, because B is less than A, D is less than C ([14]), and because C is less than A, D is less than B ([18]); therefore D is the least. But, because $A:B::C:D$, *dividendo*, $A-B:B::C-D:D$; and because in this proportion B is greater than D, $A-B$ is greater than $C-D$ ([18]. *Cor.*) Therefore, if $B+D$ be added to each, the sum of A and D will be greater than the sum of C and B. If one of the means, as B, should be greatest, then *invertendo* $B:A::D:C$; and hence, as before, the sum of B and C, that is, of the greatest and least, is greater than the sum of A and D.

Cor. 3. If three magnitudes be proportionals, the sum of the extremes will be greater than twice the mean, and therefore half the sum greater than the mean.

For, if three magnitudes be proportionals, and if the mean be greater than one of the extremes, it must be less than the other; or again, if it be less than one, it must be greater than the other ([14] or [18]). Therefore, the two extremes are the greatest and least.

Half the sum of two magnitudes, being as much greater than the one as it is less than the other (I. ax. 9.), is called an *arithmetical mean* between the two. It appears therefore, that the arithmetical mean between two magnitudes is greater than the geometrical mean.*

PROP. [21].

If four magnitudes A, B, C, D *be proportionals, they shall also be proportionals when taken* conjointly; *that is, the sum of the first and second shall be to the second, as the sum of the third and fourth to the fourth; or* compónendo $A+B:B::C+D:D$.

For the measures M, N remaining as in the preceding proposition, that is, being contained in A and C respectively 7 times, and in B and D respectively 4 times, are contained in $A+B$ and $C+D$ respectively $7+4$ or 11 times; and, therefore, $11:4$ will be the ratio of $A+B$ to B, and also of $C+D$ to D. Therefore $A+B:B::C+D:D$.

Therefore, &c.

Cor. Hence, *invertendo* and *componendo*, (after the order of *Cor.* 1. in the preceding proposition,) if four magnitudes A, B, C, D be proportionals, the first shall be to the sum of the first and second, as the third to the sum of the third and fourth; or, $A:A+B::C:C+D$.*

PROP. [22].

If one magnitude be to another as a magnitude taken from the first to a magnitude taken from the other, the remainder shall be to the remainder in the same ratio.

Let A, B be any two magnitudes, from which respectively let there be taken the magnitudes A', B', which have to one another the same ratio which A has to B; the remainders $A-A'$ and $B-B'$ shall be to one another in the same ratio.

For, because $A:B::A':B'$, *alternando*, $A:A'::B:B'$; therefore, *dividendo*, $A-A':A'::B-B':B'$; and again, *alternando*, $A-A':B-B'::A':B'$, that is ([12]) $::A:B$.

Therefore, &c.

Cor. 1. If there be any number of magnitudes, A, B, C, D, &c., in geometrical progression, the differences $A\sim B$, $B\sim C$, $C\sim D$, &c. will form a geometrical progression, in which the successive terms have the same ratio with the successive terms of the former.

For B is to C as A to B, C to D as B to C, and so on; therefore, $A\sim B$ is to $B\sim C$ as A to B, $B\sim C$ to $C\sim D$ as B to C, *i. e.* as A to B, *i. e.* as $A\sim B$ to $B\sim C$ ([12]); and so on.

Cor. 2. And conversely, any number of magnitudes A, B, C, D, &c. in geometrical progression, may be considered as the differences of other magnitudes, A', B', C', D', E', &c. forming a geometrical progression, in which the first term A' is to A as A to $A\sim B$, and the successive terms have the same ratio with the successive terms of the former.

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* The case is here excepted in which the magnitudes are equal to one another; for in that case it is manifest that the arithmetical mean is equal to the geometrical mean.

* It is usual, when two demonstrations are equally short and obvious, to prefer that which is the most elementary, or approaches nearest to first principles. Now, it may be observed of this corollary, the corresponding corollary of Prop. [28], and other parts of the present section, as the theorem *miscendo*, and Prop. [25], that, stated, as is here the case, of commensurable proportionals, they may, at least, as obviously and briefly, be referred to def. [7] as they may be derived from other propositions. The connected demonstrations have in this case been preferred with a view to the next Section, where nearly all the propositions of this Section will be re-stated, and such as have been referred to def. [7], will have to be reconsidered with reference to a more general definition. A similar consideration will be found to have directed the arrangement of the propositions.

For, if such a progression be taken, A', B', C', &c. in which A' is to A as A to $A \sim B$, and A' to B' as A to B; then because $A' : B' :: A : B$, *convertendo*, $A' : A' \sim B' :: A : A \sim B$; but $A' : A :: A : A \sim B$; therefore ([12]) $A' : A' \sim B' :: A' : A$; and because in this proportion the first term is the same with the third, the second is equal to the fourth ([18)], that is, $A' \sim B'$ is equal to A. But (by *Cor.* 1.), $A' \sim B'$, $B' \sim C'$, $C' \sim D'$, &c. form a progression in which the successive terms, as $A' \sim B'$, $B' \sim C'$, have the same ratio with A', B', that is, with A, B ([12]). Therefore, also, $B' \sim C'$ is equal to B ([18]), and hence again $C' \sim D'$ is equal to C, and so on. In fact, the progressions A, B, C, D, &c. and $A' \sim B'$, $B' \sim C'$, $C' \sim D'$, $D' \sim E'$, &c. having the same first terms, and the same common ratio of their terms, cannot but be identical.

PROP. [23].

If one magnitude be to another as a third magnitude of the same kind to a fourth, the sum of the first and third shall be to the sum of the second and fourth in the same ratio.

Let A, B be any two magnitudes, to which respectively let there be added the magnitudes A', B', which have the same ratio to one another which A has to B: the wholes $A+A'$ and $B+B'$ shall be to one another in the same ratio.

For, because $A : B :: A' : B'$, *alternando*, $A : A' :: B : B'$; therefore *componendo*, $A+A' : A' :: B+B' : B'$; and again, *alternando*, $A+A' : B+B' :: A' : B'$, that is, ([12.]) $:: A : B$.

Therefore, &c.

Cor. 1. Hence, if there be any number of magnitudes of the same kind antecedents, and as many consequents, and if every antecedent have the same ratio to its consequent, the sum of all the antecedents shall have the same ratio to the sum of all the consequents.

Cor. 2. If the ratio of A' to B' be not the same with the ratio of A to B, the ratio of $A+A'$ to $B+B'$ will not be the same with the ratio of A to B; but less, if A' be to B' in a less ratio, or greater, if A' be to B' in a greater ratio.

For if A' have to B' a less ratio than A has to B, A' must be less than a magnitude P,[*] which has to B' the same ratio as A to B ([11] *Cor.* 3.); and therefore $A+A'$ must have a less ratio ([11]) to $B+B'$ than $A+P$ has to $B+B'$; but $A+P$ has to $B+B'$ the same ratio as A to B, by the proposition: therefore $A+A'$ has to $B+B'$ a less ratio than A has to B ([12] *Cor.* 2.): and the other case in which A' has to B' a greater ratio than A to B, admits of a similar demonstration.

Cor. 3. Hence, if the ratio of A' to B be not the same with the ratio of A to B, the ratio of $A+A'$ to $B+B'$ will lie between the ratios of A to B, and of A' to B'; that is, it will be greater than the lesser of the two, and less than the greater of the two.

PROP. [24].

If there be three magnitudes of the same kind A, B, C, *and other three* A', B', C', *which, taken two and two in order, have the same ratio, viz.* A *to* B *the same ratio as* A' *to* B', *and* B *to* C *the same ratio as* B' *to* C'; *then* ex æquali in proportione directâ (*or* ex æquo) *the first shall be to the third of the first magnitudes, as the first to the third of the others; or, as it may be more briefly stated,*

$$\begin{array}{r} \textit{if } A : B :: A' : B' \\ \textit{and } \underline{B : C :: B' : C'}, \end{array}$$

then ex æquali, $A : C :: A' : C'$.

Let $3 : 4$ be the common ratio of A to B and of A' to B', and $5 : 7$ the common ratio of B to C and of B' to C'. Then, if B be divided into 4×5 or 20 equal parts, one of these parts will be contained 3×5, or 15 times in A, and 4×7 or 28 times in C, because $3 : 4$, or $3 \times 5 : 4 \times 5$ (1. *Cor.* 2) is the ratio of A to B, and $5 : 7$, or $4 \times 5 : 4 \times 7$ is the ratio of B to C. And for the like reason, if B' be likewise divided into 20 parts, one of these parts will be contained 3×5 or 15 times in A', and 4×7 or 28 times in C'. Therefore, $15 : 28$ is the ratio of A to C, and also of A' to C'; and, consequently $A : C : A' :: C'$.

Therefore, &c.

Cor. 1. The same may be stated of any number of magnitudes A, B, C, D, and A', B', C', D': that is,

$$\begin{array}{r} \text{if } A : B :: A' : B' \\ \text{and } B : C :: B' : C' \\ \text{and } \underline{C : D :: C' : D'} \end{array}$$

then, *ex æquali*, $A : D :: C' : D'$.

For by the first two proportions $A : C$

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[*] It is here assumed that to two given magnitudes of the same kind and a third there is some magnitude which is a fourth proportional: a truth obvious enough in the case of commensurable proportion here supposed; for if there be taken a common measure of the first two magnitudes and a part which is contained in the third as often as that common measure is contained in the first, a multiple containing this part as often as that common measure is contained in the second will be the fourth proportional required. The case of general proportion will be noticed at prop. 23. of the next section.

$:: A' : C'$, and from this combined with the third, $A : D :: A' : D'$; and so on for any number of proportions.

This may be stated in the following words: "ratios, which are compounded of any number of equal ratios in the same order, are equal to one another." (def. 12.)

Cor. 2. By help of this proposition another property of proportionals may be demonstrated, which is commonly cited by the word *miscendo* :* viz.

If four magnitudes A, B, C, D be proportionals, the sum of the first and second will be to their difference, as the sum of the third and fourth to their difference.

For, since $A : B :: C : D$,

componendo $A + B : B :: C + D : D$.

And again, because *invertendo* $B : A :: D : C$, *convertendo* $B : A \sim B :: D : C \sim D$. Therefore, *ex æquali* $A + B : A \sim B :: C + D : C \sim D$, which is the property in question.

Prop. [25].

If two proportions have the same consequents, the sum of the first antecedents shall be to their consequent, as the sum of the second antecedents to their consequent; that is, if $A : B :: C : D$, *and if* $A' : B :: C' : D$, *then* $A + A' : B :: C + C' : D$.

Because by the first proportion $A : B :: C : D$, and, by the second proportion, *invertendo*, $B : A' :: D : C'$; *ex æquali*, $A : A' :: C : C'$: hence, *componendo*, $A + A' : A' : C + C' : C'$, and on account of the second proportion, viz. $A' : B :: C' : D$, *ex æquali*, $A + A' : B :: C + C' : D$, which is the property in question.

Therefore, &c.

Cor. 1. The same may be stated of any number of proportions having the same consequents: that is, the sum of all the first antecedents shall be to their consequent, as the sum of all the second antecedents to their consequent.

Cor. 2. In like manner also it may shown (by "*dividendo*" instead of "*componendo*"), that if two proportions have the same consequents, the difference of the first antecedents shall be to their consequent as the difference of the second antecedents to their consequent.

Cor. 3. Hence, if there be any number of magnitudes of the same kind A, B, C, D, E, F, and as many others, A′, B′, C′, D′, E′, F′; and if the ratios of the first to the second, of the second to the third, of the third to the fourth, and so on, be respectively the same in the two series, any two combinations by sum and difference of the magnitudes of the first series, *e.g.* $A + C - E$ and $B - C + D$, shall be to one another as two similar combinations of the corresponding magnitudes of the second series, viz. $A' + C' - E'$ and $B' - C' + D'$.

For, because $A : F :: A' : F'$ (*ex æquali*); and again, $C : F :: C' : F'$, by the proposition $A + C : F :: A' + C' : F'$; but $E : F :: E' : F'$; therefore, by the preceding corollary, $A + C - E : F :: A' + C' - E' : F'$. In the same manner, it may be shown that $B - C + D : F :: B' - C' + D' : F'$; therefore, *invertendo* and *ex æquali* $A + C - E : B - C + D :: A' + C' - E' : B' - C' + D'$. And a similar demonstration may be applied to any other combinations by sum and difference.

Prop. [26].

If there be three magnitudes of the same kind A, B, C, *and other three* A′, B′, C′, *which taken two and two, but in a cross order, have the same ratio, viz.* A *to* B *the same ratio as* B′ *to* C′, *and* B *to* C *the same ratio as* A′ *to* B′; *then,* ex æquali in proportione perturbatâ (*or* ex æquo perturbato) *the first shall be to the third of the first magnitudes, as the first to the third of the others; or, as it may be more briefly stated,*

if $A : B :: B' : C'$
and $B : C :: A' : B'$

then ex æquo perturbato, $A : C :: A' : C'$.

For, if 3 : 4 be the common ratio of A to B, and of B′ to C′, and 5 : 7 the common ratio of B to C and of A′ to B′, it may be shown, exactly in the same manner as in the demonstration of Prop. 24, that the ratio of A to C is $3 \times 5 : 4 \times 7$, that is 15 : 28, and in like manner that the ratio of A′ to C′ is $5 \times 3 : 7 \times 4$, that is again 15 : 28. Therefore, 15 : 28 is the ratio of A to C, and also of A′ to C′, and consequently $A : C :: A' : C'$.

Therefore, &c.

Cor. The same may be stated of any number of magnitudes A, B, C, D, and A′, B′, C′, D′; that is, if $A : B :: C' : D'$
and $B : C :: B' : C'$
and $C : D :: A' : B'$

then *ex æquo perturbato* $A : D :: A' : D'$; for by the two first proportions, $A : C :: B' : D'$, and from this combined with the third proportion, $A : D :: A' : D'$, and so on for any number of proportions. This may be stated in the following words: "Ratios which are compounded of any number of equal ratios, but in a reverse order, are equal to one another." (*Def.* 12.)

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* Sometimes also, and more appropriately, by the English words "by sum and difference."

PROP. [27].

Ratios which are compounded of the same ratios, in whatsoever orders, are the same with one another.

The case of ratios which are compounded of the same ratios in the same order is that of [24]. *Cor.* 1. The case, again, of ratios, which are compounded of the same ratios in a reverse order, is that of [26]. *Cor.* 1.

Let the ratio of A to D, therefore, be compounded of the ratios of A to B, of B to C, and of C to D, and let the ratio of A' to D' be compounded of the ratios of A' to B', of B' to C', and of C' to D', which are the same respectively with the ratios of which the ratio of A to D is compounded, but without regard to order; thus, let $3 : 4$ be the common ratio of A to B, and of B' to C', $5 : 7$ the common ratio of B to C and of A' to B', and $11 : 8$ the common ratio of C to D and of C' to D'. Then it is evident, from the demonstration of Prop. [24], that the ratio of A to C is $3 \times 5 : 4 \times 7$, and hence again, the ratio of A to D, $3 \times 5 \times 11 : 4 \times 7 \times 8$. And on the other hand, the ratio of A' to C' is $5 \times 3 : 7 \times 4$, and hence the ratio of A' to D', $5 \times 3 \times 11 : 7 \times 4 \times 8$; which is the same with $3 \times 5 \times 11 : 4 \times 7 \times 8$, because $5 \times 3 \times 11$ is the same with $3 \times 5 \times 11$, viz. 165, and $7 \times 4 \times 8$ is the same with $4 \times 7 \times 8$, viz. 224.[*] Therefore $165 : 224$ is the ratio both of A to D, and of A' to D', and consequently $A : D :: A' : D'$. But the ratio of A to D is compounded of the same ratios with that of A' to D', without regard to order.

Therefore, &c.

Cor. 1. If there be any number of ratios as those of A to B, of C to D, of E to F, &c., magnitudes which have to one another a ratio compounded of any two of these shall have the same ratio to one another with any other magnitudes which have to one another a ratio compounded of the same two; and, in like manner, magnitudes which have to one another a ratio compounded of any three of these shall have the same ratio to one another with any other magnitudes which have to one another a ratio compounded of the same three; and so on.

Cor. 2. If the ratios of A to B, of C to D, of E to F, &c. be all equal to one another, magnitudes which have to one another a ratio compounded of any two of them will have to one another a ratio which is the same with the duplicate ratio of A to B: and in like manner, magnitudes which have to one another a ratio compounded of any three of them will have to one another a ratio which is the same with the triplicate ratio of A to B; and so on.

Cor. 3. Ratios which are the duplicate, or triplicate, &c. of the same ratio are the same with one another.

Cor. 4. In the composition of ratios any two which are reciprocals of one another may be neglected, without affecting the resulting compound ratio. (See [10.] *Cor.*)

Cor. 5. If the ratio of A to C be compounded with the ratio of C to B, that is, with the reciprocal of B to C, the ratio A to B will be obtained, which being compounded with the direct ratio B to C, reproduces the ratio of A to C: that is, if one ratio be compounded with the reciprocal of another, a ratio will be obtained, which being compounded with that other, will again produce the first. This compounding of the reciprocal is sometimes called *subducting*, or taking away the direct ratio, and the result is termed the remaining ratio.

It appears, therefore, from the proposition, that if two ratios be equal to one another (and therefore compounded of equal ratios having any order in the composition of each), and if any of the equal ratios be subducted or taken away ([15]), the remaining ratios will be equal to one another.

Scholium.

The demonstrations which have been given of Prop. [24], [26], and [27], as above stated of commensurable magnitudes, are derived from a property of numerical ratios; viz. that "if the ratio of A to B be any whatever, as $3 : 4$, and, again, the ratio of B to C any whatever, as $5 : 7$, the ratio of A to C will be determined from these by multiplying their antecedents for a new antecedent, and their consequents for a new consequent." See the demonstration of Prop. [24], which is referred to in the demonstrations of Prop. [26] and [27]. It follows that a similar rule may be observed with regard to any number of ratios which are compounded according to *Def.* 12; viz. "In all cases in which the several numerical ratios of A to B, B to C, C to D, can be assigned, *i. e.* when the magnitudes A, B, C, D, are commensurable (8. *Cor.* 1.), the compound ratio, or that of the first A to the last D, will be expressed by an antecedent which is the product of all the antecedents, and a consequent which is the product of all the consequents." Hence, also, a numerical ratio is frequently said to be compounded of two or of any other number of numerical ratios, when its antecedent is the product of all their antecedents, and its consequent the product of all their consequents: for the magnitudes whose ratio it denotes will in such a case have to one another a ratio which is compounded of the ratios expressed by those others.

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[*] See Arithmetic, art. 23.

In a geometrical progression A, B, C, D, &c. of commensurable magnitudes, the successive terms have a common numerical ratio, *e. g.* $5 : 7$; therefore, the ratio of A to C, *i. e.* the duplicate of A to B, is $5 \times 5 : 7 \times 7$, the ratio of A to D, *i. e.* the triplicate of A to B, is $5 \times 5 \times 5 : 7 \times 7 \times 7$; and so on.

Prop. [28].

If there be two fixed magnitudes, A *and* B, *which are the limits of two others,* P *and* Q, (*that is to which* P *and* Q, *by increasing together or by diminishing together, may be made to approach more nearly than by any the same given difference*), *and if* P *be to* Q *always in the same given ratio of* C *to* D, A *shall be to* B *in the same ratio.*

First, let P and Q approach to A and B respectively by a continual increase, so that P and Q can never equal, much less exceed, A and B, but may be made to approach to A and B more nearly than by any the same given difference. And let a magnitude B' be taken* such that $A : B' :: C : D$. Then, if B' is not equal to B, it must either be less than B or greater than B. First, let it be supposed less, as by any difference b. Then, because $P : Q :: C : D$, and $A : B' :: C : D$, ([12]) $A : B' :: P : Q$; but A is always greater than P; therefore, B' is always greater than Q [(18)]. Wherefore, because Q is always less than B', which is less than B by b, Q cannot approach to B within the difference b, which is against the supposition. Therefore, B' cannot be less than B.

Again, if B' be supposed greater than B, take* A' such that $A' : B' :: A : B$. Then, because B is less than B', A is less than A' ([18]), as by some difference a. And, because $A' : B' :: A : B$, and $P : Q :: A : B$, ([12]) $A' : B' :: P : Q$: but B' is always greater than Q, because it is supposed to be greater than B, which is greater than Q; therefore A' is always greater than P ([18]). Wherefore, because P is always less than A', which is less than A by a, P cannot approach to A within the difference a, which is against the supposition. Therefore B' cannot be greater than B.

Therefore, in this case, B' cannot but be equal to B; that is, $A : B :: C : D$. And the other case, in which P and Q approach to A and B respectively, by a continual decrease, may be demonstrated after the same manner; indeed in the same words, if the word "greater" be everywhere substituted for "less," and "less" for "greater."

Therefore, &c.

This proposition will be found of very extensive application in Geometry. By help of it, the lengths of plane curves, and the areas bounded by them, the curved surfaces of solids, and the contents they envelope, may in many instances be brought into comparison with little greater difficulty than right lines, rectilineal areas, and solids bounded by planes. This will be exemplified in subsequent parts of the present treatise in cases which suppose the magnitudes compared to be similar, or of the same form; but the use of the proposition is by no means confined to these. It may be regarded as one of the first steps to what is called the higher Geometry, and in this view, likewise, is well worth the attention of the student.

General Scholium

On the proportion of commensurable magnitudes.

It was shown in the first proposition of this section ([9].) that if four magnitudes be proportionals, and if any measure of the second be contained a certain number of times in the first, a like measure of the fourth shall be contained the same number of times in the third. Hence it follows, that any terms expressing the ratio of the first to the second, express also the ratio of the third to the fourth. But no terms can express the ratio of two magnitudes except the lowest, and such as are equimultiples of the lowest terms; that is, except m, n, if m, n are the lowest terms, and $l \times m$, $l \times n$, where l is a number multiplying the lowest terms (6. *Cor.* 1.). And $\frac{m}{n} = \frac{l \times m}{l \times n}$ (Arith. art. 80.). Therefore, if four magnitudes be proportionals, and if a and b be the numbers of times any common measure of the first and second is contained in them respectively, and c and d the numbers of times any common measure of the second and fourth is contained in them, $\frac{a}{b}=\frac{c}{d}$. The converse is likewise true, that is, if $\frac{a}{b}=\frac{c}{d}$, the four magnitudes must be proportionals: for it may easily be shown, that fractions do not represent the same part of the whole unit, or, which is the same thing, are not equal to one another, except they be reducible to the same lowest terms, as $\frac{m}{n}$:* and these lowest terms will express, at once, the ratio of the first *to the second*, and of the third to the fourth (1. *Cor.* 3.).

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* See note at prop. [23].

The same conclusion may be stated in other words, as follows:

If four magnitudes be proportionals, and if A, B, C, D, *represent those magnitudes numerically, that is, if* A *and* B *represent the numbers of times the unit of their kind is contained in the two first, and if* C *and* D *represent the numbers of times the unit of their kind is contained in the two last, the quotient or fraction* $\frac{A}{B}$ *shall be equal to* $\frac{C}{D}$: *and conversely.*

This equation $\frac{A}{B}=\frac{C}{D}$ is the foundation of the theory of proportion as it is treated in Arithmetic or Algebra, (see Arithmetic, art. 127 and 128.) and leads with great facility to all the theorems of the foregoing section. Thus, adding 1 to each side, $\frac{A+B}{B}=\frac{C+D}{D}$ which is the theorem cited by the Latin word *componendo*; again subtracting 1 from each, $\frac{A-B}{B}=\frac{C-D}{D}$, which is that cited by the word *dividendo*: dividing 1 by each side, $\frac{B}{A}=\frac{D}{C}$, and multiplying $\frac{B}{C}$ into each, $\frac{A}{C}=\frac{B}{D}$, which are the theorems cited by the words *invertendo* and *alternando* respectively. And with equal readiness the rest may be derived from the same equation. As they are all however sufficiently considered in their several places, the remainder of this scholium will be confined to the explanation of certain rules commonly practised in the treatment of proportions, which suppose the terms A, B, C, D to be numbers, and are at once derivable from the equation $\frac{A}{B}=\frac{C}{D}$.

Rule. 1. If four magnitudes, which are numerically represented by A, B, C, D, be proportionals, the product of the extremes will be equal to the product of the means, *i. e.* if $A : B :: C : D$, $A\times D = B\times C$: and conversely if $A\times D=B\times C$ the magnitudes represented by A, B, C, D will be proportionals.

For, since $\frac{A}{B}=\frac{C}{D}$, multiplying each side by $B\times D$, $A\times D=B\times C$: and hence, conversely, dividing each side by $B\times D$, $\frac{A}{B}=\frac{C}{D}$, *i. e.* A, B, C, D are proportionals.

Rule 2. The same things being supposed, any common factor M, which is found in both the antecedents, or in both the consequents, or in both the antecedent and consequent of the same ratio, may be expunged.

For if $\frac{K\times M}{B}=\frac{L\times M}{D}$, dividing both sides by M, $\frac{K}{B}=\frac{L}{D}$:

Or if $\frac{A}{K\times M}=\frac{C}{L\times M}$, multiplying both sides by M, $\frac{A}{K}=\frac{C}{L}$:

Or if $\frac{K\times M}{L\times M}=\frac{C}{D}$, $\frac{K}{L}=\frac{C}{D}$, for $\frac{K\times M}{L\times M}=\frac{K}{L}$.

Or the Rule may be thus stated:

Expunge all common factors, except

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* It is shown in the treatise on Arithmetic (art. 61.) that the same number can have but one set of prime factors: from this it follows, that if $a\times d=b\times c$ (as is the case when $\frac{a}{b}=\frac{c}{d}$, Arith. art 127.), and if a be prime to b, the other factor d on the first side must contain b, and therefore must be of the form $l\times b$, where l is some whole number multiplying the whole number b; and hence it is evident that c is likewise of the form $l\times a$, where l is the same whole number multiplying the whole number a. Therefore if $\frac{a}{b}=\frac{c}{d}$, and if a be prime to b, c and d must be equimultiples of a and b, that is $\frac{c}{d}$ is reducible to $\frac{a}{b}$ by dividing and d by their greatest common factor (see Prop. 5. Scholium); and hence likewise if $\frac{a}{b}=\frac{c}{d}$, and if a be not prime to b, nor c prime to d, the fractions $\frac{a}{b}$ and $\frac{c}{d}$ must be reducible to the same lowest terms by dividing a and b by their greatest common factor, and c and d by their greatest common factor.

they be common only to the two extremes, or to the two means. For, in all other cases, if, as before, $K\times M$ and $L\times M$ be the terms having a common factor, and if P and Q be the terms not having a common factor, by multiplying extremes and means, $K\times P\times M = L\times Q\times M$, and dividing each side by M, $K\times P = L\times Q$.

Of these three cases, the one which occurs most frequently is the last, viz. that in which the common factor is found in the terms of the same ratio.

Rule 3. If there be two or more proportions, the products of the corresponding antecedents and consequents shall constitute a proportion.

For if $\frac{A}{B}=\frac{C}{D}$, and if also $\frac{A'}{B'}=\frac{C'}{D'}$ multiplying together the quantities upon the left hand, and also those upon the right of the two equations,

$$\frac{A\times A'}{B\times B'}=\frac{C\times C'}{D\times D'},$$

$$\begin{array}{l} \textit{i.e.}\text{ if } A : B :: C : D \\ \text{and } A' : B' :: C' : D' \\ \hline A\times A' : B\times B' :: C\times C' : D\times D'. \end{array}$$

Again, if there be a third proportion $A'' : B'' :: C'' : D''$, the terms of this being multiplied into those of the preceding, $A\times A'\times A'' : B\times B'\times B'' :: C\times C'\times C'' : D\times D'\times D''$; and so on, if there be any number of proportions.

When this Rule is applied, the resulting proportion is said to be *compounded* of the others, and hence the rule is called, "compounding the proportions." We may observe that the compound proportion commonly admits of reduction by Rule 2., on account of the same term or terms occurring in more than one of the component proportions.

$$\begin{array}{l} \text{If, for example, } A : B :: C : D \\ \text{and } B' : A :: D : C' \\ \text{and } A'' : C' :: B' : D'', \\ \hline \end{array}$$

the proportion which is compounded of these is $A\times B'\times A'' : B\times A\times C' :: C\times D\times B' : D\times C'\times D''$: which is reducible by Rule 2. to $A'' : B :: C : D''$.

The terms are, however, seldom (or never) so intermixed as in this example. The end which is usually proposed in the compounding of proportions is to obtain the ratio of one magnitude K to another L by means of a number of intermediate magnitudes; in order to which, K is made the antecedent of the leading ratio of the first proportion, and L the consequent of the like ratio in the last.

$$\begin{array}{l} \text{Thus if } K : B :: C : D \\ \text{and } B : B' :: C' : D' \\ \text{and } B' : L :: C'' : D'' \\ \hline \end{array}$$

by compounding the proportions, $K\times B\times B : B\times B'\times L :: C\times C'\times C'' : D\times D'\times D''$, and hence by Rule 2, $K : L :: C\times C'\times C'' : D\times D'\times D''$.

SECTION 3.—*The General Theory of Proportion.*

The foregoing theorems have been established upon the supposition that the magnitudes spoken of are commensurable. This, however, is not always the case with magnitudes: there are some (examples will appear in a future page) which have no common measure, and which are therefore said to be *incommensurable.*

In the present section, the similar relations of such magnitudes will be briefly considered; a new definition will be laid down, comprehending that already given (def. [7]) of the proportion of commensurable magnitudes, at the same time that it does not require that the magnitudes which satisfy it shall be commensurable; and to this new definition the theorems of the preceding section will be shown to apply equally as to the former.

In the first place, then, it is evident that, incommensurable magnitudes having no common part, their ratio can never be exactly expressed by numbers. Numbers may nevertheless be obtained which shall serve *to compare two such magnitudes* A *and* B *to any required degree of accuracy.*

Let B be divided into any large number of equal parts, a million for example: then A will contain a certain number of these parts with an excess which is less than one of them, less, that is, than a millionth part of B; so that if we take no account of this excess in our estimate of their relative magnitude, we shall commit an error of less than one-millionth. And it is plain that, in this manner, by dividing B into a still greater number of equal parts, the error of our estimate may be made as small as we please.

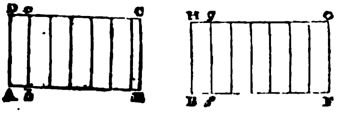
It is found, for example, (by methods which will be noticed hereafter) that, if the diameter of a circle be divided into 7 equal parts, the circumference will contain not quite 22 of those parts; if, again, the diameter be divided into 113 equal parts, the circumference will contain not quite 355 of those parts: if into 10,000,000, the circumference will contain not quite 31,415,927; and so on. Therefore the ratio of the circumference to the diameter is expressed by the ratio 22 : 7 nearly; more nearly by 355 : 133; still more nearly by 31,415,927 : 10,000,000; and so on: nor is there any limit to the accuracy of this approach, although there should (as is really the case) be no two numbers by which it can be expressed exactly.

This consideration brings us directly to *the only case in which*, consistently with the view already taken of the subject of equal ratios, *two magnitudes may be said to be similarly related (or in the same ratio) to two others of the same kind respectively, with which they are incommensurable. The ratios of the former to the latter, each to each, must admit of being approximately represented by the same numbers, to how great an extent soever the degree of approximation may be carried:* in other words, any like parts whatsoever of the two latter magnitudes, however minute they may be taken, must be contained in the two former, each in each, the same number of times, with corresponding* remainders less than the parts.

It has been already observed that this obtains with regard to the proportionals of def. [7]. When four magnitudes are proportionals by that definition, which supposes the first two and second two to be commensurable, there are, indeed, some like parts of the second and fourth which are contained in the first and third the same number of times without remainders; viz. the greatest common measures of the first two and second two, and any like parts of the greatest common measures: it is easy to perceive, however (and the same has been demonstrated at large in Prop. [9]), that any other like parts of the second and fourth will be contained in the first and third the same number of times, with corresponding less remainders. The following, therefore, is to be considered as the *general* test of two magnitudes A and C, having the same ratio to two others B and D, of the same kind with the former two respectively.

Def. 7. The first of four magnitudes is said to have the same ratio to the second which the third has to the fourth, when any like parts whatsoever of the second and fourth are contained in the first and third the same number of times exactly, or the same number of times with corresponding remainders less than the parts.

For example: let ABCD, EFGH be two rectangles having the same altitude, and let A B, E F be their bases. Let the base E F be divided into any number of equal parts E *f*, &c., and the rectangle E F G H into as many equal rectangles, E *f g* H, &c. by lines drawn through the points of division parallel to E H. Then if A *b*, &c. be taken equal to E *f*, and if straight lines be drawn through the points *b*, &c. parallel to AD; the base A B, and the rectangle A B C D, will contain, the one a certain number of parts equal to E *f*, and the other the same number of rectangles equal to E *f g* H, either exactly, or with corresponding remainders less than E *f* and E *f g* H. And this will always be the case, whatsoever be the number of parts into which E F is divided. Therefore, according to def. 7., the two rectangles and their two bases are proportionals.



Def. 8. If the first of four magnitudes contain any part of the second a greater number of times, with or without a remainder, than the third contains the like part of the fourth, the first is said to have to the second a greater ratio than the third has to the fourth also, in this case, the third is said to have to the fourth a less ratio than the first has to the second.

As from Prop. [9] with regard to commensurable proportionals, so from the terms of our new general definitions 7. and 8. with regard to the proportionals described in def. 7. it is at once evident that of four magnitudes, A, B, C, D, the first A cannot be said to have to the second B *the same ratio* which the third C has to the fourth D, according to def. 7, and at the same time a *greater* or a *less* ratio than C has to D, according to def. 8.: much less can A be said to have to B at the same time both a greater and a less ratio than C has to D.*

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* By the word "corresponding" here used, it is merely intended to point out the fact of there being two remainders, *i.e.* a remainder in the comparison of the two first magnitudes, and a remainder *corresponding* to it in the comparison of the two last. And in the same sense the word is to be understood in subsequent passages on the same subject.

* See note at Prop. [9.]

We proceed to the properties of this more general description of proportionals, which will be found the same with those already demonstrated in the preceding Section of commensurable proportionals. They will be considered accordingly in the same order, and will have the same numbers affixed to them. It will be observed, also, that they are stated in the same words, with the exception of Prop. 9., which is little more than another form of expressing def. 7., and its corollaries, which again express the same thing in different terms, the 2d. of them being, in fact, Euclid's celebrated definition of proportionals.

Prop. 9.

If four magnitudes A, B, C, D *be proportionals, the first and third shall contain respectively* $\frac{m}{n}$ths *of the second and fourth, or both more than* $\frac{m}{n}$ths, *or both less than* $\frac{m}{n}$ths, *whatsoever values may be given to* m *and* n; *and conversely, if this be the case with four magnitudes, they shall be proportionals.*

For, by def. 7., if A contain exactly $\frac{m}{n}$ths of B, C must contain exactly $\frac{m}{n}$ths of D: or, again, if A contain more than m of the n^{th} parts into which B is divided, as m' (suppose), or m' with a remainder, where m' is greater than m, C must also contain m' parts, or m' with a remainder, that is, more than m of the n^{th} parts into which D is divided: and in like manner, if A contain less than $\frac{m}{n}$ths of B, C will likewise contain less than $\frac{m}{n}$ths of D.

Next, let this be the case with four magnitudes, A, B, C, D, whatsoever numbers be substituted for m and n: A, B, C, D shall be proportionals.

For, if A and C contain exactly m of the n^{th} parts into which B and D are divided, the four A, B, C, D are commensurable proportionals, according to def. 7; and, therefore, also (by Prop. [9] of the last Section, as has been already observed) proportionals according to def. 7. Again, if A contain more than m, as m' and a remainder, that is more than m' and less than $m'+1$ parts B, C will contain, also, by the supposition, more than m' and less than $m'+1$ of the like parts of D, that is, the same number m' of these with a remainder, and this, whatever be the value of n. Therefore, (def. 7.) A, B, C, D are proportionals.

Therefore, &c.

Cor. 1. The same had been expressed by saying, "if A, B, C, D be proportionals, and if A be greater than $\frac{m}{n}$ B, C shall be likewise greater than $\frac{m}{n}$ D, if equal equal, and if less less, whatsoever values may be given to m and n: and, conversely, if this be the case with four magnitudes, they shall be proportionals."

Cor. 2. (Euc. v. def. 5.) Or, again, "if A, B, C, D be proportionals, and if n A be greater than m B, n C shall likewise be greater than m D, if equal equal, and if less less, whatsoever values may be given to m and n: and, conversely, if this be the case with four magnitudes, they shall be proportionals."

For n A, m B, n C, m D are equimultiples of A, $\frac{m}{n}$ B, C, $\frac{m}{n}$ D; and of the equimultiples of two magnitudes, one will be greater than, or equal to, or less than the other, according as the corresponding magnitude is greater than, or equal to, or less than the corresponding magnitude of the other; and conversely (ax. 1, 2, 3, 4).[*]

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[*] "The first of four magnitudes is said to have the same ratio to the second, which the third has to the fourth, when any equimultiples whatsoever of the first and third being taken, and any equimultiples whatsoever of the second and fourth; if the multiple of the first be less than that of the second, the multiple of the third is also less than that of the fourth; or, if the multiple of the first be equal to that of the second, the multiple of the third is also equal to that of the fourth; or, if the multiple of the first be greater than that of the second, the multiple of the third is also greater than that of the fourth."—Euc. v. def. 5.

This definition of proportionals has been sometimes found fault with as too abstruse and recondite for beginners;—which would not perhaps have been the case, had its connexion with the more obvious but confined view of def. [7] been always pointed out. For, we have seen that a general theory of proportion, which shall embrace indifferently all magnitudes, whether commensurable or otherwise, admits of no test essentially different from that which is here adopted. The greatest geometers in dwelling upon this part of the Elements have ever found cause to admire the profoundness and sagacity of their author. Witness the energetic testimony of Barrow, "That there is nothing in the whole body of the Elements of a more subtile invention, nothing more solidly established, or more accurately handled, than the doctrine of proportionals." Euclid has, indeed, left little in this respect, as in others, to be

Cor. 3. (Euc. v. def. 7.) In like manner, it may be shown, that, if there be four magnitudes A, B, C, D, and if a multiple of A, as m A, can be found which is greater than a multiple of B as n B, while the corresponding multiple of C, viz. m C, is not greater than the corresponding multiple of D, viz. n D, A has to B a greater ratio than C has to D. For, A will be greater than $\frac{m}{n}^{\text{th}}$ of B, but C will not be greater than $\frac{m}{n}^{\text{th}}$ of D; that is, the n^{th} part of B will be contained in A m times or more with a remainder, but the n^{th} part of D will not be contained in C so much as m times with a remainder. (See def. 8.)

Prop. 10. (Euc. v. 7.)

Equal magnitudes have the same ratio to the same magnitude: and the same has the same ratio to equal magnitudes.

For, any the same part of the same magnitude will be contained the same number of times in equal magnitudes with corresponding less remainders. And, again, any like parts of equal magnitudes, being equal to one another (ax. 2), will be contained the same number of times with the same remainder in the same magnitude.

Therefore, &c.

Cor. If a ratio which is compounded of two ratios be a ratio of equality, one of these must be the reciprocal of the other. (See [10] Cor.)

Prop. 11. (Euc. v. 8.)

Of two unequal magnitudes, the greater has a greater ratio to the same magnitude: and the same magnitude has a greater ratio to the lesser of the two.

The first is evident: for, of any the same magnitude a part may be found less than the difference of two unequal magnitudes, which part must evidently be contained a greater number of times in the greater than in the lesser of the two.

In the second place, therefore, let A and B be any two magnitudes of which A is the greater, and let C be any third magnitude: C shall have to the greater A a less ratio than the same C has to the other B.

Let D be the difference of A and B: then, whether B and D be or be not both of them greater than C, multiples m B, m D may be taken of them which are both greater than C. And, because C is less than m B, let multiples of C be taken, as 2 C, 3 C, &c. until a multiple be found, as $\overline{p+1}$.C, which is the first greater than m B. Then, because m B is not less than the preceding multiple p C, and because m D is greater than C, the two m B, m D together are greater than p C and C together, that is, than $\overline{p+1}$.C. But, because A is equal to B and D together, it is evident that m A is equal to m B and m D together. Therefore m A is greater than $\overline{p+1}$.C, and m B less than $\overline{p+1}$.C. Therefore the $\overline{p+1}^{\text{th}}$ part of A is contained in C, not so much as m times, and the $\overline{p+1}^{\text{th}}$ part of B is contained in C, m times with a remainder. Therefore (def. 8.) C has a less ratio to A than it has to B, or a greater ratio to B than it has to A. (See also 9. *Cor.* 3.)

Therefore, &c.

Cor. 1. (Euc. v. 9.) Magnitudes which have the same ratio to the same magnitude are equal to one another: as likewise those to which the same magnitude has the same ratio.

Cor. 2. A ratio which is compounded of two ratios, one of which is the reciprocal of the other, is a ratio of equality: for, A having to C a ratio which is compounded of the ratios of A to B and of B to C, if the latter be the same with the ratio of B to A, A must be equal to C.

Cor. 3. (Euc. v. 10.) If one of two magnitudes have a greater ratio to the same magnitude than the other has, the first must be greater than the other: and if the same magnitude have a greater ratio to one of two magnitudes than it has to the other, the first must be less than the other.

Prop. 12. (Euc. v. 11.)

Magnitudes A, B *and* C, D, *which have the same ratio with the same magnitudes* P, Q, *have the same ratio with one another.*

For any part of B will be contained in A exactly or with a remainder, as often as a like part of Q is contained in P exactly or with a remainder, because A : B :: P : Q, that is, as often as a like part of D, is contained in C exactly, or with a remainder, because C : D :: P : Q.

Therefore, &c.

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done by succeeding writers, but to follow his steps as closely as they are able: the principles, the theorems, the demonstrations they are in search of are all to be found in this one masterpiece, and for the most part under the simplest form. From this it will, of course, be easily understood how large a portion of the present Section is borrowed from the 5th Book of the Elements.

Cor. 1. (Euc. v. 13.) If A have to B the same ratio as C to D, and C to D a greater or a less ratio than E to F, A shall have to B a greater or a less ratio than E has to F.

Cor. 2. If A have to B a greater or a less ratio than C to D, and C to D the same ratio as E to F, A shall have to B a greater or a less ratio than E has to F.

Prop. 13. (Euc. v. D and C.)

If four magnitudes be proportionals, and if the first be any multiple or part of the second, the third shall be the same multiple or part of the fourth: and conversely, if one magnitude be the same multiple or part of another, that a third magnitude is of a fourth, the four magnitudes shall be proportionals.

The first is evident from def. 7.; for it is supposed in that definition that if any part of the second be contained a certain number of times in the first exactly, a like part of the fourth will be contained the same number of times in the third exactly. And, in the second place, if A be the same multiple or part of B that C is of D, A, B, C, D will be commensurable proportionals, and therefore ([9]) also proportionals by def. 7.

Therefore, &c.

Prop. 14. (Euc. v. A.)

If four magnitudes A, B, C, D *be proportionals, and if the first be greater than the second, the third shall be greater than the fourth; if equal, equal; and if less, less.*

For, if A be greater than B, a part of the latter may be found which shall be less than their difference, and which shall therefore be contained a greater number of times exactly, or with a remainder, in A, than in B: therefore, because A, B, C, D are proportionals, the like part of D will also be contained a greater number of times exactly or with a remainder in C than in D, that is, C will be greater than D.

And, in the same manner it may be shown, that if A be equal to B, C will be equal to D; and if less, less.

Therefore, &c.

Prop. 15. (Euc. v. B.)

If four magnitudes A, B, C, D *be proportionals, they shall also be proportionals when taken* inversely: *that is,* invertendo, $B : A :: D : C$.

Of A, C take any equimultiples whatever $m\,A$, $m\,C$, and of B, D, any equimultiples whatever $n\,B$, $n\,D$. Then, according as $n\,B$ is equal to, or greater than, or less than $m\,A$, it is plain that $m\,A$ will be equal to, or less than, or greater than $n\,B$; and, therefore, on account of the proportion, (9. *Cor.* 2.) $m\,C$ will be likewise equal to, or less than, or greater than $n\,D$, that is, $n\,D$ will be equal to, or greater than, or less than $m\,C$. And this will be the case, whatever be the numbers n and m. Therefore (9. *Cor.* 2.) B, A, D, C are proportionals.

Therefore, &c.

Prop. 16. (Euc. v. 4.)

If four magnitudes A, B, C, D *be proportionals, and if there be taken any equimultiples of the first and third; and also any equimultiples of the second and fourth: these equimultiples shall likewise be proportionals.*

Let $m\,A$, $m\,C$ be any equimultiples whatever of A, C, and $n\,B$, $n\,D$ any equimultiples whatever of B, D. Take K, M any equimultiples whatever of $m\,A$, $m\,C$, and therefore (2.) likewise equimultiples of A, C; also L, N any equimultiples whatever of $n\,B$, $n\,D$, and therefore likewise equimultiples of B, D. Then, because $A : B :: C : D$, if K be greater than L, M will be greater than N; if equal, equal; and if less, less (9. *Cor.* 2.). But K, M are any equimultiples whatever of $m\,A$, $m\,C$, and M, N any equimultiples whatever of $n\,B$, $n\,D$. Therefore (9. *Cor.* 2.) $m\,A : n\,B :: m\,C : n\,D$.

Therefore, &c.

Cor. If A, B, C, D be proportionals, and if any like parts of the first and third be taken, and also any like parts of the second and fourth, these like parts will likewise be proportionals.

Prop. 17. (Euc. v. 15.)

Magnitudes have the same ratio to one another which their equimultiples have.

For, if $m\,A$, $m\,B$ be any equimultiples whatever of A, B, it is evident, that any part P of B will be contained in A exactly, or with a remainder, as often as m times that part or $m\,P$ is contained in $m\,A$ with a remainder. Also, whatever part P is of B, $m\,P$ is the like part of $m\,B$. Therefore, $A : B :: m\,A : m\,B$ (def. 7).

Therefore, &c.

Cor. 1. Magnitudes have the same ratio to one another which their like parts have.

Cor. 2. If A, B, C, D be proportionals, and if any equimultiples whatever be taken of the first and second, and also any equimultiples whatever of the third and fourth, these equimultiples will likewise be proportionals (12).

The same, it is evident, may be stated with regard to any like parts taken of the first and second, and also of the third and fourth.

Prop. 18. (Euc. v. 14.)

If four magnitudes of the same kind be proportionals, and if the first be greater than the third, the second shall be greater than the fourth; if equal, equal; and if less, less.

Let A, B, C, D, be proportionals, and first, let A be greater than C; B shall be greater than D.

For A, being greater than C, has to B a greater ratio (11.) than C has to B: therefore (12. *Cor.* 1.) C has also to D a greater ratio than the same C has to B: therefore, (11. *Cor.* 3.) B is greater than D.

And, in like manner it may be shown, that if A be equal to C, B must be equal to D; and if less, less.

Therefore, &c.

Cor. Hence, also, if four magnitudes of the same kind be proportionals, and if the second be greater than the fourth, the first will be greater than the third; if equal, equal; and if less, less.

Prop. 19. (Euc. v. 16.)

If four magnitudes A, B, C, D *of the same kind be proportionals, they shall also be proportionals, when taken* alternately; *that is,* alternando A : C : : B : D.

Of A, B take any equimultiples whatever *m* A, *m* B, and of C, D any equimultiples whatever *n* C, *n* D: then (17. *Cor.* 2.) *m* A, *m* B, *n* C, *n* D are proportionals; and, therefore, (18.) if *m* A be greater than *n* C, *m* B will also be greater than *n* D; if equal, equal; if less, less. But *m* A, *m* B are any equimultiples whatever of A, B, and *n* C, *n* D any equimultiples whatever of C, D. Therefore, (9. *Cor.* 2.) A : C : : B : D.

Therefore, &c.

Prop. 20. (Euc. v. 17.)

If four magnitudes A, B, C, D, *be proportionals, they shall also be proportionals, when taken* dividedly: *that is, the difference of the first and second shall be to the second as the difference of the third and fourth to the fourth;—or* dividendo A~B : B : : C~D : D.

For, if any like parts whatever, as the n^{th}, of B and D be contained in A and C the same number of times, *m*, exactly, or with corresponding remainders, they will be contained in A~B and C~D, the same number of times, *m*~*n*, exactly, or with the same remainders. Therefore, (*def.* 7.) A~B : B : : C~D : D.

Therefore, &c.

Cor. 1. (Euc. v. E.) If four magnitudes A, B, C, D be proportionals, then *convertendo*, A : A~B : : C : C~D (See [20] *Cor.* 1.)

Cor. 2. (Euc. v. 25.) If four magnitudes of the same kind be proportionals, the greatest and least of them together shall be greater than the other two together. (See [20] *Cor.* 2.)

Cor. 3. If three magnitudes be proportionals, half the sum of the extremes shall be greater than the mean: in other words, the arithmetical mean between two magnitudes is greater than the geometrical mean between the same two. (See [20] *Cor.* 3.)

Prop. 21. (Euc. v. 18

If four magnitudes A, B, C, D, *be proportionals, they shall also be proportionals when taken* conjointly; *that is, the sum of the first and second shall be to the second, as the sum of the third and fourth to the fourth: or,* componendo, A+B : B : : C+D : D.

For, if any like parts whatever, as the n^{th}, of B and D be contained in A and C the same number of times, *m*, exactly, or with corresponding remainders, they will be contained in A+B and C+D the same number of times, *m*+*n*, exactly, or with the same remainders. Therefore, (*def.* 7.) A+B : B : : C+D : D

Therefore &c.

Cor. If four magnitudes A, B, C, D be proportionals, A : A+B : : C : C+D. (See [21] *Cor.*)

Prop. 22. (Euc. v. 19.)

If one magnitude be to another as a magnitude taken from the first to a magnitude taken from the other, the remainder shall be to the remainder in the same ratio.

See the demonstration of [22]

Cor. 1. If there be any number of magnitudes A, B, C, D, &c. in geometrical progression, the differences A~B, B~C, C~D, &c. will form a geometrical progression in which the successive terms have the same ratio with the successive terms of the former (See [22] *Cor.* 1.)

Cor. 2. And conversely, any number of magnitudes A, B, C, D, &c. in geometrical progression may be considered as the differences of other magnitudes A', B', C', D', E', &c. forming a geometrical progression, in which the first term A' is to A, as A to $A \sim B$, and the successive terms have the same ratio with the successive terms of the former. (See [22]. *Cor.* 2.)

Prop. 23.

If one magnitude be to another as a third magnitude of the same kind to a fourth, the sum of the first and third shall be to the sum of the second and fourth in the same ratio.

See the demonstration of [23].

Cor. 1. (Euc. v. 12.) Hence, if there be any number of magnitudes of the same kind antecedents, and as many consequents, and if every antecedent have the same ratio to its consequent, the sum of all the antecedents shall have the same ratio to the sum of all the consequents.

Cor. 2. If the ratio of A' to B' be not the same with the ratio of A to B, the ratio of $A + A'$ to $B + B'$ will not be the same with the ratio of A to B; but less, if the magnitudes added be to one another in a less ratio, or greater, if the magnitudes added be to one another in a greater ratio.* (See [23]. *Cor.* 2.)

Cor. 3. Hence, if the ratio of A' to B' be not the same with the ratio of A to B, the ratio of $A + A'$ to $B + B'$ will lie between the ratios of A to B and of A' to B'; that is, it will be greater than the lesser of the two, and less than the greater of the two.

Prop. 24.

If there be three magnitudes of the same kind, A, B, C, *and other three* A', B', C', *which, taken two and two in order, have the same ratio; viz.* A *to* B *the same ratio as* A' *to* B', *and* B *to* C *the same ratio as* B' *to* C'; *then* ex æquali in proportione directâ (*or* ex æquo) *the first shall be to the third of the first magnitudes, as the first to the third of the others; or, as it may be more briefly stated,*

$$\begin{array}{r} \textit{if } A : B :: A' : B' \\ \textit{and } B : C :: B' : C' \\ \hline \textit{then}, \text{ ex æquali } A : C :: A' : C'. \end{array}$$

Of A, A' take any equimultiples mA, mA', of B, B' any equimultiples nB, nB', and of C, C' any equimultiples pC, pC': then, because A, B, A', B are proportionals, mA, nB, mA', nB' are also proportionals, (16.): and, in like manner, because B, C, B', C' are proportionals, nB, pC, nB', pC' are also proportionals.

Now, if mA be greater than pC, it will have to nB a greater ratio than pC has (11.); and therefore, (12. *Cor.* 1.) mA' will also have to nB' a greater ratio than pC has to nB; that is, on account of the proportionals pC, nB, pC', nB' (15.), a greater ratio than pC' has to nB' (12. *Cor.* 2.); therefore, also, mA' will be greater than pC' (11. *Cor.* 3.). In the same manner it may be shown, that if mA be equal to pC, mA' will also be equal to pC'; and, if less, less. And the numbers m, p may be any whatever. Therefore (9. *Cor.* 2.) A, C, A', C' are proportionals, and $A : C :: A' : C'$.

Therefore, &c.

Cor. 1. (Euc. v. 22.) Hence, also,

$$\begin{array}{r} A : B :: A' : B' \\ \text{and } B : C :: B' : C' \\ \text{and } C : D :: C' : D' \\ \hline \end{array}$$

ex æquali $A : D :: A' : D'$; and the same is true whatever be the number of magnitudes A, B, C, D, &c. A', B', C', D', &c.

This may be stated in the following words: "Ratios, which are compounded of any number of equal ratios in the same order, are equal to one another."

Cor. 2. If four magnitudes A, B, C, D be proportionals, then *miscendo* $A + B : A \sim B :: C + D : C \sim D$. (See [24] *Cor.* 2.)

Prop. 25. (Euc. v. 24.)

If two proportions have the same consequents, the sum of the first antecedents shall be to their consequent, as the sum of the second antecedents to their consequent; that is, if $A : B :: C : D$, *and if* $A' : B :: C' : D$, *then* $A + A' : B :: C + C' : D$.

See the demonstration of [25].

Cor. 1. The same may be stated of

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* It is required in the demonstration of this Corollary, by help of the proposition, as in [23]. *Cor.* 2., that to two given magnitudes of the same kind, and a third, there may be found a fourth proportional. The case in which the two first magnitudes are commensurable has been already noticed in the note upon [23]. *Cor.* 2.: in the other case, *i. e.* when they are incommensurable, we can only approximate to the fourth proportional as we approximate to the ratio of the two magnitudes numerically, as in the opening of this section. Since, however, such approximation may be continued without limit, and magnitudes can be obtained which are in a greater and less ratio to the given magnitude, according to def. 8., we presume that there is some magnitude between them which is to the given third magnitude in the same ratio which the second has to the first—that is, some magnitude which is a fourth proportional to the three.

any number of proportions having the same consequents; that is, the sum of all the first antecedents shall be to their consequent as the sum of all the second antecedents to their consequent.

Cor. 2. In like manner, also, it may be shown (by "*dividendo*" instead of "*componendo*") that if two proportions have the same consequents, the difference of the first antecedents shall be to their consequent as the difference of the second antecedents to their consequent.

Cor. 3. Hence if there be any number of magnitudes of the same kind A, B, C, D, E, F, and as many others A′, B′, C′, D′, E′, F′, and if the ratios of the first to the second, of the second to the third, of the third to the fourth, and so on, be respectively the same in the two series; any two combinations by sum and difference of the magnitudes of the first series, *e. g.* $A + C - E$ and $B - C + D$ shall be to one another as two similar combinations of the corresponding magnitudes of the second series, viz. $A' + C' - E'$ and $B' - C' + D'$.

PROP. 26.

If there be three magnitudes of the same kind A, B, C, *and other three* A′, B′, C′, *which taken two and two, but in a cross order, have the same ratio, viz.* A *to* B *the same ratio as* B′ *to* C′, *and* B *to* C *the same ratio as* A′ *to* B′; *then,* ex æquali in proportione perturbatâ *(or* ex æquo perturbato), *the first shall be to the third of the first magnitudes as the first to the third of the others; or, as it may be more briefly stated,*

$$\begin{array}{r} \textit{if } A : B :: B' : C' \\ \textit{and } \underline{B : C :: A' : B'} \\ \text{ex æquo perturbato } A : C :: A' : C'. \end{array}$$

Of A, B and A′ take any equimultiples $m\,A$, $m\,B$ and $m\,A'$, and of C, B′ and C′ any equimultiples $n\,C$, $n\,B'$ and $n\,C'$: then, because A, B, B′, C′, are proportionals, $m\,A$, $m\,B$, $n\,B'$, $n\,C'$ are also proportionals (17. *Cor.* 2.); and, because B, C, A′, B′ are proportionals, $m\,B$, $n\,C$, $m\,A'$, $n\,B'$ are also proportionals (16).

Now, if $m\,A$ be greater than $n\,C$, it will have to $m\,B$ a greater ratio than $n\,C$ has to $m\,B$ (11.); and therefore, (12. *Cor.* 1.) $n\,B'$ will also have to $n\,C'$ a greater ratio than $n\,C$ has to $m\,B$, that is, on account of the proportionals $n\,C$, $m\,B$, $n\,B'$, $m\,A'$ (15.), a greater ratio than $n\,B'$ has to $m\,A'$ (12. *Cor.* 2.): therefore, also, $m\,A'$ will be greater than $n\,C'$ (11. *Cor.* 3.). In the same manner, it may be shown that if $m\,A$ be equal to $n\,C$, $m\,A'$ will also be equal to $n\,C'$, and if less, less. And the numbers m, n may be any whatever. Therefore (9. *Cor.* 2.), A, C, A′, C′ are proportionals, and $A : C :: A' : C'$.

Therefore, &c.

Cor. 1. (Euc. v. 23.) Hence, also,

$$\begin{array}{r} \text{if } A : B :: C' : D' \\ \text{and } B : C :: B' : C' \\ \text{and } \underline{C : D :: A' : B'} \\ \textit{ex æquo perturbato } A : D :: A' : D'; \end{array}$$

and the same is true whatever be the number of magnitudes A, B, C, D, &c. A′, B′, C′, D′, &c.

This may be stated in the following words: "ratios which are compounded of any number of equal ratios, but in a reverse order, are equal to one another."

PROP. 27. (EUC. v. F.)

Ratios which are compounded of the same ratios, in whatsoever orders, are the same with one another.

The case of ratios which are compounded of the same ratios in the same order is that of 24. *Cor.* 1.

The case, again, of ratios which are compounded of the same ratios in a reverse order, is that of 26. *Cor.* 1.

Let the ratio of A to D, therefore, be compounded of the ratios of A to B, of B to C, and of C to D: and let these ratios be the same respectively with the ratios of C′ to D′, of A′ to B′ and of B′ to C′, of which the ratio of A′ to D′ is compounded, but in a different order: the ratio of A to D shall be the same with the ratio of A′ to D′.

$$\begin{array}{r} \text{For, because } B : C :: A' : B' \\ \text{and } \underline{C : D :: B' : C'} \\ \textit{ex æquali, } B : D :: A' : C' \\ \text{but again, } \underline{A : B :: C' : D'} \end{array}$$

$$\begin{array}{l} \text{Therefore } \textit{ex æquo} \\ \textit{perturbato} \;\ldots\; A : D :: A' : D'. \end{array}$$

And, in a similar manner, the case of ratios which are compounded of the same three ratios in any other order, may be demonstrated. For, if K, L, M represent the three ratios in one order, in whatever other order they may be arranged, two of them will be found which are contiguous in both arrangements; commencing with which two, the demonstration will differ little from the above.

Hence, again, it may be shown, that ratios which are compounded of the same four ratios K, L, M, N in whatsoever orders, are the same with one another; as for instance, in the orders K, L, M, N, and M, K, N, L: for the latter ratio is the same with the ratio which is compounded of the same ratios in the order M,K,L,N, because the ratio which is compounded of K, N, L, is the same with that which is compounded of K, L, N; and, for a similar reason, the ratio which is compounded of M, K, L, N, is the same with that which is compounded of K, L, M, N.

And the same reasoning may be extended to five, six, or any other number of ratios.

Therefore, &c.

Cor. 1 (Euc. v. G.) If there be any number of ratios as those of A to B, of C to D, of E to F, &c. magnitudes which have to one another a ratio compounded of any two of these shall have the same ratio to one another with any other magnitudes which have to one another a ratio compounded of the same two; and, in like manner, magnitudes which have to one another a ratio compounded of any three of these shall have the same ratio to one another with any other magnitudes which have to one another a ratio compounded of the same three; and so on.

Cor. 2. If the ratios of A to B, of B to D, of E to F, &c. be all equal to one another, magnitudes which have to one another a ratio compounded of any two of them will have to one another a ratio which is the same with the duplicate ratio of A to B: and in like manner, magnitudes which have to one another a ratio compounded of any three of them will have to one another a ratio which is the same with the triplicate ratio of A to B; and so on.

Cor. 3. Ratios which are the duplicate, or triplicate, &c. of the same ratio are the same with one another.

Cor. 4. In the composition of ratios any two which are reciprocals of one another may be neglected, without affecting the resulting compound ratio. (See 10. *Cor.*)

Cor. 5. Hence, if two ratios be equal to one another (and therefore compounded of equal ratios, having any order in the composition of each), and if any of the equal ratios be subducted or taken away, the remaining ratios will be equal to one another. (See [27] *Cor.* 5.)

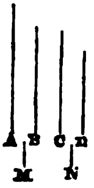
PROP. 28.

If there be two fixed magnitudes A *and* B, *which are the limits of two others* P *and* Q (*that is, to which* P *and* Q, *by increasing together, or by diminishing together, may be made to approach more nearly than by any the same given difference*) *and if* P *be to* Q *always in the same given ratio of* C *to* D; A *shall be to* B *in the same ratio.*

See the demonstration of [28.] and the note at 23. *Cor.* 2.

It has been already observed, that this proposition is of extensive application in Geometry, and the uses have been mentioned to which it will be found applied in subsequent parts of this treatise. The present section, or rather, the passing from a property demonstrated of commensurable proportionals to the demonstration of the same property with regard to incommensurable proportionals, offers an immediate illustration of it; which seems also the rather not to be passed by in this place, as we here take leave of the abstract theory of proportion, to consider its application to the proper subjects of Geometry, viz. lines, surfaces and solids; and a theory of proportion would scarcely appear complete without some notice of the equality existing between the *products of the extremes and means.* In what sense this expression is to be interpreted with regard to a proportion of four magnitudes of any kind A, B, C, D, was pointed out in the General Scholium at the end of Section II.; viz. that it supposes the magnitudes to be commensurable, and is to be understood of the numbers which stand for them. But we have seen that the term *product* is sometimes also, for the sake of brevity, used synonymously for *rectangle,*—as whenever the product of two lines is spoken of, meaning the rectangle which they contain.

We propose then to demonstrate generally, that, *if four straight lines* A, B, C, D *be proportionals,* (*whether commensurable or otherwise,*) *the rectangle under the extremes will be equal to the rectangle under the means;* and that in such a manner as may serve to illustrate the use of Prop. 28.



First, let A and B be commensurable, and therefore also C and D: and let their common ratio be any whatever, as 7 : 5; that is, let there be common measures M, N, the first of A and B, and the second of C and D, which are contained in A and C respectively 7 times, and in B and D respectively 5 times. Then because A contains M 7 times, and that D contains

N 5 times, the rectangle $A \times D$ contains the rectangle $M \times N$, 7×5 (I. 29. *Cor.* 1.) that is 35 times: and for a like reason, the rectangle $B \times C$ contains the rectangle $M \times N$, 5×7 or 35 times. Therefore (ax. 1.) the rectangle $A \times D$ is equal to the rectangle $B \times C$.

In the next place let the straight lines, A, B, and therefore also C, D, be incommensurable, A, B, C, D, being proportionals according to def. 7. Then (def. 7.) straight lines P and Q may be found, which approach nearer to A and C, than by any assigned difference, and contain like parts of B and D; so that by what has been just demonstrated, $P \times D = Q \times B$. But $P \times D$, and $Q \times B$ may be made, by increasing together, to approach more nearly to $A \times D$, and $C \times B$, than by any assigned difference, because, by taking the like parts of B, D continually less and less, P and Q may be made to increase towards A and C within any assigned difference. Therefore (Prop. 28.) $A \times D = C \times B$ or $B \times C$.

Therefore, &c.

The same will be demonstrated after a different manner in Section 5, to the subject of which Section the theorem properly belongs.[*]

SECTION 4.—*Proportion of the Sides in Triangles.*

Def. 13. Two straight lines are said to be *similarly divided*, when any two adjoining parts of the one have to one another the same ratio with the corresponding parts of the other.

It is evident (*ex æquali*), that when this is the case, any two parts whatsoever of the one will have to one another the same ratio as the corresponding parts of the other.

14. (Euc. vi. *Def.* 1.) Two rectilineal figures are said to be *similar* when they have all their angles equal, each to each, and the sides about the equal angles proportionals.[*]

(Euc. vi. 21.) It is evident from this definition that rectilineal figures, which are similar to the same rectilineal figure, are similar to one another. (I. ax. I. and II. 12.)

15. *Homologous sides* of similar figures are those which lie between equal angles, and are antecedents, or consequents, of the same proportion.

16. Two straight lines are said to be *similarly placed* in two similar figures, when they cut corresponding sides of the figures proportionally.

PROP. 29. (EUC. vi. 2.)

If a straight line be drawn parallel to the base of a triangle, it shall cut the sides, or the sides produced, proportionally: and, conversely, if a straight line lat *the sides of a triangle, or the sides roduced, proportionally, it shall be parallel to the base of the triangle.*

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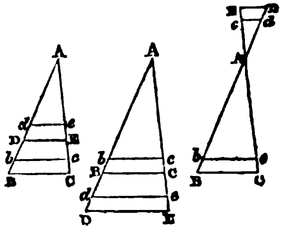
[*] Upon quitting this subject, the consideration of which has carried us to so great a length, it may be well to observe why the common arithmetical or algebraical method of treating the same appears insufficient for the purposes of Geometry. In the first place, that method does not represent the magnitudes themselves as constituting the proportion, but only the numbers by which the magnitudes are measured; upon which account, should the method be adopted, the task would still remain of explaining its application to magnitudes in general, before it was applied to the magnitudes treated of in Geometry. In the next place, it does not furnish a general test of proportion, but distinguishes, in every instance, the two cases of rational numbers and such as are irrational, corresponding to our cases of commensurable magnitudes and such as are incommensurable—a separation which, though it has been, for the sake of simplicity, admitted into the theory of the foregoing sections, is avoided in their application both by the definition of Euclid, and by our similar definition in Section 3. Lastly, the arithmetical method is so concise, that it becomes necessary to develope it in its application, and to state anew the important rules it furnishes for the treatment of proportions; whether with the view of deriving a new proportion from one taken singly, or with that of combining two or more proportions in order to obtain from them a single proportion, which could result from neither of them taken alone. The manner in which it has been attempted to supply these desiderata in the present treatise, and to furnish a theory which, at the same time that it should be complete and general, might be easily apprehended, and easily applied, is undoubtedly prolix, though not more so than the importance of the subject seemed to demand. The embarrassments of the beginner have been obviated and (it is hoped) in a great measure removed by the more simple theory of commensurable proportion of Section 2., inserted chiefly with this view; nor is the subject left so lame and incomplete as would have been the case had the general discussion of Section 3. been entirely omitted. (See note at prop. [9]).

[*] This definition, like some others to be found in the Elements, is excessive. To contain no more than is strictly necessary (or, indeed, than as yet has appeared to be possible) it should be modified as follows. *Two rectilineal figures are said to be similar, when the first has all its sides but one proportional to the sides of the other, and the angles included by those sides equal to the angles included by the corresponding sides of the other.* For a figure of any number of sides, as 6, being determined by 5 of those sides placed at given angles, it is sufficiently evident, that a figure of 6 sides may be constructed, which shall have 5 of those sides in the same ratio respectively to 5 sides of a given six-sided figure, and the four included angles in order, equal to the four corresponding angles of that figure: but it is not sufficiently evident that, being so constructed, it will have its 6th side in the same ratio to the 6th side of the other figure, and the two adjacent angles equal, respectively, to the corresponding angles of the other.

This kind of excess in a definition is, in the accurate language of mathematics, always more or less objectionable; especially as instances are not wanting, in which it is assumed, without proof, as the foundation of certain properties of the thing defined. Such is the case, for example, when parallel straight lines are defined to be such, that any two points whatsoever in the one are at equal distances from the other; whereas, it cannot be stated *a priori* of two straight lines, that more than two points of the one shall be at equal distances from the other. When speaking, therefore, for the first time of similar figures in 32. *Cor.* 1, it will be our care to demonstrate that part of *Def.* 14, which is assumed, retaining it in the text as allowedly more concise and obvious to apprehension than under the restricted form.

Let A B C be a triangle, and let the straight line D E, which is drawn parallel to the base B C, cut the sides A B, A C, or the sides A B, AC produced, in the points D, E: A D shall be to A B as A E to A C.



Let A B be divided into any number of equal parts B *b*, &c.: then it may be shown, as in I. 49, that AC will be divided into as many equal parts C *c*, &c. by lines drawn parallel to B C: also, if in A D there be taken as many parts D *d*, &c. equal to B *b*, as can be found in it, exactly, or with an excess less than B *b*, and if through the points *d*, &c. lines be drawn parallel to B C or D E, there will be found in AE, the same number of parts E *e*, &c. equal to C *c*, exactly, or with a corresponding excess less than C *c*. And this will be the case, how great soever be the number of parts into which A B, A C are divided; therefore, (*Def.* 7.) A D : A B : : A E : A C.

Next, let A D be to A B as A E to A C: D E shall be parallel to B C. For, if there be any line whatever, B C′, parallel to D E, then, by the first part of the proposition, A E is to A C′ as A D to A B, that is, as A E to A C: therefore, (18.) A C′ is equal to A C, that is, the points C′ and C coincide, and B C is parallel to D E.

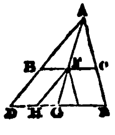
Therefore, &c.

Cor. In the same manner it may be shown, that if the sides of an angle be cut by any number of parallels, any two parts of the one will have to one another the same ratio as the corresponding parts of the other, that is, the sides will be similarly divided in the points of section (def. 13.)

Prop. 30.

If a straight line be drawn parallel to the base of a triangle, and be terminated by the sides, or by the sides produced, any straight line which passes through the vertex, and a point of the base, shall divide the base and its parallel in the same ratio: and, conversely, if the base and its parallel be divided in the same ratio, the straight line which joins the points of division, or that straight line produced, shall pass through the vertex of the triangle.

Let A B C be a triangle, and let the straight line D E, which is drawn parallel to the base B C, cut the sides A B, A C in the points D, E respectively: then if any straight line A F be drawn through the vertex, to cut the base in F and its parallel in G, B F shall be to F C as D G to G E.



Through F draw F H parallel to A D, to meet D E in H. Then, (29.) D G is to H G, as A G to F G, and, *convertendo*, D G is to D H, as A G to A F. But D H is equal to B F (I. 22.) therefore, D G is to B F as A G to A F. In the same manner it may be shown, that G E is to F C as A G to A F. Therefore, (12.) D G : B F : : G E : F C; and *alternando*, D G : G E : : B F : F C.

In the next place let B F be to F C as D G to G E; and let the points F, G be joined: the straight line F G, or F G produced, shall pass through the point A. For, let A F (produced if necessary) cut D E in any point whatever G′: then, by the former part of the proposition, D G′ : G′ E : : B F : F C, but D G : G E : : B F : F C, therefore, (12.) D G′ : G′ E : : D G : G E, Therefore the straight line D E is divided in the same ratio in the points G′ and G. But two points cannot divide the same straight line in the same ratio except they coincide: for if H, for example, be any other point than G in the line D E, and if D H be less than D G, the ratio of D H to G E will be less (11.) than the ratio of D G to G E; and the ratio of D H to H E still less than the ratio of D H to G E, because HE is greater than G E: much more then is the ratio of D H to H E less than the ratio of D G to G E. And in the same manner it may be shown, that if D H be greater than D G, D H will be to H E in a greater ratio than D G to G E. Therefore, as was before stated, two points cannot divide the same straight line in the same ratio, except they coincide. Therefore the points G′, G coincide, *i. e.* A F G is no other than a straight line, and F G produced passes through A.

Therefore, &c.

Cor. 1. If two parallel straight lines be cut by any number of straight lines which pass through the same point, they shall be similarly divided in the points of section.

Cor. 2. It has been seen in the course of the demonstration that the parts of the base are to the parts of the parallel respectively in the same ratio, viz. as A G to A F, or, which is the same thing (29.), as A B to A D: therefore also the whole base (23.) is to the whole parallel as A B to A D; that is, as the side of the triangle to the segment next the vertex which is cut off by the parallel from the side, or from the side produced.

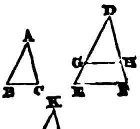
The same is at once made evident by drawing through C a parallel to A D.

Cor. 3. It has also been seen in the course of the demonstration that a straight line D E can be divided in the same ratio in one point only. Here D E and E D are considered as different straight lines, viz. differing in their first points.

Prop. 31. (Euc. vi. 4. 5.)

If two triangles are equiangular, they shall have the sides about the equal angles proportionals, those which are opposite to the equal angles, being the antecedents or consequents of the ratios and, conversely.

Let A B C, D E F be two triangles having the three angles A, B, C of the one equal to the three angles D, E, F of the other, each to each: the sides about any two equal angles A, D, shall be proportionals, that is, A B shall be to A C as D E to D F.



In D E, D F, or in D E, D F produced, take D G, D H equal to A B, A C respectively; and join G H. Then, because the triangles A B C, D G H, have two sides of the one equal to two sides of the other, each to each, and the included angles equal, they are equal in every respect (I. 4.), and the angle D G H is equal to A B C, that is, to D E F. Therefore, (I. 15.) G H is parallel to E F, and (29.) D G : D E :: D H : D F; therefore, *alternando*, D G : DH :: D E : D F, that is, A B : A C :: D E : D F.

In the same manner it may be shown that A B is to B C as D E to E F; and again, A C to C B as D F to F E.

Next, let the triangles A B C, D E F have their sides, each to each, in the same ratio, that is, A B to A C, as D E to D F, A C to C B, as D F to F E, and therefore, *ex æquali*, (24.) A B to B C, as D E to E F: the triangle A B C shall have its three angles A, B, C equal to the three angles D, E, F of the triangle D E F, each to each.

Let K L M be a triangle having its three angles K, L, M equal to the three angles D, E, F respectively, and its side K L equal to A B (I. 50.). Then, by the first part of the proposition, because the triangles K L M, D E F are equiangular, K L : L M :: D E : E F, but D E : E F :: A B : B C, therefore, (12.) K L : L M :: A B : B C : but K L is equal to A B; therefore, (18.) L M is equal to B C. And in the same manner it may be shown that K M is equal to A C. Therefore, the triangles A B C, K L M have the three sides of the one equal to the three sides of the other, each to each, and are, consequently (I. 7.), equal in every respect. Therefore A B C is equiangular with K L M, that is, with D E F.

Therefore, &c.

Cor. 1. Equiangular triangles are similar (*Def.* 14.)

Cor. 2. Triangles which have the sides of the one parallel to the sides of the other, or perpendicular to the sides of the other, are similar. For such triangles are equiangular (I. 18.).

Cor. 3. The same may be said of triangles which have the sides of the one making any equal angles with the sides of the other towards the same parts (I. 18. *Cor.*).

Scholium.

This is not necessarily the case with other rectilineal figures; that is, it cannot be said, that if any two rectilineal figures be equiangular, the sides about the equal angles will be proportionals. And the reason of this difference between a triangle, and a rectilineal figure of four, or a greater number of sides, is obvious. For, in the triangle, D E F, a line G H, which is drawn parallel to one of the sides E F, will cut off from the other sides, or from those sides produced, parts less or greater than them in the same proportion in which it is itself less or greater than E F: and, accordingly, the new triangle D G H, which is equiangular with D E F (I. 15.) has its sides respectively less or greater than the corresponding sides of the triangle D E F, in the same ratio. But, in a quadrilateral figure as G E F H, or generally any rectilineal figure, which has more than three sides, a straight line drawn parallel to one of the sides, as E F, and therefore cutting, as before, the two adjoining sides, or those two sides produced, will diminish or increase those two sides of the figure only, and even those not necessarily in the proportion in which it is itself less or greater than E F; and, accordingly, the new rectilineal figure, which is equiangular with the former, has its sides, with the exception of three, the same with the sides of the former; while these three are diminished or increased. Therefore, although equiangular triangles have, necessarily, their corresponding sides, each to each, in the same ratio, equiangular rectilineal figures of a greater number of sides have not, necessarily, their sides so related.

Prop. 32. (Euc. vi. 6.)

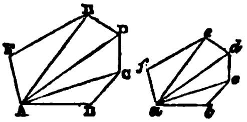
If two triangles have one angle of the one equal to one angle of the other, and the sides about the equal angles proportionals the two triangles shall be similar.

Let the triangles A B C, D E F (see the last figure,) have the angle at A equal to the angle at D, and the sides about the equal angles, that is, A B, A C, D E, D F, proportionals: the triangle A B C shall be similar to D E F.

In D E, D F, or in D E, D F produced, take D G, D H equal to A B, A C respectively, and join G H. Then the triangles A B C, D G H are equal in every respect (I. 4.). And, because D G : D H :: D E : D F, *alternando* D G : D E :: D H : D F. Therefore (29.) G H is parallel to E F: and (I. 15.) the angles D G H and D H G are equal to the angles at E and F respectively: but D G H is equal to B, and D H G to C: therefore the angles at B and C are equal to the angles at E and F, each to each; and the triangle A B C (31.*Cor.*) is similar to D E F.

Therefore, &c.

Cor. 1. Hence it may easily be shown, that, if any two rectilineal figures, as A B C D E F, *a b c d e f*, have all their angles but two equal in order, viz. B, C, D, E, to *b*, *c*, *d*, *e*, each to each, and the corresponding sides about the equal angles proportionals, their remaining angles shall be equal, each to each, viz. the angles A and F to the angles *a* and *f*, and their remaining sides A F, *a f*, in the same ratio with any other two corresponding sides A B, *a b*: that is, the rectilineal figures, A B C D E F, *a b c d e f*, shall be similar according to *Def.* 14.



Join A C, A D, A E, *a c*, *a d*, *a e*. Then, because the triangles A B C, *a b c* have one angle of the one equal to one angle of the other, and the sides about the equal angles proportionals, they are similar (32.); therefore, the angle A C B is equal to *a c b*, and A C : C B :: *a c* : *c b*. And, because the whole angle B C D is equal to the whole angle *b c d*, and that the angles A C B, *a c b* are equal to one another, the remaining angle A C D is equal to the remaining angle *a c d*: also, because A C : C B :: *a c* : *c b* and C B : C D :: *c b* : *c d*, *ex æquali* A C : C D :: *a c* : *c d*. Therefore (32.) the triangles A C D, *a c d* are similar: and in the same manner it may be shown that the triangles A D E, A E F are similar to the triangles *a d e*, *a e f* respectively. Therefore, the angles at F, *f* are equal to one another, and because the several angles making up the angle F A B are equal respectively to those which make up the angle *f a b*, the angles F A B, *f a b* are likewise equal. Also, because A F is to A E, as *a f* to *a e*, and A E to A D as *a e* to *a d*, and A D to A C, as *a d* to *a c*, and A C to A B as *a c* to *a b*, *ex æquali* A F is to A B as *a f* to *a b*, and *alternando* A F : *a f* :: A B : *a b*.

Cor. 2. It appears from the preceding corollary, that similar rectilineal figures may be divided into the same number of similar triangles, and that their corresponding or homologous sides are to one another in the same ratio, each to each.

Cor. 3. It appears also from *Cor.* 1., that lines similarly drawn in similar figures cut the sides at equal angles, and are to one another as the homologous sides of the figures.

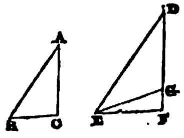
For if the sides A B, *a b* of the similar figures A B C D E F and *a b c d e f*, be divided in the same ratio in the points P, *p*, it may easily be shown (23.) that

P B is to *p b* as A B to *a b*, or as B C to *b c*; that is, *alternando*, P B : B C :: *p b* : *b c*. And, in the same manner, if any other corresponding sides, as E F and *e f* be divided in the same ratio in the points Q, *q*, it may be shown that Q E : E D :: *q e* . *e d*, or, *invertendo*, that D E : E Q :: *d e* : *e q*. Therefore, by *Cor.* 1., the straight lines P Q and *p q*, which join the points P, Q and *p*, *q*, make equal angles with A B, *a b* respectively, and with E F, *e f*, respectively, and P Q is to *p q* as P B to *p b*, or as A B to *a b*.

Prop. 33. (Euc. vi. 7.)

If two triangles have one angle of the one equal to one angle of the other, and the sides about two other angles proportionals; and, if one of the remaining angles be a right angle, or if they be both greater, or both less than right angles; the two triangles shall be similar.[*]

Let A B C, D E F be two triangles, which have the angle at A equal to the angle at D, and the sides about two other angles, B and E, proportionals: then, if one of the remaining angles, as C, be a right angle, or if both of them, C, F, be greater, or both less, than right angles, the triangle A B C shall be similar to the triangle D E F.



At the point E make the angle D E G equal to A B C, and let the line E G meet D F, or D F produced in G. Then, because the triangles A B C, D E G have two angles of the one equal to two angles of the other, each to each, they are equiangular: therefore (31.) D E : E G :: A B : B C, but A B : B C :: D E : E F: therefore (12.) D E : E G :: D E : E F, and (18.) E G is equal to E F.

And first, if one of the angles A C B, D F E be a right angle, whether it be A C B or D F E, one of the lines E G, E F will be at right angles to D F; and therefore, if E G do not coincide with E F, there will be an isosceles triangle E F G, which has one of the angles at the base a right angle, and therefore (I. 6.) the other likewise a right angle, and the two together equal to two right angles, which is impossible (I. 8.): therefore, in this case, E G cannot but coincide with E F, that is, the angle D E F cannot but be equal to the angle D E G, that is, to A B C.

Again, if the angles A C B, D F E be both greater than right angles, they will be together greater than two right angles;—or, if they be both less than right angles, they will be together less than two right angles: but, if E G do not coincide with E F, these same two angles will be together equal to the angles D G E, D F E, that is, to the angles D G E, E G F, (I. 6.) because E G is equal to E F, that is, to two right angles (I. 2.),—which is impossible: therefore in these cases, also, E G cannot but coincide with E F, that is, the angle D E F cannot but be equal to the angle D E G, that is, to A B C.

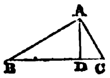
And, because the triangles A B C, D E F have two angles of the one equal to two angles of the other, each to each, they are equiangular (I. 19. *Cor.* 1.), and therefore similar (31. *Cor.* 1.).

Therefore, &c.

Prop. 34. (Euc. vi. 8.)

In a right-angled triangle, if a perpendicular be drawn from the right angle to the hypotenuse, the triangles upon either side of it shall be similar to the whole triangle, and to one another.

Let A B C be a right-angled triangle, and from the right angle A, let there be drawn to the hypotenuse the perpendicular A D: the triangles D B A, D A C shall be similar to the whole triangle A B C, and to one another.



Because the angles B D A, B A C are right angles, and because the angle at B is common to the two triangles D B A, A B C, their third angles (I. 19. *Cor.* 1.) are equal to one another; and therefore the triangle D B A, being equiangular with the triangle A B C, is similar to it (31. *Cor.* 1.). In the same manner it may be shown that D A C is equiangular with A B C, and therefore similar to it. And, because D B A, D A C are equiangular with the same triangle, they are equiangular with, and therefore similar to, one another.

Therefore, &c.

Cor. (Euc. vi. 8. *Cor.*) The perpendicular A D is a mean proportional between the segments B D, D C of the hypotenuse; and either side, as A B, is a mean proportional between the

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[*] Angles which are both greater or both less than right angles are said to be *of the same affection*: hence, instead of saying "both greater or both less than right angles," this proposition is sometimes enunciated by saying "both of the same affection."

whole hypotenuse B C, and the segment B D, which is adjacent to it.

For, because the angles BAD, ACD, are equal to one another, the sides B D, A D of the similar triangles BDA, ADC are homologous, and the angles B D A, A D C are equal to one another; therefore (31.) B D : DA :: AD or DA : DC: and, in like manner, because the sides B D, B A of the similar triangles B D A, B A C are homologous, and the angles at B equal, B D : B A :: BA : B C.

SECTION 5.—*Proportion of the Surfaces of Rectilineal Figures.*

PROP. 35.

Rectangles which have the same or equal altitudes, are to one another as their bases.

For, if the base of one of the rectangles be divided into any number of equal parts, the rectangle itself will be divided into as many equal rectangles by straight lines drawn parallel to its side through the points of division. Also, the base of the other rectangle will contain a certain number of parts equal to those into which the first base is divided, exactly or with an excess less than one of those parts, and that rectangle will contain as many rectangles equal to those into which the first rectangle is divided, exactly or with a corresponding excess less than one of them. And this will be the case whatsoever be the number of parts in the first base and rectangle. Therefore, (def. 7.) the first rectangle is to the second as the base of the first is to the base of the second; that is, rectangles of the same altitude are to one another as their bases.

Therefore, &c.

The same demonstration has been already given, with a figure, to illustrate the definition of Proportionals in Section 3. (See def. 7.)

Cor. (Euc. vi. 1 part of.) In the same manner it may be shown, that any two parallelograms which have the same or equal altitudes are to one another as their bases; or, the same may be directly inferred from this proposition, for every parallelogram is equal to a rectangle, having the same base and altitude.

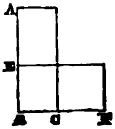
Scholium.

Proportions of this kind, in which the two first terms stand in the same relation to the two last respectively, as, in the above instance, in the relation of a rectangle of given altitude to its base, are of perpetual occurrence in the Mixed Sciences, and are commonly enunciated, as in the proposition, by the word *as*. They constitute, indeed, a peculiar class, which has received the name of *Variations;* one quantity being said to *vary* as another when it increases and decreases in the same proportion with that other. We have already had one instance in a rectangle of given altitude and its base. To take another perhaps more familiar, we say commonly, that the weight of a mass of lead is in proportion to its magnitude; and the same is understood, when it is said, that the weight of such a mass *is as* its magnitude, or *varies as* its magnitude. Again, the distance a horse runs in a given time "is in proportion to his speed," or "is as his speed," or "varies as his speed." It is evident that the use of the word *as* in these and the like cases enables us to avoid the following and similar enunciations. "The weight of one mass of lead is to the weight of another, as the magnitude of the first to the magnitude of that other." "The distance a horse runs in a given time with one velocity, is to the distance he would run in the same time with another velocity, as the first velocity to that other." (See Arithmetic, art. 136.)

PROP. 36.

Any two rectangles are to one another in the ratio which is compounded of the ratios of their sides.

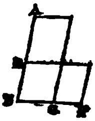
Let A C, E F be two rectangles, and let an angle of the one be made to coincide (I. 1. and ax. 11.) with an angle of the other, as at B: the rectangle A C shall be to the rectangle E F in the ratio which is compounded of the ratios of A B to E B, and of B C to B F.



For the rectangle A C is to the rectangle E F in the ratio which is compounded of the ratios of A C to E C, and of E C to E F. (def. 12.) But, because the rectangles A C, E C have the same altitude B C, A C is to E C as A B to E B (35.); and, in like manner, because the rectangles E C, E F have the same altitude E B, E C is to E F as BC to BF. Therefore, (27. *Cor.* 1.) the ratio which is compounded of the ratios of A C to E C, and of E C to E F, is the same with the ratio, which is compounded of the ratios of A B to E B, and B C to B F; that is, the rectangle A C is to the rectangle E F in the ratio which is compounded of the ratios of A B to E B, and of B C to E F.

Therefore, &c.

Cor. 1. (Euc. vi. 23.) In the same manner it may be shown that any two parallelograms AC, EF, which have one angle of the one equal to one angle of the other, are to one another in the ratio which is compounded of the ratios of the sides about the equal angles. For if the equal angles be made to coincide, as at B, the parallelogram AC is to the parallelogram E F in the ratio which is compounded of the ratios of A C to E C, and of E C to E F, that is (35. *Cor.*) in the ratio which is compounded of the ratios of A B to E B, and of B C to BF.



Cor. 2. It appears, also, from the proposition, that any two parallelograms whatever are to one another in the ratio which is compounded of the ratio of their bases and altitudes; for the rectangles, to which (I. 24. *Cor.*) they are equal, are in that ratio.

Prop. 37

If the straight line A *be to the straight line* B *in one ratio, and the straight line* A′ *to the straight line* B′ *in another ratio, the rectangle* A A′ *shall be to the rectangle* B B′ *in the ratio which is compounded of these two ratios.*

This proposition is the same with the preceding, which is here stated under a different form, as well for the sake of rendering more obvious such references as may be made to it under this form, as to separate the following corollaries from those of Prop. 36. Its demonstration is accordingly contained in that of Prop. 36.

Cor. 1. If A be to B as A′ to B′, the rectangle A A′ will be to the rectangle B B′ in the duplicate ratio of A to B. For, if C be taken a third proportional to A and B, then, because B is to C as A to B, that is as A′ to B′, the ratio which is compounded of the ratios of A to B and of A′ to B′ is the same with the ratio of A to C, that is (Def. 11.) with the duplicate ratio of A to B.

Cor. 2. Hence if three straight lines A, B, C, be proportionals, the square of the first shall be to the square of the second as the first is to the third; that is, the duplicate ratio of two straight lines is the same with the ratio of their squares.

Cor. 3. If A, B, C, D be four straight lines, and A′ B′, C′, D′ four others,

and if A : B :: C : D

and A′ : B′ :: C′ : D′

then AA′ : BB′ :: CC′ : DD′; that is, if four straight lines A, B, C, D be proportionals, and likewise four others A′, B′, C′, D′; the rectangles A A′, B B′, C C′, D D′, which are contained under the corresponding antecedents and consequents, shall also be proportionals. For ratios which are compounded of the same ratios are the same with one another (27). And hence it is evident, conversely, that if AA′, BB′, CC′, DD′ be proportionals, and A′, B′, C′, D′ likewise proportionals; A, B, C, D must also be proportionals.

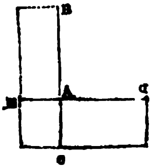
Cor. 4. If four straight lines A, B, C, D be proportionals, their squares A^2, B^2, C^2, D^2 shall likewise be proportionals; and conversely.

Prop. 38. (Euc. vi. 16.)

If four straight lines be proportionals the rectangle contained by the extremes shall be equal to the rectangle contained by the means; and conversely, if the rectangle contained by two straight lines be equal to the rectangle contained by other two, the four straight lines shall constitute a proportion, in which the sides of one rectangle are extremes, and the sides of the other rectangle means.

The first part of this proposition has been already demonstrated in the *Scholium* on Prop. 28., Sect. 3.; which demonstration the reader is here desired to consult, as depending immediately upon the definition of proportionals in Sect. 3.

So important a theorem, however, cannot be considered in too many points of view. The following proof, by help of Prop. 35., is therefore added, being in effect the same with that of Euclid.



Let AB, AC, AD, A E be the four proportionals, of which the extremes AB, AE are placed at right angles to one another, and contain the rectangle B E, and the means A C, A D are placed in BA, EA produced, and, being for that reason (I. 3.) likewise at right angles to one another, contain the rectangle C D. Complete the rectangle C E. Then, because the rectangles B E, C E have the same altitude AE, the rectangle BE is to the rectangle CE as AB to AC (35.): and, in like manner, because the rectangles CD, CE have the same altitude AC, the rectangle CD is to the rectangle CE as AD to AE; but the line AB is to AC as AD to AE; therefore, the rectangle BE is to CE as CD to CE. And, because the rectangles BE, CD have the same ratio to the same rectangle CE, they are equal to one another (11. *Cor.* 1.).

Next, let the rectangle under AB, AE, be equal to the rectangle under AC, AD; the lines AB, AC, AD, AE, shall be proportionals, AB, AE being extremes, and AC, AD means.

For, the rectangles being placed as before, and the rectangle CE being completed, BE is to CE as CD to CE (10.); but BE is to CE as AB to AC (35.), and CD to CE as AD to AE; therefore (12.) AB is to AC as AD to AE.*

Therefore, &c.

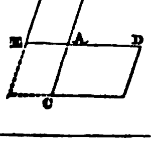
The principle of this demonstration is contained in 10. *Cor.* and 11. *Cor.* 2., by the use of which corollaries it may consequently be put under a more concise form.

Cor. 1. (Euc. i. 17.) If three straight lines be proportionals, the rectangle under the extremes shall be equal to the square of the mean; and, conversely, if the rectangle under the extremes be equal to the square of the mean, the three straight lines shall be proportionals.

Cor. 2. It appears from the proposition that rectangles which have their sides about the right angles reciprocally proportional, are equal; and, conversely, that equal rectangles have their sides about the right angles reciprocally proportional.

Cor. 3. (Euc. vi. 14.) And, in the same manner, it may be shown, that any two parallelograms which have one angle of the one equal to one angle of the other, and their sides about the equal angles reciprocally proportional, are equal to one another; and conversely.

For, if the equal angles be placed vertically as in the figure, and the parallelogram CE be completed, the demonstration of this more general case will be the same with that of the proposition; 35. *Cor.* being cited instead of 35.



Scholium.

By help of this proposition the proportion of four lines is convertible into the equality of two rectangles, and the reverse. Thus it appears, that I. 36. *Cor.* 2. and 3., in which it is inferred that *the squares of the sides of a right angled triangle are respectively equal to the rectangles under the hypotenuse and its adjoining segments*, and that *the square of the perpendicular is equal to the rectangle under the segments of the hypotenuse*, may be stated in the words of 34. *Cor.*, in which it is inferred that *either side is a mean proportional between the hypotenuse and segment adjacent to it*, and that *the perpendicular is a mean proportional between the segments of the hypotenuse.* Again, I. 38. in which it is demonstrated that, in every triangle, if a perpendicular be drawn from the vertex to the base, or to the base produced, *the difference of the squares of the sides is equal to the difference of the squares of the segments of the base, or of the base produced*, may be stated thus:—*the base is to the sum of the sides as the difference of the sides to the difference of the segments of the base, or sum of the segments of the base produced.* For it is shown (I. 38.) that the difference of the squares of two straight lines is equal to the rectangle under their sum and difference.

Many other instances will occur in the remaining part of this treatise, in which the demonstrations are considerably abridged by the use of this very important theorem.

We shall conclude the present *Scholium* by applying it to the demonstration of the following *Lemmas* (or auxiliary theorems) which will be found of service in such problems as have the sum or the difference of the sides of a triangle among their data. (See Book III. Sect. 7.) It will be seen that they belong, according to our arrangement, to the subject of the preceding Section; to which they should have been subjoined, had they admitted of an easy demonstration without the aid of this 38th Proposition.

Lemma 1.

If a perpendicular be drawn from the vertex of a triangle to the base, and if the base be equally produced both ways, so that the base produced may be a third proportional to the base and the

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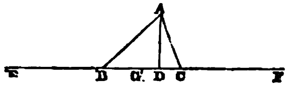
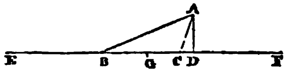
* Or we may say, "AD : AE :: AB : AC," in which case AC and AD will be extremes, and AB, AE means.

sum of the sides, the sides shall be to one another as the corresponding segments of the base produced.

Let ABC be a triangle, and from the vertex A to the base B C let there be drawn the perpendicular A D; also let the base BC be equally produced both ways to the points E and F; so that BC may be to BA + AC as BA+AC to E F. B A shall be to A C as ED to DF.





Bisect BC in G: then because EB is equal to C F, E F is likewise bisected in G. Therefore the difference of ED, DF is equal to twice GD. But twice GD is equal to the difference or to the sum of BD, DC, according as the point D is in BC, or in BC produced. Therefore the difference of ED, D F, is equal to the difference of BD, DC, or to the sum of BD, DC, according as the point D lies in BC, or in BC produced. Again, because AD is perpendicular to BC, the difference of the squares of BA, AC is equal to the difference of the squares of B D, D C (I. 38.) or, which is the same thing (I. 34.), the rectangle under the sum and difference of BA, AC is equal to the rectangle under the sum and difference of BD, DC. Therefore (38.) $BA \sim AC : BD \mp DC :: BD \pm DC : BA + AC$;* that is, since the second term is equal to $ED \sim DF$, and the third to BC, $BA \sim AC : ED \sim DF :: BC : BA + AC$, that is, $:: BA + AC : ED + DF$, by the supposition. Therefore, *alternando* (19.) $BA \sim AC : BA + AC :: ED \sim DF : ED + DF$: and, *by sum and difference* (24. *Cor.* 2.) $2\,BA : 2\,AC :: 2\,ED : 2\,DF$, that is, (17. *Cor.* 2.) $BA : AC :: ED : DF$.

Therefore, &c.

Cor. If DE be taken to AB as BA+AC to BC, or, which is the same thing, in a ratio which is the subduplicate of GE to GB, then the other segment DF of the base equally produced in the opposite direction shall be to the other side AC in the same ratio.

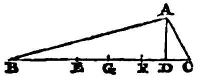
Lemma 2.

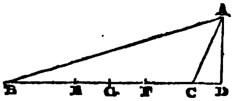
If a perpendicular be drawn from the vertex of a triangle to the base, and if the base be equally reduced both ways, so that the base reduced may be a third proportional to the base and the difference of the sides, the sides shall be to one another as the corresponding segments of the base reduced.

The demonstration is so similar to that of *Lemma* 1., that it will be readily apprehended from the following outline, with reference to the adjoined figures.

ABC is the triangle, AD the perpendicular upon the base, EF the base equally reduced both ways, so that $EF : BA \sim AC : BC$, and G is the middle point of the base. Then, because $BE = CF$, the point G bisects also E F, and $ED + DF = 2\,GD$, that is, $= BD \mp DC$, according as the point D lies in BC, or in BC produced. Now, converting $BA^2 \sim AC^2 = BD^2 \sim DC^2$ into a proportion, as in *Lem.* 1, $BA + AC : BD \mp DC :: BD \pm DC : BA \sim AC$, or $BA + AC : ED + DF :: BC : BA \sim AC$, that is, $:: BA \sim AC : EF$ (or $ED \sim DF$). Therefore *alternando*, $BA + AC : BA \sim AC :: ED + DF : ED \sim DF$: and hence, *by sum and difference*, $BA : AC :: ED : DF$.

Therefore, &c.



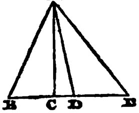


PROP. 39. (EUC. vi. 1., first part of.)

Triangles which have the same or equal altitudes are to one another as their bases.

For such triangles are the halves of rectangles which are upon the same bases respectively, and have the same or equal altitudes: and because these rectangles are as the bases (35.), the triangles, which are their halves, have to one another the same ratio (17. *Cor.* 1.).

Otherwise: As in 35. it is demonstrated of the rectangles, so here it may, after the same manner, be demonstrated of the two triangles A B C, A D E, and the two bases B C, D E, that they are propor-



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* The sign ± denotes that the sum or difference is to be taken according to one or other of two supposed cases, that is, in the present instance, according as the point D lies in B C, or in B C produced; the sign ∓ that the difference or sum is to be taken according to those cases.

tionals according to def. 7.; I. 27. being cited instead of I. 25.

Therefore, &c.

Cor. 1. In like manner it may be shown that triangles which have the same or equal bases are to one another as their altitudes.

Cor. 2. Also any two triangles are to one another in the ratio which is compounded of the ratios of their bases and of their altitudes.

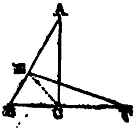
Cor. 3. Wherefore if the base of one triangle be to the base of another as the altitude of that other to the altitude of the first, the two triangles will be equal to one another (11. *Cor.* 2.).

Prop. 40.

Triangles which have one angle of the one equal to one angle of the other, are to one another in the ratio which is compounded of the ratios of the sides about the equal angles.

For, if the triangles be completed into parallelograms having the same equal angles and the same sides containing them, these parallelograms (36. *Cor.* 1.) will be to one another in the ratio which is compounded of the ratios of the sides; therefore the triangles, which are their halves, will be to one another in the same ratio. (17. *Cor.* 1.)

Otherwise: As in 36. it is demonstrated of the rectangles, and in 36. *Cor.* 1. of the parallelograms, so here it may, after the same manner, be demonstrated of the two triangles A B C, E B F, by making the equal angles coincide, as at B, and completing the triangle E B C, that the triangle A B C is to the triangle E B F in the ratio which is compounded of the ratios of A B to E B and of B C to B F; 39. being cited instead of 35.



Therefore, &c.

Cor. Triangles, which have one angle of the one equal to one angle of the other, are to one another as the rectangles under the sides about the equal angles (37.).

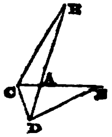
Prop. 41. (Euc. vi. 15.)

Triangles which have one angle of the one equal to one angle of the other, and their sides about the equal angles reciprocally proportional, are equal to one another; and, conversely, equal triangles which have one angle of the one equal to one angle of the other, have their sides about the equal angles reciprocally proportional.

For, if the triangles be completed into parallelograms having the same equal angles and the same sides containing them, these parallelograms (38. *Cor.* 3.) will be equal to one another, because they have the sides about the equal angles reciprocally proportional; and therefore the triangles, which are their halves, are likewise equal (I. ax. 5.).

And in like manner the converse from the converse part of the same 38. *Cor.* 3.

Otherwise: As of the rectangles in Prop. 38., so here it may, after the same manner, be demonstrated of the two triangles A B C, A D E, by making the equal angles vertical, as at A, and completing the triangle A C D, that if the sides be reciprocally proportional, that is, if A B be to A D as A E to A C, the triangles A B C, A D E will have the same ratio to the triangle A C D, and therefore will be equal to one another: and, conversely, that if A B C, A D E be equal to one another, and therefore have the same ratio to the triangle A C D, A B will be to A D as A E to A C, that is, the sides about the equal angles will be be reciprocally proportional; 39. being cited instead of 35.

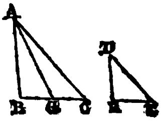


Therefore, &c.

Prop. 42. (Euc. vi. 19.)

Similar triangles are to one another in the duplicate ratio of their homologous sides.

Let A B C, D E F be similar triangles, and let the sides B C, E F be homologous; the triangle A B C shall have to the triangle D E F the duplicate ratio of that which B C has to E F.



Because the angle at B is equal to the angle at E, the triangle A B C is to the triangle D E F in the ratio which is compounded of the ratios of A B to D E, and of B C to E F (40.). But, because the triangles are similar, A B is to B C as D E to E F, and *alternando* A B : D E :: B C : E F; therefore the ratio which is compounded of the ratios of A B to D E, and of B C to E F, is the duplicate of the ratio of B C to E F (37. *Cor.* 1.). Therefore the triangle A B C has to the triangle D E F the duplicate ratio of that which B C has to E F.

Otherwise: Take B G a third proportional to B C and E F, and join A G. Then the triangle A B C is to the triangle A B G as B C to B G, that is, (def. 11.) in the duplicate ratio of B C to E F. But because A B is to D E as B C to E F, that is, (12.) as E F to BG, the triangles A B G, D E F have their sides about the equal angles B and E reciprocally proportional: therefore the triangle A B G is equal to the triangle D E F (41.). Therefore the triangle A B C is to the triangle D E F in the duplicate ratio of B C to E F.

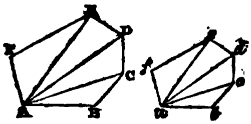
Therefore, &c.

Cor. Since the duplicate ratio of two straight lines is the same with the ratio of their squares (37. *Cor.* 2.) it appears that similar triangles are to one another as the squares of their homologous sides.

PROP. 43. (EUC. vi. 20.)

Similar rectilineal figures are to one another in the duplicate ratio of their homologous sides; and their perimeters are as those sides.

For it has been seen (32. *Cor.* 2.) that any two similar rectilineal figures A B C D E F, *a b c d e f*, may be divided into the same number of similar triangles by straight lines drawn from corresponding angles A, *a*; and the homologous sides of these triangles, which are the same with the homologous sides of the figures, are to one another, each to each, in the same ratio. But similar triangles are to one another in the duplicate ratio of their homologous sides. Therefore the triangles into which the figure A B C D E F is divided, are to the similar triangles into which the figure *a b c d e f* is divided, each to each, in the same ratio (37. *Cor.* 4.) viz. in the duplicate ratio of that which A B has to *a b*. Therefore the sum of all the former is to the sum of all the latter (23. *Cor.* 1.) that is, the figure ABCDEF is to the figure *abcdef*, in the same ratio.



Again, because the sides A B, B C, &c. of the one figure are to the homologous sides *a b*, *b c*, &c. of the other figure in the same ratio, the sum of the former is to the sum of the latter in the same ratio; that is, the perimeter of the one figure is to the perimeter of the other as A B to *a b*.

Therefore, &c.

Cor. 1. Similar rectilineal figures are to one another as the squares of their homologous sides (37. *Cor.* 2.).

Cor. 2. (Euc. vi. 22.). If four straight lines be proportionals, any similar rectilineal figures described upon the first and second shall be to one another as any similar rectilineal figures described upon the third and fourth; and conversely (37. *Cor.* 4.).

PROP. 44. (EUC. vi. 31.)

In a right-angled triangle, if similar rectilineal figures be similarly described upon the hypotenuse and the two sides, the figure upon the hypotenuse shall be equal to the sum of the figures upon the two sides.

For the figure upon one of the sides is to the similar figure upon the hypotenuse, as the square of that side to the square of the hypotenuse (43. *Cor.* 1.), and the similar figure upon the other side is to the figure upon the hypotenuse as the square of that other side to the square of the hypotenuse—proportions having the same consequents: therefore (25.) the sum of the figures upon the two sides is to the figure upon the hypotenuse as the sum of the squares of the two sides to the square of the hypotenuse, that is (I. 36.), in a ratio of equality.

Therefore, &c

SECTION 6.—*Of Lines in Harmonical Progression.*

Def. 17. Three straight lines are said to be *in harmonical progression* when the first is to the third as the difference of the first and second to the difference of the second and third.*

Of three lines A, B, C, which are in this progression, B is said to be an *harmonical mean* between A and C, and

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* This progression was called *harmonical* from its having been first noticed (it is said, by Pythagoras) in the lengths of chords which, having the same thickness and tension, produce the sounds of a certain note, its fifth and its octave. These lengths are as 1, $\frac{2}{3}$, and $\frac{1}{2}$, of which it is plain that the first is to the third as the difference of the first and second to the difference of the second and third,

It is observable that if harmonical means be inserted between the numbers above mentioned, lengths will be found among them producing the other notes of the major scale.

"If a musical string C O and its parts D O, E O, F O, G O, A O, B O, c O, be in proportion to one another as the numbers

$$1,\ \tfrac{8}{9},\ \tfrac{4}{5},\ \tfrac{3}{4},\ \tfrac{2}{3},\ \tfrac{3}{5},\ \tfrac{8}{15},\ \tfrac{1}{2},$$

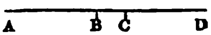
their vibrations will exhibit the system of 8 sounds, which musicians denote by the letters C, D, E, F, G, A, B, c." Smith's Harmonics, Sect. II. Art. 1, C a *third harmonical progressional* to A and B.

After the same manner, also, three magnitudes of any other kind are said to be in harmonical progression, viz. when the first is to the third as the difference of the first and second to the difference of the second and third; and the terms *harmonical mean* and *third harmonical progressional* are applied to them in the same sense.

18. Any number of straight lines, or other magnitudes, A, B, C, D, &c. are said to be in harmonical progression, when every consecutive (or following) three are in harmonical progression.

19. A straight line is said to be *harmonically divided,* when the distances of one extremity of the line and of the points of division from the other extremity, are in harmonical progression.

Thus, A D is said to be harmonically divided, when the points B, C are so taken that A B, A C, A D are in harmonical progression.



It will be shown in Prop. 45., that, if the line A D be harmonically divided in the points B, C, the line D A will likewise be harmonically divided in the same points: that is, that if A B, A C, and A D be harmonical progressionals, D C, D B, and D A shall likewise be harmonical progressionals.

20. Four straight lines are said to be *harmonicals*, when they pass through the same point, and divide any one straight line harmonically.

Prop. 45.

If A B, A C, A D *be harmonical progressionals in the same straight line,* D C, D B, D A *shall likewise be harmonical progressionals.* (See figure of def. 19.).

Because A B, A C, A D are in harmonical progression, (def. 17.) A B : A D :: B C : C D; therefore, *alternando,* A B : B C :: D A : C D, and, *invertendo,* D C : D A :: B C : A B. Therefore D C, D B, D A are three straight lines, such, that the first is to the third as the difference of the first and second to the difference of the second and third; that is, (def. 17.) D C, D B, D A are in harmonical progression.

Therefore, &c.

Cor. If a given line A C be divided in any ratio in the point B, and if A C produced be divided in the same ratio in the point D (so that DA may be to DC as A B to B C), the whole line A D will be harmonically divided in the points B and C. For it is obvious that D A is harmonically divided in the points B and C; that is, that D A, D B, D C are harmonical progressionals: therefore, also, A D, A C, A B are harmonical progressionals, and A D is divided harmonically in the points B and C.

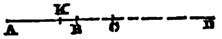
Prop. 46.

If A B, A C, A D *be harmonical progressionals in the same straight line, and if the mean* A C *be bisected in* K, K B, K C, K D *shall be in geometrical progression: and conversely.*

Now $\frac{4}{5}$ is an harmonical mean between 1 and $\frac{2}{3}$, $\frac{3}{4}$ the second of two harmonical means between 1 and $\frac{2}{3}$, and $\frac{8}{9}$ the first of three harmonical means between 1 and $\frac{2}{3}$. Again, $\frac{3}{5}$ is the first of two harmonical means between $\frac{2}{3}$ and $\frac{1}{2}$, and $\frac{8}{15}$ the third of three harmonical means between $\frac{2}{3}$ and $\frac{1}{2}$. In fact, taking the original progression 1, $\frac{2}{3}$, $\frac{1}{2}$, and inserting first one harmonical mean between its terms, we get the progression 1, $\frac{4}{5}$, $\frac{2}{3}$, $\frac{4}{7}$, $\frac{1}{2}$; secondly, two harmonical means between its terms, 1, $\frac{6}{7}$, $\frac{3}{4}$, $\frac{2}{3}$, $\frac{3}{5}$, $\frac{6}{11}$, $\frac{1}{2}$; and thirdly, three harmonical means between its terms, 1, $\frac{8}{9}$, $\frac{4}{5}$, $\frac{8}{11}$, $\frac{2}{3}$, $\frac{8}{13}$, $\frac{4}{7}$, $\frac{8}{15}$, $\frac{1}{2}$; from which progressions, rejecting such fractions as admit 7, 11, and 13 in the denominator, that is, such as have other numbers entering into their terms, besides 2, 3, 5, and their products, those which remain will represent the lengths of strings producing, with the same thickness and tension, the sounds denoted by C, D, E, F, G, A, B, c.

The above observation, striking and ingenious as it is, must not, however, lead the student to suppose that the theory of Harmonics has any mysterious connexion with the properties of lines harmonically divided. Why such lengths only as are related by the numbers 2, 3, 5, and their products, produce a gradation of sounds pleasing to the ear as those of the gamut, it is for that theory to explain; but the discovery of these relations, by taking harmonical means, is attributable to the simple property, that the reciprocals of numbers in harmonical progression are in arithmetical progression. Thus the reciprocals of 1, $\frac{8}{9}$, $\frac{4}{5}$, $\frac{8}{11}$, $\frac{2}{3}$, $\frac{8}{13}$, $\frac{4}{7}$, $\frac{8}{15}$, $\frac{1}{2}$, that is of $\frac{8}{8}$, $\frac{9}{8}$, $\frac{10}{8}$, $\frac{11}{8}$, $\frac{12}{8}$, $\frac{13}{8}$, $\frac{14}{8}$, $\frac{15}{8}$, $\frac{16}{8}$, are fractions having the common denominator 8, and 8, 9, 10, 11, &c. for their numerators, that is, are in arithmetical progression. And, generally, if a, b, c be in harmonical progression, *i. e.* if $a : c :: a \sim b : b \sim c$, multiplying extremes and means, $a\,b \sim a\,c = a\,c \sim b\,c$, and dividing by $a\,b\,c$, $\frac{1}{c} \sim \frac{1}{b} = \frac{1}{b} \sim \frac{1}{a}$, *i. e.* $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in arithmetical progression. It follows as a necessary inference, that, if the lengths of the strings which produce harmonious sounds, bear to each other a ratio which can be expressed in whole numbers, however great, they may be made terms in *some* harmonical series; the singular result which arises from this ratio being expressed in terms involving only the numbers 2, 3, 5, and their products, is that the whole series is obtained by the interposition only of two and of three harmonical means between the note and its fifth and the fifth and octave; for, $\frac{4}{5}$ being comprised in the series which results from the interposition of three means, that of one mean may be neglected.

In the first place, because A B : A D :: B C : C D, and that A D is greater than C D, A B is also greater than BC, (18. *Cor.*). Wherefore the point K, which bisects A C, lies between A and B. Again, *alternando*, A B : B C :: A D : C D: therefore, by *sum and difference*, A B ∼ B C : A B + B C :: A D ∼ C D : A D + C D. But the first term of this proportion is equal to 2 K B, the second to 2 K C, the third likewise to 2 K C, and the fourth to 2 K D. Therefore 2 K B : 2 K C :: 2 K C : 2 K D; and hence, (17. *Cor.* 2.) K B : K C :: K C : K D, that is, K B, K C, K D are in geometrical progression.



Next, let K B, K C, K D be in geometrical progression, and let K A be taken equal to K C: A B, A C, A D shall be in harmonical progression.

For since K B : K C :: K C : K D, by *sum and difference* K B + K C : K B ∼ K C :: K C + K D : KC ∼ K D, that is, AB : BC :: AD : CD. Therefore, *alternando*, A B : A D :: B C : C D, or A B, A C, A D are in harmonical progression.

Therefore, &c.

Prop. 47.

The same being supposed, D A, D K, D B, D C *shall be proportionals.*

Because D A is equal to the sum, and D C to the difference of D K, K C, the rectangle under D A, D C is equal to the difference of the squares of D K, K C (I. 34.). Again, because K C is a mean proportional between K B and K D, the square of K C is equal to the rectangle under K B, K D, (38. *Cor.* 1.). Therefore the rectangle under D A, D C is equal to the difference of the square of D K, and the rectangle under KB, K D; that is, (I. 30. *Cor.*) to the rectangle D K, DB. Therefore (38.) D A, D K, D B, DC are proportionals.

Therefore, &c.

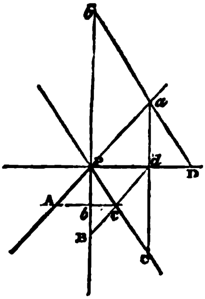
Cor. 1. If K B, K C, K D be proportionals in the same straight line, and if K A be taken in the opposite direction equal to the mean K C; D A, D K, D B and D C shall be proportionals (46.).

Cor. 2. From this proposition it appears that the harmonical mean D B between two straight lines D A and D C is a third proportional to the arithmetical mean D K, and the geometrical mean M between the same two. For $DB \times DK = DA \times DC = M^2$ if M be a geometrical mean between DA and DC: and because $DB \times DK = M^2$, D K, M, and D B are proportionals.

Prop. 48.

If four straight lines pass through the same point; to whichsoever of the four a parallel be drawn, its parts intercepted by the other three, shall be to one another in the same ratio.

Let the four straight lines P A, P B, P C, P D pass through the same point P; through A, any point in P A, draw A C parallel to P D, and let it be divided by the other three P A, P B, P C into the parts A *b*, *b c*; through *c* draw B *d* parallel to P A, and let it be divided by the other three into the parts B *c*, *c d*; through *d* draw C *a* parallel to P B, and let it be divided by the other three into the parts C *d*, *d a*; lastly, through *a* draw D *b'* parallel to P C, and let it be divided by the other three into the parts D *a*, *a b'*: then, A *b* shall be to *b c*, as *c d* to B *c*, as C *d* to *d a*, and as *a b'* to D *a*.



Because A *c* is parallel to P D, and *c d* to P A, A *d* is a parallelogram. therefore (I. 22.) P *d* is equal to A *c*. And, by *similar triangles* B P *d*, B *b c*, P *d* : *b c* :: B *d* : B *c*, (31.): but P *d* is equal to A *c*; therefore, *dividendo*, A *b* : *b c* :: *c d* : B *c*. In the same manner it may be shown that *c d* : B *c* :: C *d* : *d a*; and again, that C *d* : *d a* :: *a b'* : D *a*. Therefore the ratio of A *b* to *b c* is the same with the ratio of *c d* to B *c*, which is the same again with that of C *d* to *d a*, which is the same with that of *a b'* to D *a*. And any straight lines parallel to these (30.) will be divided in the same ratio.

Therefore, &c.

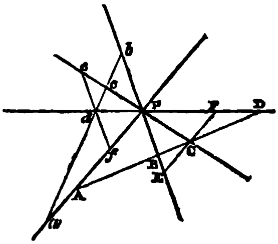
It will be observed that, if the parts be considered as proceeding in a particular direction, viz. from A towards B, C, D, the proportional parts are continually in an inverted order: thus, A *b*, is to *b c* not as B *c* to *c d*, but as *c d* to B *c*, and so on.

Cor. If four straight lines pass through the same point, and if a parallel to one of them has equal parts of it intercepted by the other three, a parallel to any of the others shall likewise have equal parts of it intercepted by the other three.

Prop. 49.

If four straight lines pass through the same point, and divide any one straight line harmonically; they shall divide every straight line, which is cut by them, harmonically.

Let the four straight lines P A, P B, P C, P D pass through the same point P, and let the straight line A D be harmonically divided by them in the points B, C: then, if any other straight line *a b* be cut by the same four lines, the straight line *a b* shall likewise be divided harmonically.



Through C draw E F parallel to P A, and let it cut P B, P D in the points E, F respectively. Then, by *similar triangles* B P A, B E C, (31.) the ratio of A P to E C is the same with that of A B to B C. Again, by *similar triangles* D P A, D F C, the ratio of A P to C F is the same with the ratio of A D to D C. But, because A D is harmonically divided, A B has to B C the same ratio as A D to D C, (def. 19.): therefore, (12.) A P has to E C the same ratio as A P to C F, and (11.*Cor.*1.) E C is equal to C F.

And, because E F, which is parallel to P A, has equal parts of it intercepted by P B, P C, P D, if through the point *d* in which *a b* cuts P D, the straight line *e f* be drawn parallel to P B and terminated by P C, P A, the line *e f* will likewise be divided equally in *d* (48.*Cor.*). But, by *similar triangles* *c* P *b*, *c e d*, *c b* : *c d* :: P *b* : *e d*, and by *similar triangles* *a* P *b*, *a f d*, *a b* : *a d* :: P *b* : *d f* (or *e d*): therefore *c b* : *c d* :: *a b* : *a d*, (12.) that is, *a b* is harmonically divided in the points *c, d*.

Therefore, &c.

Cor. 1. If there be four straight lines harmonicals (def. 20), and if a parallel be drawn to any one of them, equal parts of the parallel shall be intercepted by the other three.

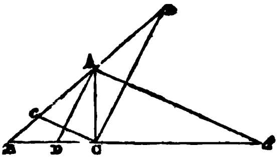
Cor. 2. And, conversely, if four straight lines pass through the same point, and if, a parallel being drawn to any one of them, equal parts of the parallel be intercepted by the other three, those four straight lines shall be harmonicals.

Thus, the two sides of a triangle, a line drawn from the vertex to the bisection of the base, and a line drawn through the vertex parallel to the base, are harmonicals.

Prop. 50. (Euc. vi. 3. & A.)

If the vertical and exterior-vertical angles of a triangle be bisected by straight lines which cut the base and the base produced, the base and likewise the base produced shall be divided in the ratio of the sides: and conversely.

Let A B C be a triangle; and first, let the vertical angle B A C be bisected by the straight line A D which cuts the base B C in D: B D shall be to D C as B A to A C.



Through C draw C E parallel to A D, and let it meet B A produced in E: then, because the angles A E C, A C E are (I. 15.) equal, respectively, to the halves of the bisected angle, they are equal to one another: wherefore A C is equal to A E, (I. 5.). But, again, because C E is parallel to D A, B D : D C :: B A : A E, (29.): therefore B D : D C :: B A : A C.

And, conversely, if B D be to D C as B A to A C, A D shall bisect the vertical angle. For, C E being drawn (as before) parallel to A D, because B A is to A C as B D to D C, that is (because C E is parallel to D A) as B A to A E (29.), A C is (11. *Cor.* 1.) equal to A E. Therefore the angle A E C is equal to A C E (I. 6.), and the parts of the angle in question being equal to A E C, A C E respectively (I. 15.), are equal to one another; that is, it is bisected by the line A D.

And, in the same words, it may be demonstrated, that if the exterior vertical angle be bisected by the line A *d*, which cuts the base produced in *d*, B *d* shall be to *d* C as B A to A C; and that, conversely, if B *d* be to *d* C as B A to A C, A *d* shall bisect the exterior vertical angle: the letters *d*, *e*, being substituted for D, E.

Therefore, &c.

When the sides AB, AC are equal to one another, A D bisects the base B C at right angles (I. 6. *Cor.* 3.); and A *d* is parallel to the base B C; for A *d* is always at right angles to A D, because the angle D A *d* is equal to the halves of the two angles B A C, C A E together, that is, to the half of two right angles.

Cor. Since the ratios of B D to D C, and of B *d* to *d* C, are each of them the same with the ratio of B A to A C, they are the same with one another (12.), and B *d* is harmonically divided in the points D, C. Therefore the two sides of a triangle, and the lines which bisect the vertical and exterior-vertical angles, are harmonicals (def. 20.).

SECTION 7.—*Problems.*

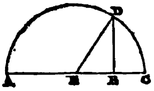
Def. 21. A straight line is said to be divided *in extreme and mean ratio*, when the whole line is to the greater segment as the greater segment is to the less.

A straight line so divided is also said to be divided *medially;* and the ratio of its segments is called *the medial ratio.*

PROP. 51. Prob. 1. (EUC. vi. 13.)

To find a mean proportional between two given straight lines A B *and* B C.

Let the straight lines A B, B C be placed in the same straight line: from the point B (I. 44.) draw B D at right angles to A C: bisect A C in E (I. 43.), and from the centre E, with the radius E C, describe a circle cutting B D in D: B D shall be the mean proportional required.



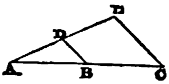
For, the angle E B D being a right angle, the square of B D is (I. 36. *Cor.* 1.) equal to the difference of the squares of E B, E D. But, because E A and E C are, each of them, equal to E D, A B is equal to the sum, and B C to the difference of E D, E B: therefore (I. 34.) the square of B D is equal to the rectangle under A B, B C, and (38. *Cor.* 1.) B D is a mean proportional between A B, B C.

Therefore, &c.

PROP. 52. Prob. 2. (EUC. vi. 11.)

To find a third proportional to two given straight lines A B, B C.

Let A B, B C be placed in the same straight line: from the point A draw A D, making any angle with A C, and make A D equal to B C: join B D, and through C draw (I. 48.) C E parallel to B D, to meet A D produced in E.



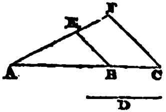
Then, because (29.) A B is to B C as A D (or B C) to D E, D E is the third proportional required.

Therefore, &c.

PROP. 53. Prob. 3. (EUC. vi. 12.)

To find a fourth proportional to three given straight lines A B, B C, *and* D.

Let the straight lines A B, B C be placed in the same straight line: from the point A draw A E, making any angle with A C, and make AE equal to D: join B E, and through C draw C F parallel to B E (I. 48.), to meet AE produced in F.



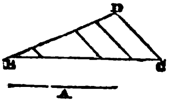
Then, because (29.) A B is to B C as A E (or D) to E F, E F is the fourth proportional required.

Therefore, &c.

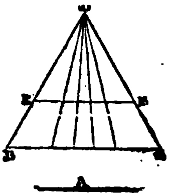
PROP. 54. Prob. 4. (EUC. vi. 10.)

To divide a given straight line A, *similarly to a given divided straight line* BC.

Method 1. Draw B D making any angle with B C, and make B D equal to A: join C D, and through the several points in which B C is divided, draw lines parallel to CD (I. 48.): then (29. *Cor.*), because these lines are parallel to C D, the straight line D B, that is, A, is divided by them similarly to the given divided straight line B C.



Method 2. Upon BC describe (I. 42.) the equilateral triangle D B C: take D E, D F each of them equal to A, and join E F; and from D through the several points in which B C is divided, draw straight lines cutting E F.

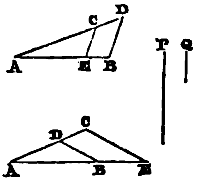


Then, because D E is equal to D F, and D B to D C, the triangles D E F, D B C are similar (32.): but B C is equal to B D; therefore E F is equal to E D, that is, to the given straight line A; and, for the same reason, the angle D E F is equal to the angle D B C: therefore (I. 15.) E F is parallel to B C. And, because E F, B C are parallels, E F, that is, A, is divided similarly to B C by lines drawn from the point D (30. *Cor.* 1.).

Therefore, &c.

Prop. 55. Prob. 5.

To divide a given straight line, or a given straight line produced, in a given ratio.*



Let AB be the given straight line, and let the given ratio be that of the straight line P to the straight line Q.

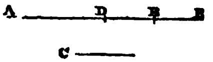
Draw AC making any angle with A B; make A C equal to P and C D equal to Q; join D B, and through C draw C E parallel to D B to meet A B, or A B produced in E. Then because A E is to E B as A C to C D (29.), that is, as P to Q, A B, or A B produced, is divided in the point E in the given ratio.

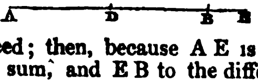
Therefore, &c.

Prop. 56. Prob. 6.

To divide a given straight line A B, *or* A B *produced, so that the rectangle under the segments may be equal to a given square, viz. the square of* C.

Bisect A B in D, and take D E such that the square of D E may be equal to the difference or to the sum of the squares of D B and C (I. 59. or 60.), according as A B is to be divided or A B produced; then, because A E is equal to the sum, and E B to the difference of D B, D E, the rectangle under A E, E B is equal to the difference of the squares of D B, D E (I. 34.), that is, to the square of C; and A B is divided, as required, in the point E.





Therefore, &c.

Cor. Hence also a given straight line A B or A B produced may be so divided, that the rectangle under the segments may be equal to a given rectangle (I. 58.).

N. B. In the first case, viz. when the point E is to be found between A and B, the problem will be impossible if C be greater than the half of AB; for AE × E B is equal to the difference of the squares of D B, D E, which is never greater than the square of D B. The second case is not thus limited.

We may remark also, that in both cases, two points E may be found satisfying the given conditions, viz. one upon each side of the point D: in the first case, when C is equal to the half of A B, these two points coincide at D.

Prop. 57. Prob. 7.

To find an harmonical mean between two given straight lines A B *and* A C.

Divide BC in the point D in the ratio of B A to A C (55.): then, because AB : A C :: B D : D C, the three lines AB, A D, and A C are in harmonical progression (def. 17.), and A D is an harmonical mean between A B and AC

Therefore, &c.

Prop. 58. Prob. 8.

To find a third harmonical progressional to two given straight lines A B, A C.

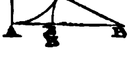
Divide AC produced in the point D in the ratio of A B to B C (55.): then, because A D : C D :: A B : B C,

alternando A D : A B :: C D : B C; therefore (def. 17.) A B, A C, A D, are in harmonical progression, and A D is a third progressional to A B, A C.

Therefore, &c.

Prop. 59. Prob. 9. (Euc. vi. 30.)

To divide a given straight line A B *in extreme and mean ratio.*



From A draw A C perpendicular to A B (I. 44.), and make it equal to the half of A B: join CB: from the centre C with the radius C A describe a circle cutting CB in D, and from the centre B with the radius B D describe a circle cutting A B in E: A B shall be divided in extreme and mean ratio in the point E.

---

* A line or a line produced is said to be divided in a given ratio, when the segments of the line or of the line produced are to one another in that ratio. By the segments of a line produced are meant the whole line produced and the part produced.

Because (I. 36.) the squares of C A, A B are together equal to the square of C B, that is (I. 32.), to the squares of C D, D B, together with twice the rectangle C D, D B, and that the square of C A is equal to the square of C D (I. 25. *Cor.*); the remaining square of A B is equal to the square of D B, together with twice the rectangle C D, D B, or, which is the same thing, to the square of B E, together with the rectangle A B, B E, for B E is equal to D B, and A B to twice C D.

But (I. 30. *Cor.*) the square of A B is also equal to the rectangle A B, A E, together with the rectangle A B, B E. Therefore the square of B E, together with the rectangle A B, B E, is equal to the rectangle A B, A E, together with the rectangle A B, B E. Therefore the rectangle AB, AE is equal to the square of B E (1. ax. 3); and (38. *Cor.* 1.) $AB : BE :: BE : AE$, that is, A B is divided in the point E in extreme and mean ratio. (def. 21.)

Therefore, &c.

Scholium.

The parts of a line thus divided belong to a certain class of incommensurable magnitudes, described in the following general theorem.

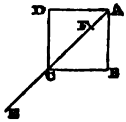
If P, Q be two magnitudes of the same kind, and such that P is contained in Q a certain number of times with a remainder which is to P as P is to Q; the magnitudes P and Q shall be incommensurable.

For, let P be contained in Q 5 times, and let R be the remainder; then, because $R : P :: P : Q$, $5R : 5P :: P : Q$ (17. *Cor.* 2.); therefore, *alternando*, $5R : P :: 5P : Q$, and *dividendo*, $5R \sim P : P :: 5P \sim Q : Q$. But $P : R :: Q : P$: therefore, *ex æquali*, $5R \sim P : R :: 5P \sim Q$ (or R) $: P$; in which proportion the third term is less than the fourth: therefore the first term is less than the second (14.), that is, R is also contained in P 5 times, with a remainder $5R \sim P$ or R_2, which is to R as R to P. Therefore, again, R_2 is contained in R 5 times with a remainder R_3 which is to R_2 as R_2 to R; and so on: *i. e.* every following remainder is always contained in the preceding 5 times with a new remainder, and there is no remainder which is contained in the preceding a certain number of times exactly; therefore P and Q have no common measure (5. *Cor.* 1.).

Therefore, &c.

Now it is evident that the segments of a line medially divided are magnitudes of this description, for the greater segment E B being contained in the whole line A B once with a remainder A E which is to E B as E B to A B, it follows, as in the preceding demonstration, that A E likewise is contained in E B once with a remainder, which is to A E as A E to E B; and therefore A E, E B are incommensurable.

Another instance of incommensurables of this class is afforded by the side and diagonal of a square.



Let A B be a side, and A C one of the diagonals of the square ABCD. Produce AC to E so that C E may be equal to C B; and from C A cut off C F likewise equal to C B. Then because A E is equal to the sum, and A F to the difference of A C, C B, the rectangle under A F, A E is (I. 34.) equal to the difference of the squares of A C, C B, that is (I. 36. *Cor.* 1.), to the square of A B. Therefore (38. *Cor.* 1.), A F is to A B as A B to A E. But, because A B is equal to B C, and that E F is equal to 2 B C, A B is contained in A E twice with the remainder A F. Therefore A B, A E are magnitudes of the same kind with P, Q in the above theorem, and are incommensurable. And because A E is incommensurable with A B, and that its part C E is equal to A B, the remainder, that is, the diagonal A C, is incommensurable with A B.

We shall conclude this Scholium with an easy method of approximating numerically to all ratios of this description.

To explain it, let us suppose, as in the latter instance, that there are two magnitudes, P and Q, and that P is contained in Q twice with a remainder R, which is to P as P to Q: the ratio of P to Q shall lie between any two consecutive ratios of the series $1 : 2$, $2 : 5$, $5 : 12$, $12 : 29$, $29 : 70$, &c. the terms of which are formed from the two first, by making every new term equal to twice the last, together with the last but one; that is, it shall lie between $1 : 2$ and $2 : 5$; again between $2 : 5$ and $5 : 12$, &c., the ratios of each successive pair approaching always more nearly to one another, and therefore to the ratio sought.

For, since P is contained in Q twice with a remainder R, $Q = 2P + R$: therefore $R : P :: P : 2P + R$; that is, a ratio the same with that of R to P is formed by taking P for a new antecedent, and twice P together with R for a new consequent.

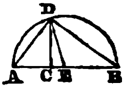
The same may be shown of the ratio $2P + R : 5P + 2R$, which is formed from this last according to the same rule, viz. that it is likewise the same with the ratio from which it is formed, that is, with the original ratio; and so on. Let these successive ratios be written one after another in the order in which they are derived: (*a*) $R : P$; (*b*) $P : 2P + R$, (*c*) $2P + R : 5P + 2R$; (*d*) $5P + 2R : 12P + 5R$; (*e*) $12P + 5R : 29P + 12R$, &c. Now, of these ratios, each of which is equal to the ratio sought, the ratio (*c*) lies between (23. *Cor.* 3.) $2P : 5P$ and $R : 2R$, that is (def. 5.) between $2 : 5$ and $1 : 2$; in like manner, (*d*) lies between $5 : 12$ and $2 : 5$, (*e*) between $12 : 29$ and $5 : 12$, &c. Again, $5 : 12$ is formed by adding the terms of $2 \times 2 : 2 \times 5$ to those of $1 : 2$, and therefore lying between $2 : 5$ (1. *Cor.* 3.) and $1 : 2$ approaches more nearly to $2 : 5$ than $1 : 2$ approaches; in like manner, $12 : 29$ is formed by adding the terms of $2 \times 5 : 2 \times 12$ to those of $2 : 5$, and therefore lying between $5 : 12$ and $2 : 5$, approaches more nearly to $5 : 12$ than $2 : 5$ does; and so on. Not only, therefore, does the ratio sought lie between the successive ratios of the series $1 : 2$, $2 : 5$, $5 : 12$, $12 : 29$, &c.; but these approach continually more and more nearly to one another, and therefore to the ratio sought.

In the same manner, if P were contained in Q 5 times, we might approximate to their ratio by means of the series 1, 5, 26, 135, &c. which is formed from the two first terms by making every following term equal to 5 times the last, together with the last but one: whence the application of the Rule to any given case is sufficiently apparent.

In the case of the medial ratio the series becomes 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, &c. which is formed from the two first terms by making every following term equal to the sum of the two last terms; and this is the simplest case possible.*

PROP. 60. Prob. 10.

Upon a given hypotenuse A B, *to describe a right angled triangle, which shall have its three sides proportionals.*



Divide AB medially (59.) in the point C, and from C draw CD (I. 44.) perpendicular to AB; bisect AB in E, and from the centre E, with the radius E A describe a circle cutting C D in D; join D A, D B, DE; ADB shall be the triangle required.

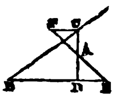
For the triangle A D B is right-angled at D, because ED is equal to EA or EB (I. 19. *Cor.* 4.). Also AD^2 is to DB^2 as the rectangle (I. 36. *Cor.* 2.) under A C, A B, to the rectangle under CB, AB, that is, as A C to CB (35.); and DB^2 is to BA^2 as the rectangle under C B, B A, to the square of B A, that is (35.) as $CB : BA$. But because A B is medially divided in C, $AC : CB :: CB : BA$; therefore (12.) $AD^2 : DB^2 :: DB^2 : BA^2$, and $AD : DB :: DB : BA$ (37. *Cor.* 4.); that is, AD, DB, BA are proportionals.

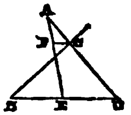
Therefore, &c.

PROP. 61. Prob. 11.

Through a given point A, *to draw a straight line such that the parts of it intercepted by the legs,* B C, B D, *of a given angle* B, *may be to one another in a given ratio.*

Let the given ratio be that of the two straight lines P, Q, of which the latter is greater than the distance of A from BE. From A to BD draw the straight line AE equal to Q, and take AF equal to P; through F draw F C parallel to B D (I. 48.), and draw the straight line A C D. Then, because A C is to A D as A F to A E (29.), that is, as P to Q, C D is the straight line required.





Therefore, &c.

In the solution of this problem the ratio is supposed to be that of two given straight lines P, Q. If Q be less than the perpendicular drawn from A

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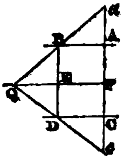
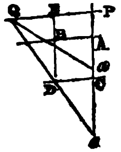
[*] The converging ratios in the text are also derivable from the doctrine of Continued Fractions. The series are of the kind called *recurring* series, because every term has a given relation to a certain number of the terms preceding it. It may also be remarked, that the incommensurable ratios of the text are all included in the surd formula $\frac{1}{2}(m + \sqrt{m^2 + 4})$, where m is the number of times P is contained in Q; and may therefore be readily obtained by the process for extracting the square root.

to B E, take any line Q′ which is not less than that perpendicular, and P′ such that P′ : Q′ :: P : Q (53.), and proceed with P′, Q′, as above.

Prop. 62. Prob. 12.

Through a given point P, *to draw a straight line which shall pass through the intersection of two given straight lines* A B, C D, *that intersection being without the limits of the draught.**





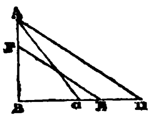
Through P draw any straight line cutting the given lines in the points A, C; and through any point B in A B draw BD parallel to A C to meet C D in D. (I. 48.) Take P *a* equal to twice P A, and P *c* equal to twice P C: join *a* B, *c* D, and let them be produced to meet one another in Q; and join P Q; P Q shall be the line required.

For, let B D cut P Q in E; then, because BD is parallel to *a c*, B E is to ED as *a* P to P *c* (30.), that is, as A P to P C (17.); and therefore the line P Q produced (30.) passes through the intersection of A B, C D.

Therefore, &c.

Prop. 63. Prob. 13.

To find two straight lines which shall contain a rectangle equal to a given rectangle, and have one to the other a given ratio.



Let the rectangle AB, BC be the given rectangle, and, A B making any angle with B C, let B C be produced to D, so that A B may be to BD in the given ratio (53.). Join AD; take B E a mean proportional (51.) between B C and B D, and through E draw E F parallel to A D (I. 48.), to meet A B in F; B E, B F shall be the straight lines required.

For (29.) they are evidently in the given ratio; and because BF is to BA as BE to BD, the rectangle under BF, BE is to the rectangle under BA, BD as the square of B E to the square of B D (37. *Cor.* 1. and 2.), that is, as B C to B D. But the rectangle under A B, B C is likewise to the rectangle under A B, B D, as B C to BD (35.); therefore the rectangle under E B, B F is equal to the rectangle under A B, B C (11. *Cor.* 1.).

Therefore, &c.

Prop. 64. Prob. 14.

Given any number of straight lines A, A_2, A_3, A_4, *antecedents, and as many* B, B_2, B_3, B_4, *consequents, to find a straight line such that* A *shall bear to it the ratio which is compounded of the ratios of* A *to* B, A_2 *to* B_2, A_3 *to* B_3, *and* A_4 *to* B_4.

Find (53.) the straight line P such that B : P :: A_2 : B_2, Q such that P : Q :: A_3 : B_3, and R such that Q : R :: A_4 : B_4; R shall be the straight line required.

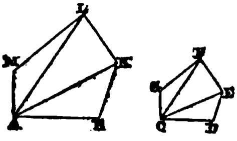
For the ratio of A to R is compounded (def. 12.) of the ratios of A to B, B to P, P to Q, and Q to R, that is, of the ratios of A to B, A_2 to B_2, A_3 to B_3, and A to B_4.

Therefore, &c.

Prop. 65. Prob. 15. (Euc. vi. 18.)

Upon a given straight line A B, *to describe a rectilineal figure similar to a given rectilineal figure* C D E F G.

Method 1. Join C E, C F; at the points A, B (I. 47.) make the angles B A K, A B K, equal to the angles D C E, C D E respectively; and because these two angles are, together, less than two right angles (I. 8.) A K, B K will meet, if produced, in some point K (I. 15. *Cor.* 4.). Again, at the points A, K, make the angles KAL, A K L equal to the angles E C F, C E F respectively, and (as before) let A L, K L meet one another in the point L: lastly, at the points A, L make the angles L A M, A L M equal to F C G, C F G respectively, and let A M, L M meet one another in the point M: the rectilineal figure A B K L M shall be similar to C D E F G.



For the angles of the two figures are evidently equal, each to each, because they are the sums of corresponding angles of equiangular triangles. Also, because the triangles K B A, E D C are similar, B K is to K A as D E to E C; and for the like reason, K A is to K L as E C to E F: therefore, *ex æquali*, B K : K L :: D E : E F. And in the same manner it may be demonstrated, that the sides about the other equal angles of the two figures are proportionals. Therefore, the figure described upon A B is similar to the given figure C D E F G. (def. 14.)

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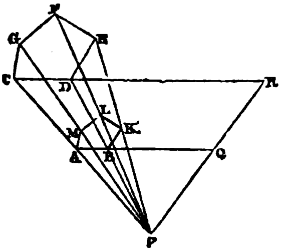
* This operation occurs so frequently in perspective drawings, that it is convenient to have a ruler expressly constructed for the purpose, called a *centrolinead.* Such an instrument was a desideratum among draughtsmen, until it was supplied by the ingenuity of Mr. P. Nicholson.

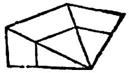
Method 2. Place A B parallel to C D, and join C A, D B. Then, if A B be not equal to C D, C A will not be parallel to D B (I. 21. and 14. *Cor.* 2.), and C A, D B will meet, if produced, in some point P: join P E, P F, P G: through B draw B K parallel to D E (I. 48.), and let it meet P E in K: through K draw K L parallel to E F, and let it meet P F in L: through L draw L M parallel to F G, and let it meet P G in M; and join A M: the figure A B K L M shall be similar to the given figure C D E F G.



For, because P C : P A :: P D : P B (29.), that is, :: P E : P K, that is :: P F : P L, that is, :: P G : P M, A M is parallel to C G; and because the sides of the two figures are parallel, each to each, they are equiangular (I. 18.) with one another. Also, because the triangles C D P, A B P are similar, C D : D P :: A B : B P, and for the like reason D P : D E :: B P : B K; therefore, *ex æquali*, C D : D E :: A B : B K. And in the same manner it may be demonstrated, that the sides about the other equal angles of the two figures are proportionals. Therefore the figure upon A B is similar to the given figure. When A B is equal to C D, C A is parallel to B D, and the lines E K, F L, G M, instead of being drawn to a point P, must be drawn parallel to C A or B D. The demonstration that the figure ABKLM, so constructed, will be both similar and equal to C D E F G, may then be easily derived from I. 21.

Therefore, &c.

The last method will appear under a still more simple and convenient form, if A B be made to coincide with C D, and B with D, as in the adjoined figure.



Prop. 66. Prob. 16.

To describe a rectilineal figure, which shall be similar to a given rectilineal figure CDEFG, *and shall have its perimeter equal to a given straight line* AQ.

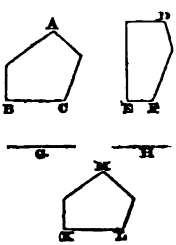
Produce C D to R, (see the second figure of *Prop.* 65.) so that C R may be equal to the perimeter of the given figure C D E F G; take A B (53.) a fourth proportional to C R, A Q, and C D, and upon A B (65.) describe a rectilineal figure A B K L M similar to the given figure. Then, because the figures A B K L M and C D E F G are similar, the perimeter of the first is to the perimeter of the other (43.), as A B to C D, that is (12.), as A Q to C R: but the perimeter of C D E F G is equal to C R: therefore (18. *Cor.*) the perimeter of A B K L M is equal to A Q, that is, to the given perimeter. Therefore, AB K L M is the figure required.

Therefore, &c.

Prop. 67. Prob. 17. (Euc. vi. 25.)

To describe a rectilineal figure which shall be similar to a given rectilineal figure A B C, *and shall have its area equal to the area of another given rectilineal figure* DEF.

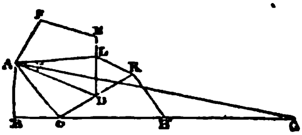
Find (I. 58. *Cor.*) the straight lines G, H such that their squares may be equal to the figures ABC, DEF respectively: take K L a fourth proportional (53.) to G, H, and B C, and upon K L (65.) describe the figure K L M similar to the figure B C A. Then, because the figure A B C is to the figure M K L as the square of B C to the square of K L (43.), that is, (37. *Cor.* 4.) as the square of G to the square of H, and that the figure A B C is equal to the square of G, the figure K L M is (18.) equal to the square of H, that is, to the figure D E F: and it is similar to ABC; therefore it is the figure required.



Prop. 68. Prob. 18.

To divide a given rectilineal figure A B C D E F *in a given ratio, by a straight line drawn from one of its angles, or from a given point in one of its sides.*

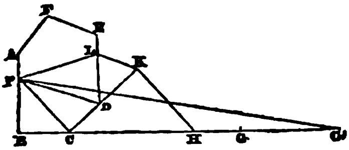
In the first place, let A be the given angle from which the line of division is to be drawn.



Describe (I. 54.) the triangle A B G, equal to the figure A B C D E F, and naving the side A B, and angle A B C the same with it: divide (55.) the base B G in the given ratio in the point H: join A C and A D: through H draw H K parallel to A C, to meet C D produced in K; through K draw K L parallel to A D to meet D E in L, and join A L: A L shall be the line required.

For, by the construction, which is similar to that of I. 55., it is evident, that the figure A B C D L is equal to the triangle A B H:* but the whole figure is equal to the triangle A B G: therefore, the part A L E F is equal to the triangle A H G. Therefore, the parts of the figure are as the triangles A B H, A H G, that is (39.) as the bases B H, H G, that is, in the given ratio.

Next, let P be the given point in the side A B, from which the line of division is to be drawn.



Describe, as before, (I. 54.), the triangle A B G equal to the given figure, and, by drawing A G' parallel to P G, make the triangle P B G' equal to A B G (as in I. 56.): divide (55.) B G' in the given ratio in the point H: join P C and P D: through H draw H K parallel to P C, to meet C D produced in K: through K draw K L parallel to P D to meet D E in L, and join P L. Then, as before, the part P B C D L of the given figure, is equal to the triangle P B H, and the remaining part P A F E L to the triangle P H G'. Therefore, the parts of the figure are as the bases B H, H G' (39.), that is, in the given ratio.

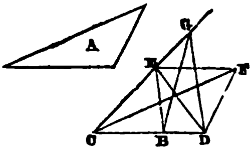
Therefore, &c.

Prop. 69. Prob. 19.

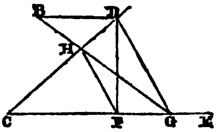
Given a triangle A, *a point* B, *and two straight lines* C D, C E, *forming an angle* D C E; *to describe a triangle which shall be equal to the triangle* A, *so that it may have the angle* D C E *for one of its angles, and the opposite side passing through the point* B.

In the solution of this problem we have three cases to consider; first, when the point B is in one of the given lines as C D; secondly, when the point B is without the given angle D C E; and thirdly, when B is within the angle D C E.

Case 1. Let the point B be in the line C D. Take C D equal to a side of the triangle A, and upon C D (I. 50.) describe a triangle C D F, which shall have its two remaining sides equal to the two remaining sides of the triangle A, each to each, and therefore (I. 7.) shall be equal to the triangle A in every respect: through F (I. 48.) draw F E parallel to D C: join D E, B E: through D (I. 48.) draw D G parallel to B E to meet C E in G, and join G B: the triangle C B G shall be the triangle required.



For, because E B is parallel to G D, the triangle G E B (I. 27.) is equal to D E B: therefore, adding the triangle E C B to each, the whole triangle G C B is equal to E C D, that is (because E F is parallel to C D) to F C D, that is, to the given triangle A.



Case 2. Let the point B be without the angle D C E. Through B (I. 48.) draw B D parallel to C E to meet C D in D, and by the construction pointed out in *Case* 1. describe a triangle D C F having the given side D C and the given angle D C E, and equal to the given triangle A: in C F produced take the point G such that C G × G F may be equal to B D × C F (56.): join B G, and let B G cut C D in H: C H G shall be the triangle required.

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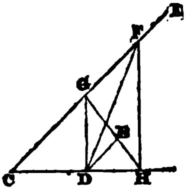
* The line A H is wanting in this figure; and P H is in like manner wanting in the figure below, as also A G, A G'.

Join HF. Then, because C G × G F is equal to B D × C F, C F : FG :: C G : BD (38.), that is, since D H B and C H G are (I. 15.) equiangular, :: C H : H D. Therefore (29.) H F is parallel to DG; and hence, as before, the triangles DHF and GHF are (I. 27) equal to one another, and CHG is equal to CDF, that is to A.

Case 3. Let the point B be within the angle D C E. Through B (I. 48.) draw BD parallel to E C to meet C D in D, and, by the construction pointed out in *Case* 1, describe a triangle D C F, having the given side CD and the given angle DCE, and equal to the given triangle A : in CF (if it be possible) take the point G such that C G × G F may be equal to B D × C F (56. *Cor.*): join B G, and let G B produced cut C D in H; C H G shall be the triangle required.



Join HF. Then, by a demonstration which may be given in the same words as that of *Case* 2, the triangle C H G is equal to C D F, that is to the given triangle A.

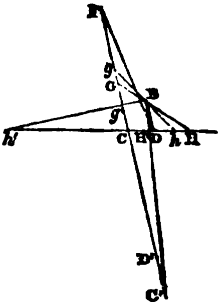
In this last case a solution will be impossible if B D exceed a fourth of C F; for then B D × C F will exceed a fourth of the square of C F, that is (I. 29. *Cor.* 2.) the square of half C F; and no point G can be taken in C F such that C G × G F may exceed the square of half C F (56. *N.B.*).

We may remark also that, whenever the solution is possible in the last case, two points G may be found such that C G × G F = B D × C F, (56.) and therefore two lines G H may be drawn satisfying the given conditions. When B D is exactly a fourth of C F, these two points coincide with one another and with the middle point of C F, and therefore the two solutions become identical.

If B be in one of the lines D C, E C produced, or within the angle which is vertical to D C F, the solution will be manifestly impossible.

Scholium.

Had the problem been proposed under the following form: "through a given point B to draw a straight line which shall form with two given straight lines CD C E cutting one another in C a triangle equal to a given triangle A;" we should have had four solutions, two of them corresponding to the angle in which B lies, and one for each of the adjacent angles; as is apparent from the foregoing constructions applied to the adjoined figure. When B D is equal to a fourth of C F, two of these become identical; when B D exceeds a fourth of C F, the same two become impossible, but the other two, viz. those which correspond to the adjacent angles, are always possible, except when B is in one of the lines as C D. This last-mentioned position is peculiar: it has likewise, however, two solutions, one for each of the adjoining angles.

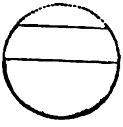


BOOK III.

§ 1. *First Properties of the Circle*—§ 2. *Of Angles in a Circle*—§ 3. *Rectangles under the segments of Chords*—§ 4. *Regular Polygons, and Approximation to Circular Area*—§ 5. *Circle a Maximum of Area, and Minimum of Perimeter*—§ 6. *Simple and Plane Loci*—§ 7. *Problems.*

SECTION 1.—*First Properties of the Circle.*

Def. 1. Any portion of the circumference of a circle is called an *arc;* and the straight line which joins the extremities of an arc is called the *chord* of that arc.



When the chord passes through the centre, it is a diameter, and the arcs upon either side of it, being equal to one another, are called, each of them, a *semi-circumference.*[*]

2. The figure which is contained by an arc and its chord is called a *segment.*

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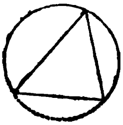
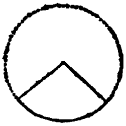
[*] That every diameter divides a circle and its circumference into two equal parts, is evident from the symmetrical character of the circle. The same may be proved by doubling the figure upon the diameter; for, every point of the circumference being at the same distance from the centre, the parts so applied (whether of circumference or area) will coincide.

A segment which is contained by a semicircumference and a *diameter*, is equal to half the circle, and is therefore called a *semicircle*.

3. *An angle in a segment* is the angle contained by two straight lines drawn from any point of the arc of the segment to the extremities of its base or chord.



4. A *sector* of a circle is the figure contained by any arc, and the radii drawn to its extremities.



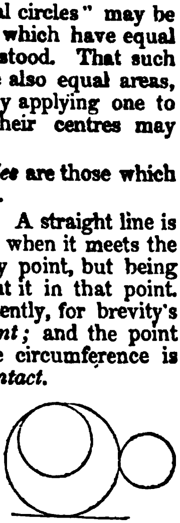
5. *Equal circles* are those which have equal radii.

By this is intended no more than that when the term "equal circles" may be hereafter used, those which have equal radii are to be understood. That such circles, however, have also equal areas, is at once evident by applying one to the other, so that their centres may coincide.

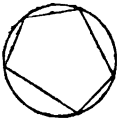
6. *Concentric circles* are those which have the same centre.

7. (Euc. iii. def. 2.) A straight line is said to *touch* a circle, when it meets the circumference in any point, but being produced does not cut it in that point. Such a line is frequently, for brevity's sake, called *a tangent;* and the point in which it meets the circumference is called the *point of contact.*

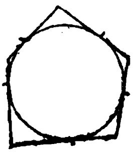
8. *Circles* are said to *touch one another*, when they meet, but do not cut one another.



9. A rectilineal figure is said to be *inscribed in a circle*, when all its angular points are in the circumference of the circle. Also, when this is the case, the circle is said to be *circumscribed about the rectilineal figure.*



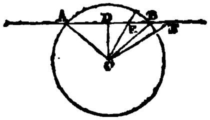
10. A rectilineal figure is said to be *circumscribed about a circle*, when all its sides touch the circle. Also, when this is the case, the circle is said to be *inscribed in the rectilineal figure.*



Prop. 1. (Euc. iii. 2.)

If a straight line meet the circumference of a circle in two points, it shall cut the circle in those points; and the part of the straight line which is between them shall fall within the circle.

Let the straight line A B meet the circumference of a circle having the centre C in the points A and B: it shall cut the circle in those points, and the part A B shall fall within the circle.



Join C A, C B: bisect A B in D, and join C D: take also the points E and F in the line A B, the former between A and B, the latter upon the other side of B, and join C E, C F.

Then, because C A is equal to C B, C A B is an isosceles triangle; therefore (I. 6. *Cor.* 3.) C D, which is drawn from the vertex to the bisection of the base, is perpendicular to A B. And, because C E is nearer to the perpendicular than C B is, (I. 12. *Cor.* 2.) C E is less than C B; therefore the point E is within the circle: on the other hand, because C F is farther from the perpendicular than C B is, C F is greater than C B; and therefore the point F is without the circle. And the same may be said of the parts about A. Therefore the straight line in question cuts the circle in the points A and B.

Also, because every point, as E, of A B, is at a less distance from C than the radius, the part A B falls within the circle.

Therefore, &c.

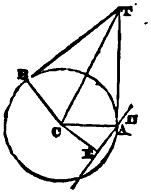
Cor. 1. A straight line cannot meet a circle in more than two points.

Cor. 2. A straight line which touches a circle meets it in one point only.

Cor. 3. A circle is concave towards its centre.

Prop. 2. (Euc. iii. 16.)

The straight line, which is drawn at right angles to the radius of a circle from its extremity, touches the circle; and no other straight line can touch it in the same point.



From the point A in the circumference of a circle having the centre C, let the straight line A T be drawn at right angles to the radius C A: A T shall touch the circle.

For, C A being shorter (I. 12. *Cor.* 3.) than any other line which can be drawn from C to A T, every other point of A T lies without the circle: therefore A T meets the circle in the point A but does not cut it, that is, it touches the circle.

In the next place, let D E be any other straight line passing through the same point A: D E shall cut the circle.

For, C A not being perpendicular to D E, let C E be perpendicular to it. Then, because C E is less than C A (I. 12. *Cor.* 3.), the point E is within the circle. But if D be a point in D E upon the other side of A, C D will be farther from the perpendicular than C A; wherefore C D being (I. 12. *Cor.* 2.) greater than C A, the point D will be without the circle. Therefore the straight line D E cuts the circle; and the same may be demonstrated of every other straight line which passes through A, except the straight line A T only, which is at right angles to A C.

Therefore, &c.

Cor. 1. (Euc. iii. 18.) If a straight line touches a circle, the straight line drawn from the centre to the point of contact, shall be perpendicular to the line touching the circle.

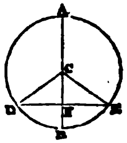
Cor. 2. (Euc. iii. 19.) If a straight line touches a circle, and from the point of contact, a straight line be drawn at right angles to the touching line, the centre of the circle shall be in that line.

Cor. 3. Tangents T A, T B which are drawn to a circle from the same point T, are equal to one another. For, C A T, C B T being right-angled triangles which have the common hypotenuse C T, and their sides C A, C B equal to one another, their remaining sides T A, T B are likewise equal. (I. 13.)

Cor. 4. Tangents which are at the extremities of the same diameter are parallel to one another.

Prop. 3. (Euc. iii. 3.)

If a diameter cut any other chord at right angles, it shall bisect it; and conversely, if a diameter bisect any other chord, it shall cut it at right angles.



Let A B be a diameter of the circle A D E, the centre of which is C; and, first, let it cut the chord D E at right angles in the point F: D E shall be bisected in F.

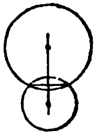
Join C D, C E. Then, because C D E is an isosceles triangle, the straight line C F which is drawn from the vertex C at right angles to the base D E, bisects the base, that is, D F is equal to F E. (I. 6. *Cor.* 3.)

Next let D E be bisected in F by the diameter A B; the angles D F A, E F A shall be right angles. For in the isosceles triangle C D E, the straight line which is drawn from the vertex C to the middle point of the base D E is at right angles to the base. (I. 6. *Cor.* 3.)

Therefore, &c.

Cor. 1. A diameter bisects all chords which are parallel to the tangent at either extremity of the diameter.

Cor. 2. The straight line which bisects any chord at right angles passes through the centre of the circle.



Cor. 3. If two circles have a common chord, the straight line which bisects it at right angles shall pass through the centres of both the circles.

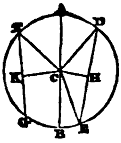
Cor. 4. (Euc. iii. 4.) It appears from the proposition, that two chords of a circle cannot bisect one another except they both of them pass through the centre.

For, if only one of the chords pass through the centre, it cannot be bisected by the other which does not pass through the centre.

And if, when neither of them passes through the centre, it were possible that they should bisect one another, the diameter passing through the supposed point of mutual bisection would be at right angles to each of them, which is absurd.

Prop. 4. (Euc. iii. 15.)

The diameter is the greatest straight line in a circle; and, of others, that which is nearer to the centre is greater than the more remote: also the greater is nearer to the centre than the less.



Let A B be a diameter of the circle A B D, the centre of which is C: let D E, FG be any two chords to which perpendiculars C H, C K are drawn; and let the distance C H be less than C K: the diameter A B shall be greater than the chord D E, and the chord D F shall be greater than the chord F G.

Join C D, C E, C F. Then, because C A is equal to C D, and C B to C E, the whole A B is equal to C D and C E together: but C D and D E together (I. 10.) are greater than D E: therefore A B is greater than D E. Again, because C H D, C K F are right-angled triangles, and that C D-square is equal to C F-square, the squares of C H, H D together (*I. 36.*) are equal to the squares of C K, K F together: but the square of C H is less than the square of C K; therefore the square of H D is greater than the square of K F, that is, H D is greater than K F. And D E, F G are double of H D, K F respectively, because the perpendiculars C H, C K pass through the centre (3.): therefore D E is greater than F G.

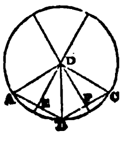
Next, let the chord D E be greater than F G; it shall also be nearer to the centre. For, C H and C K being drawn as before, the squares of C H, H D together are equal to the squares of C K, K F together; but the square of H D, which is half of D E, is greater than the square of K F, which is half of F G: therefore the square of C H is less than the square of C K, and C H is less than C K, that is, D E is nearer to the centre than F G is.

Therefore, &c.

Cor. (Euc. iii. 14.) Equal straight lines in a circle are equally distant from the centre; and those which are equally distant from the centre are equal to one another.

PROP. 5. (EUC. iii. 9.)

If a point be taken, from which to the circumference of a circle there fall more than two equal straight lines, that point shall be the centre of the circle.



Let ABC be a circle, and let D be a point taken, such that the three straight lines D A, D B, D C drawn from the point D to the circumference, are equal to one another: the point D shall be the centre of the circle A B C.

Join A B and B C, and bisect them in the points E and F respectively; and join D E, D F. Then, because D A B is an isosceles triangle, the straight line D E, which is drawn from the vertex D to the bisection of the base A B, is at right angles to A B (I. 6. *Cor.* 3.); and, because D E bisects the chord A B at right angles (3. *Cor.* 2.), it passes through the centre of the circle. In the same manner it may be shown that the straight line D F passes through the centre of the circle. But the only point through which each of the straight lines D E, D F passes, is their point of intersection D. Therefore D is the centre of the circle.

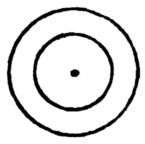
Therefore, &c.

Cor. 1. From any other point than the centre there cannot be drawn to the circumference of a circle more than two straight lines that are equal to one another, whether the point be within or without the circle. (Euc. iii. 7 and 8 parts of.)

Cor. 2. It appears from the demonstration that if three points A, B and C be given which are not in the same straight line, a circle may be found, the circumference of which shall pass through the three points A, B and C; the circle, namely, which has for its centre the intersection of the two lines which bisect A B and B C at right angles.

PROP. 6.

If two circles have the same centre, either they shall coincide, or one of them shall fall wholly within the other.



For if the radii of two concentric circles be equal to one another, it is manifest that every point in the circumference of the one must be at the same distance from their common centre, with every point in the circumference of the other; and therefore the two circumferences cannot but coincide, But if the radii be unequal, every point in the circumference of that which has the lesser radius is at a less distance from the common centre, and therefore must fall within the circumference of the greater circle.

Therefore, &c.

Cor. (Euc. iii. 5 and 6.) If two circles cut or touch one another, they cannot have the same centre.

Prop. 7. (Euc. iii. 10.)

The circumferences of two circles cannot intersect one another in more than two points.

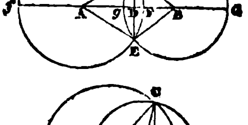
For if they should have three points in common, those three points could not be (1.) in the same straight line. Therefore a point might be found (5. *Cor.*) equally distant from the three, which point would (5.) be the centre of each of the circles; that is, there would be two circles cutting one another and having the same centre, a thing impossible.

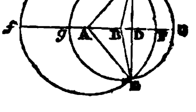
Therefore, &c.

Prop. 8.

If the circumferences of two circles meet one another in a point which is not in the straight line joining their centres, or in that straight line produced; they shall meet one another in a second point upon the other side of that straight line, and shall cut one another.

Let A, B be the centres of two circles, the circumferences of which meet one another in the point C, which is not in A B, nor in A B produced: from C draw C D perpendicular to A B or to A B produced, and produce C D to E so that D E may be equal to D C: the circumferences shall meet one another in the point E, and shall cut one another in each of the points C, E.





Join A C, A E, B C, B E. Then because the triangles A D C, A D E have two sides of the one equal to two sides of the other, and have also the included angles A D C, A D E equal to one another, the base A C (I. 4.) is equal to A E: therefore, the point E is in the circumference of the circle which has the centre A. In the same manner it may be shown, that the same point is in the circumference of the other circle. Therefore, the two circumferences meet one another in the point E.

Again, let the circumference of the first circle cut the line A B produced in the points F, *f*, and let the circumference of the other circle meet the same line A B, in the points G, *g*, the points F and G being towards the same parts, as also *f* and *g*: then, A G is equal to the sum, and A *g* to the difference of A B, B C. But, because A B C is a triangle (I. 10.), the sum of the sides A B, B C is greater, and their difference is less, than the side A C, that is, than A F or A *f*. Therefore, A G is greater than A F, and A *g* is less than A *f*. But, because (7.) the circumferences cannot have more than the two points C, E in common, it is evident that if the arcs C G E, C F E do not coincide, the one must be wholly without or wholly within the other: and the same may be said of the arcs C *g* E, C *f* E. Consequently, the arc C G E of the second circle is without the first, and the arc C *g* E of the same circle within the first, that is, the circumferences cut one another in the points C and E.

Therefore, &c.

Cor. 1. (Euc. iii. 11, 12, and 13.) Circles that touch one another meet in one point only, which point is in the straight line that joins their centres, or in that straight line produced.

For, should they meet in two points, they would have a common chord, which common chord would (3. *Cor.* 3.) be bisected by the straight line joining their centres; and, therefore, the points of meeting being upon either side of this straight line, the circles, as in the proposition, would not only meet, but cut one another.

Cor. 2. Hence, if two circles touch one another, the distance between their centres must be equal to the sum or to the difference of their radii; the sum if they touch externally, the difference if they touch internally.

Scholium.

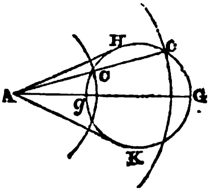
We may remark that from the second part of the demonstration of this proposition, it likewise follows that—

1. If a point A be taken in the diameter of a circle C G E, which is not the centre (see the lower figure), of all the straight lines which can be drawn from that point to the circumference, the greatest is that which passes through the centre, viz. A G, and the other part A *g* of that diameter is the least: also

of any others that which is nearer to the greatest is greater than the more remote. (Euc. iii. 7. part of.)

2. If a point A be taken without a circle C G E (see the upper figure), and straight lines be drawn from it to the circumference, whereof one A G passes through the centre; of those which fall upon the concave circumference, the greatest is that which passes through the centre, viz. A G; and of the rest, that which is nearer to the one passing through the centre is always greater than one more remote: but of those which fall upon the convex circumference, the least is that between the point without the circle and the diameter; and of the rest, that which is nearer to the least is always less than the more remote. (Euc. iii. 8. part of.)

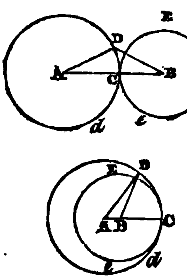
The parts of the circumference which are here termed concave and convex towards the point A, are determined by the points H and K, in which tangents drawn from A meet the circumference,—the part H G K being concave, and H g K convex.



Prop. 9.

If the circumferences of two circles meet one another in a point which is in the straight line joining their centres, or in that straight line produced, they shall meet in no other point; the circumference of that which has the greater radius shall fall wholly without the circumference of the other; and the two circles shall touch one another.

Let A, B be the centres of two circles, the circumferences of which meet one another in the point C, which is in the line A B, or in A B produced: and let the radius of the first circle be greater than the radius of the other: the circumference of the first shall fall wholly without the circumference of the other, and the circles shall touch one another in C.



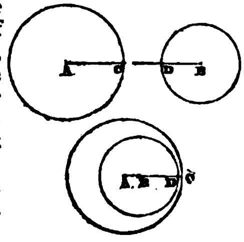
Let D be any point in the circumference of the circle which has the centre A; and join A D, D B. Then, because A D B is a triangle, the side D B (I. 10.) is greater than the difference of A B, A D, that is, greater than B C: but B C is the radius of the circle which has the centre B: therefore, the point D lies without the latter circle. And the same may be demonstrated of every point in the circumference of the greater circle. Also, because the arcs E C, e C of the one circle, lie upon the same side of the arcs D C, d C of the other, the circles meet, but do not cut one another in the point C; that is, they touch one another. Therefore, &c.

Cor. 1. Circles that cut one another meet in two points, one upon either side of the straight line which joins their centres. For circles meeting in a point which is in that straight line do not cut, but touch one another, as is shown in the proposition: and such as meet in a point which is not in that straight line, meet also (8.) in a second point upon the other side of it.

Cor. 2. Hence, if two circles cut one another, the straight line which joins their centres must be less than the sum, and greater than the difference of their radii. (I. 10.)

Prop. 10.

If the circumferences of two circles do not meet one another in any point, the distance between their centres shall be greater than the sum, or less than the difference of their radii, according as each of the circles is without the other or one of them within the other.



Let A, B be the centres of the two circles, and let the line A B, or that line produced, cut the circumferences in the points C, D. Then, it is evident that A B is equal to the sum, or to the difference of A C, B C, according as each of the circles is without the other, or one of them within the other. If it be equal to the sum, then, because B C is greater than B D, the sum of A C, B C is greater than the sum of A C, B D; that is, the distance of the centres is greater than the sum c the radii: and if it be equal to the difference, then, for the same reason, the difference of A C, B C is less than the difference of A C, B D, that is, the distance of the centres is less than the difference of the radii

Therefore, &c.

Cor. 1. Hence it appears, conversely, that two circles will, 1°, cut one another; or 2°, touch one another; or 3°, one of them fall wholly without the other; according as the distance between their centres is, 1°, less than the sum, and greater than the difference of their radii; or 2°, equal to the sum, or to the difference of their radii; or 3°, greater than the sum or less than the difference of their radii.

Cor. 2. Therefore, 1°, if two circles cut one another, the distance of their centres must be at the same time less than the sum and greater than the difference of their radii; and conversely, if this be the case, the circles will cut one another.

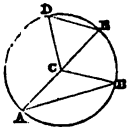
2°. If two circles touch one another, the distance of their centres must be equal to the sum or to the difference of their radii, according as the contact is external or internal; and conversely, if either of these be the case, the circles will touch one another.

3°. If two circles do not meet one another, the distance of their centres must be greater than the sum or less than the difference of their radii, according as each is without the other, or one of them within the other; and conversely, if either of these be the case, the circles will not meet one another.

SECTION 2.—*Of Angles in a Circle.*

PROP. 11.

In the same, or in equal circles, the greater chord subtends the greater angle at the centre: and conversely, the greater angle at the centre is subtended by the greater chord.



Let C be the centre of a circle A B D, and let A B, D E be two chords in the same circle, of which A B subtends a greater angle at C than D E does: A B shall be greater than D E.

For, the radii A C, C B being equal to the radii D C, C E respectively, C A B and C D E are triangles having two sides of the one equal to two sides of the other, each to each, but the angle A C B greater than D C E: therefore, the base A B (I. 11.) is likewise greater than the base D E.

And, conversely, if A B be greater than D E, it shall subtend a greater angle at C: for C A B and C D E are, in this case, two triangles having two sides of the one equal to two sides of the other, each to each, but the base A B greater than the base D E: therefore the angle A C B (I. 11.) is likewise greater than the angle D C E.

The same demonstration may be applied to the case of equal circles.

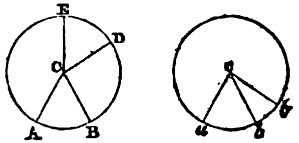
Therefore, &c.

Cor. In the same or in equal circles, equal chords subtend equal angles at the centre; and conversely.

PROP. 12. (EUC. iii. 26 and 27, first parts of.)

In the same or in equal circles, equal angles at the centre stand upon equal arcs; and conversely.

Let C, *c* be the centres of two equal circles, and let A C B, *a c b* be equal



angles at the centres; the arc AB shall be equal to the arc *a b*.

For if the circles be applied one to the other, so that the centre C may be upon *c*, and the radius CA upon *c a*, the radius CB will coincide with *c b*, because the angle ACB is equal to *a c b*. Also the points A B will coincide with the points *a*, *b* respectively, because the radii C A, C B are equal to the radii *c a*, *c b*. Therefore the arc A B coincides with the arc *a b*, and is equal to it.

And conversely, if the arcs A B, *a b* be equal to one another, the angles A C B, *a c b* shall be likewise equal. For, if not, let any other angle *a c b'* be taken equal to A C B; then, by the former part of the proposition, the arc *a b'* is equal to A B, that is, to *a b*, which is absurd; therefore, the angle *a c b* cannot but be equal to A C B.

In the next place, let A C B, D C E be equal angles in the same circle: then, if *c* be the centre of a second circle equal to it, and if the angle *a c b* be made equal to A C B or DCE, the arc *a b* will be equal to AB or D E; therefore, the arcs A B, D E are equal to one another. And, in like manner, the converse.

Therefore, &c.

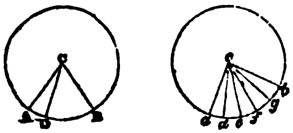
Cor. 1. (Euc. iii. 28 and 29.) In the same or in equal circles, equal arcs are subtended by equal chords; and conversely (11. *Cor.*).

Cor. 2. By a similar demonstration it may be shown that in the same or in equal circles, equal sectors stand upon equal arcs; and conversely.

Prop. 13. (Euc. vi. 33, part of.)

In the same or in equal circles, any angles at the centre are as the arcs upon which they stand; so also are the sectors.

Let C, *c* be the centres of two equal circles; and let A C B, *a c b*, be any angles at the centre: the angle ACB shall be to the angle *a c b* as the arc A B to the arc *a b*.



Let the angle *a c b* be divided into any number of equal angles by the radii *c d*, *c e*, *c f*, *c g*, and therefore the arc *a b* into as many equal parts (12.) by the points *d*, *e*, *f*, *g*. Then, if the arc AD be taken equal to *a d*, and if C D be joined, the angle A C D will be (12.) equal to *a c d;* and if the arc A D be contained in A B a certain number of times with a remainder less than A D, the angle A C D will be found in the angle A C B the same number of times with a remainder less than A C D: and this, whatsoever be the number of parts into which the arc *a b* is divided. Therefore, (II. def. 7.) the angle A C B is to the angle *a c b* as the arc A B to the arc *a b*.

And in the same manner it may be shown that the sector A C B is to the sector *a c b* as the arc AB to the arc *a b* (12. *Cor.* 2.)

The case of arcs or sectors occurring in the same circle has a similar demonstration.

Therefore, &c.

Scholium.

Hence the angle at the centre of a circle is said to be *measured* by the arc upon which it stands: and generally, any angle in a circle is said to be measured by that part of the circumference which measures an equal angle at the centre.

In Book I. def. 9. an angle was stated to have its origin in the meeting of two straight lines in a point, and to be greater or less according to the extent of the opening between those lines: a *right angle* was then defined; an angle "less" than which was said to be *an acute angle*, and an angle "greater" *an obtuse angle.* We ought rather however to have defined the obtuse angle to be greater than one, and less than two right angles: for if the opening between the legs of such an angle be increased to a still further degree, it becomes equal to two right angles—greater than two—equal to three—greater than three—and, by a still increasing separation of one leg from the other in the same direction, equal to four right angles—greater than four—and so on.

An angle which is greater than two and less than four right angles is frequently called a *reverse* or *re-entering* angle.*

These angles (right, acute, obtuse, and re-entering) are all that have place in elementary Geometry, or in the subjects to which it is commonly applied; the angles spoken of being *understood* never to exceed four right angles. But where no such limitation, confining the magnitude of the angle, is supposed, it is plain from the considerations abovementioned, that the magnitude of any angle, in general, cannot be estimated from the apparent opening between the legs. Besides this opening, there is to be considered the direction in which it is supposed to have been generated, and yet further, the number of times the revolving leg may have coincided with and passed by the other; for the same apparent opening is the result of different angular revolutions: just as the hand of a watch is at the same apparent distance from any given position, whether it has made fifty and a quarter, or a hundred and a quarter, or a hundred more circuits. The traversed space being made up of parts which coincide, and which do not therefore distinctly appear, the number of these parts must be specified if we would form an estimate of the whole.

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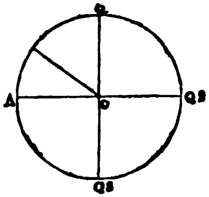
* It has been already observed (I. 2. note) that an angle is sometimes said to be *supplementary*, viz. when it is considered as the supplement of the adjacent angle to two right angles: in like manner, an angle takes the name of an *explementary* angle, when together with the adjoining and opposite angle it fills up the whole space about the angular point.

This complete definition of angular magnitude is of the greatest importance in the higher parts of the mathematics, and may be well illustrated by help of the measuring circumference.



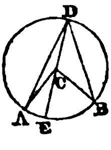
With the centre C, and radius C A, let there be described a circle A Q Q_2 Q_3, and let the diameters A Q_2, Q Q_3 be drawn at right angles to one another, dividing the whole angular space about the centre into 4 equal angles, each of which will be measured by a *quadrant*,* or fourth part of the circumference.

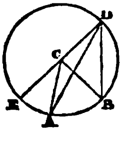
Let us now suppose that the radius of the circle, being made to revolve about its centre from the original position C A, is brought successively into the positions C Q, C Q_2, C Q_3, and thence again, continuing its revolution, a second time into the same positions C A, C Q, C Q_2, C Q_3, and so on. Then it is evident that the angular space through which the radius will have revolved, will be, in these successive positions, one, two, three right angles; upon returning to A four right angles, which is the whole angular space about the point C: and thence again, coming a second time to the same positions C Q, C Q_2, C Q_3, five, six, seven right angles, and so on: which angular spaces will be measured respectively by one, two, three quadrants, a whole circumference; five, six, seven quadrants, and so on: and any angular spaces intermediate to these will be measured by corresponding arcs intermediate, that is, of magnitudes between one and two, two and three, three and four, &c. quadrants.

PROP. 14. (EUC. iii. 20.)

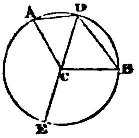
The angle at the centre of a circle is double of the angle at the circumference upon the same base, that is, upon the same part of the circumference.

Let A C B be any angle at the centre C of the circle A B D, and let A D B be an angle at the circumference upon the same base A B: the angle A C B shall be double of the angle A D B.





Join D C, and produce it to E. Then, because C A is equal to C D (I. 6.), the angle C A D is equal to C D A: therefore the angle A C E, which is equal to C A D, C D A together (I. 19.) is double of C D A. In like manner it may be shown that the angle B C E is double of C D B. Therefore the sum or difference of the angles E C A, E C B is also double of the sum or difference of the angles C D A, C D B, that is, the angle A C B is double of the angle A D B.



The adjoined figure shows that this proof is equally applicable when the angle ACB is re-entering.

Therefore, &c.

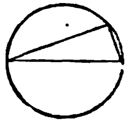
Cor. 1. Any angle at the circumference is measured by half the arc upon which it stands.

Cor. 2. (Euc. iii. 26 and 27, second parts of, and Euc. vi. 33 part of). In the same or in equal circles, equal angles at the circumference stand upon equal arcs, and conversely; and, generally, any angles at the circumference are as the arcs upon which they stand. (13. and II. 17.)

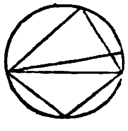
PROP. 15. (EUC. iii. 21.)

Angles in the same segment of a circle are equal to one another.

For they are halves of the same angle, viz. the angle at the centre which stands upon their common base; or, which is the same thing, they are measured by the same arc, viz. the half of their common base.



Cor. 1. (Euc. iii. 31., first part of.) The angle which is in a semicircle is a right angle, for it is measured by half the semi-circumference, that is, by a quadrant.



Cor. 2. (Euc. iii. 31 second part of.) The angle, which is in a segment greater than a semicircle, is less than a right angle; and the angle, which is in a segment less than a semicircle, is greater than a right angle; for the one is measured by an arc which is greater, and the other by an arc which is less than a quadrant.

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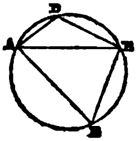
* From the Latin word *quadrans*, a fourth part.

Cor. 3. If upon the base of a triangle there be described a segment of a circle, the vertex of the triangle shall fall without, or within, or upon the arc of the segment, according as the vertical angle of the triangle is less than, or greater than, or equal to the angle in the segment.

For it may easily be shown, (I. 8 *Cor.* 1.) that if the vertex fall within the arc of the segment, the vertical angle must be greater than the angle of the segment, and if without it, less.

PROP. 16.

If any chord be drawn in a circle, the angles contained in the two opposite segments shall be together equal to two right angles.



Let A D B be a circle, and let it be divided by the chord AB into the segments A D B, A E B: the angles ADB, AEB contained in these segments shall be together equal to two right angles

For the angle A D B is measured by half the arc A E B upon which it stands (14. *Cor.* 1.), and in like manner the angle A E B is measured by half the arc A D B. Therefore the angles A D B, AEB together are measured by half the circumference, that is, by two quadrants, and are consequently equal to two right angles.

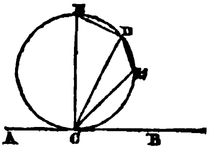
Therefore, &c.

Cor. 1. (Euc. iii. 22.) If a quadrilateral figure be inscribed in a circle, either pair of its opposite angles shall be equal to two right angles.

Cor. 2. And conversely, if the opposite angles of a quadrilateral be together equal to two right angles, a circle may be described about it. For, if the circle described through the three points A, D, B (5. *Cor.*) were to cut the side B E in any other point than E, suppose F, the angles A F B, A D B being equal to two right angles, would be equal to the angles A E B, A D B, and therefore the angle A F B to the angle A E B; whereas one of them, being exterior, must (I. 8. *Cor.* 1.) be greater than the other.

PROP. 17. (EUC. iii. 32.)

If a straight line touch a circle, and if from the point of contact a straight line be drawn cutting the circle, the angles which it makes with the tangent shall be equal to the angles which are contained in the alternate segments of the circle.



Let the straight line A B touch the circle C D E in the point C, and from C let there be drawn the straight line C D cutting the circle: the angle D C A shall be equal to the angle in the segment D F C, and the angle D C B to the angle in the segment D E C.

From C draw C E at right angles to the tangent A B, and therefore (2. *Cor.* 2.) passing through the centre of the circle; let C E meet the circumference in E: take any point F in the arc of the opposite segment, and join C E, E D, D F, F C. Then, because C D E is a semicircle, the angle C D E is a right angle (15. *Cor.* 1.): therefore the remaining angles of the triangle C D E (I. 19.), that is, the angles DEC and D C E, are together equal to a right angle. But the angles D C B and D C E are likewise together equal to a right angle: therefore the two latter angles are equal to the two former, and the angle D C B is equal to D E C, that is, to the angle in the alternate segment.

And because (16.) the angles in the two segments are together equal to two right angles, that is (I. 2.), to the angles D C B, D C A, the angle D C A is equal to the angle in the other segment D F C.

Therefore, &c.

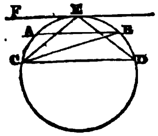
Cor. The converse is also true: that is, if from the extremity of a chord there be drawn a straight line, such that the angles which it makes with the chord are equal to the angles in the alternate segments of the circle, that straight line must touch the circle.

Scholium.

The theorem which has been just demonstrated, states no more than is contained in Prop. 15., if the tangent be considered as a chord in which the points of section are coincident. For, if the point F be supposed to move up to the point C, the chord C F will tend more and more to coincide in position with the tangent C B. But if E F be joined, then, by Prop. 15, the angle D C F is always equal to the angle D E F. Therefore, when F coincides with C, that is, when the chord C F becomes a tangent at C, the angle D C B is equal to the angle D E C. As this, however, was a case not contemplated in the demonstration given of that proposition, the inference could not have been directly drawn from it. The proposition (2.) that the tangent is at right angles to the radius is an instance of the same kind. Others may be seen in the corollaries of the two following propositions, and in certain properties of tangents which will be found in the next section.

Prop. 18.

Parallel chords intercept equal arcs; and conversely.



Let A B C be a circle, and let the chords A B, C D be parallel to one another: the arc A C shall be equal to the arc B D.

Join B C. Then, because A B is parallel to C D, the angle A B C (I. 15.) is equal to the angle B C D: therefore (14. *Cor.* 2.) the arc A C is equal to the arc B D.

And conversely, if the arc A C be equal to the arc B D, the angle A B C will (14. *Cor.* 2.) be equal to the angle B C D, and therefore (I. 15.) A B will be parallel to C D.

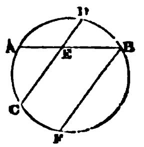
Therefore, &c.

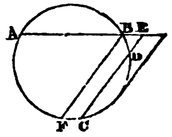
Cor. If one of the chords, as A B, be supposed to move parallel to itself until the points A and B in which it cuts the circle coincide, as at E, the same and its converse will be true: that is, if a chord and tangent be parallel, they shall intercept equal arcs; and conversely.

For, because E F is parallel to C D, the angle F E C is equal (I. 15.) to the angle E C D, which stands upon the arc E D: but, because E F is a tangent, (17.) the same F E C is equal to E D C which stands upon the arc E C. Therefore the arc E C is equal to E D, (14. *Cor.* 2.) And the proof of the converse is similarly varied.

Prop. 19.

If two chords of a circle meet one another, the angle contained by them shall be measured by half the sum, or by half the difference of the intercepted arcs, according as the point in which they meet is within or without the circle.





Let ABC be a circle, and let the chords A B, C D meet one another in the point E: the angle A E C shall be measured by half the sum or by half the difference of the arcs, A C, B D, according as the point E is within or without the circle.

Through B draw B F parallel to D C, and let it meet the circumference in F: then (18.) the arc F C is equal to B D, and therefore the arc A F is equal to the sum or to the difference of A C, B D, according as the point E is within or without the circle. But, because B F is parallel to D C, the angle A E C is equal to A B F (I. 15.); and A B F is measured by half the arc A F, (14. *Cor.* 1.): therefore the angle A E C is measured by half the sum or by half the difference of the arcs A C, B D, according as the point E is within or without the circle.

Therefore, &c

When the point E is in the circumference, the result of this proposition coincides with that of 14. *Cor.* 1.

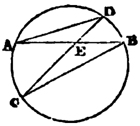
Cor. By a similar demonstration (18. *Cor.*) if a chord meet a tangent in a point which is not the point of contact, the angle contained by them will be measured by half the difference of the intercepted arcs.

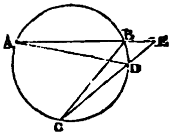
The case of a chord meeting a tangent in the point of contact, has been already contemplated in Prop. 17. It may be considered, however, as included under the above rule, the measuring arc in this case being the same by this corollary as by Prop. 17.

Section 3.—*Rectangles under the Segments of Chords.*

Prop. 20. (Euc. iii. 35.)

If two chords of a circle cut one another, the rectangles under their segments terminating in the points of section shall be equal, whether they cut one another within or without the circle.





Let A B C be a circle, and let the chords A B, C D cut, or be produced to cut, one another in the point E: the rectangle under A E, E B shall be equal to the rectangle under C E, E D.

Join A D, B C. Then, because the angle EAD is equal to the angle E C B in the same segment (15.), and that the angles at E, which are vertical (I. 3.) as in the upper figure, or coincide as in the lower, are equal to one another, the triangles A E D, C E B are equiangular. Therefore (II. 31.) A E : E D :: E C : E B, and (II. 38.) the rectangle under A E, E B is equal to the rectangle under C E, E D.

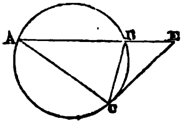
Therefore, &c.

We may remark that an easy demonstration of this proposition is likewise afforded by I. 39.; for the rectangles in question are each of them equal to the difference of the squares of the radius, and of the distance of the point E from the centre of the circle.

Cor. And hence, conversely, if two straight lines A B, C D cut one another in a point E, and if the points A, B and C, D be so taken, that the rectangle under A E, E B be equal to the rectangle under C E, E D; the points A, B, C, D shall lie in the circumference of the same circle.

Prop. 21. (Euc. iii. 36.)

If any chord of a circle be produced to cut a tangent to the same circle, the square of the tangent shall be equal to the rectangle under the segments of the chord.



Let A B C be a circle, and let the chord A B be produced to meet the tangent C E in E: the square of C E shall be equal to the rectangle under A E, E B.

Join C A, C B. Then, because the angle E C B is equal to the angle E A C in the alternate segment of the circle, (17.) and that the angle at E is common to the two triangles E C B, E A C, these two triangles are equiangular. Therefore (II. 31.) A E : E C :: E C : E B, and (II. 38. *Cor.* 1.) the square of E C is equal to the rectangle under A E, E B.

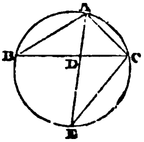
Therefore, &c.

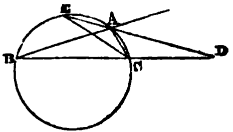
The same remark may be made here as at the end of the preceding proposition: viz., that an easy demonstration is likewise afforded by I. 39. and I. 36. *Cor.* 1.

Cor. And hence, conversely, if two straight lines A B, C E cut one another in a point E, and if the points A, B and C, be so taken, that the square of E C be equal to the rectangle under A E, E B, the straight line E C shall touch the circle which passes through the points A, B, C

Prop. 22. (Euc. vi. B.)

*If the vertical or exterior-vertical angle of a triangle be bisected by a straight line, which cuts the base, or the base produced, the square of that straight line shall be equal to the difference of the rectangles under the two sides, and under the segments of the base, or of the base produced.**





Let A B C be a triangle, and let the vertical or exterior-vertical angle be bisected by the straight line A D, which meets the base or the base produced in D: the square of A D shall be equal to the difference of the rectangles B A, A C, and B D, D C.

Let A E C be the circle which (5. *Cor.*) passes through the points A, B, C, and let A D be produced to meet the circumference in E, and join E C.

Then, because the angles B A D, E A C are halves, or supplementary to the halves of the bisected angle, they are equal to one another: also the angle A B D is equal to the angle A E C in the same segment (15.): therefore, the triangles B A D, E A C being equiangular, (II. 31.) B A : A D :: E A : A C, and (II. 38.) the rectangle under B A, A C is equal to the rectangle under E A, A D. Again, because the chords B C, E A cut one another in D (20.), the rectangle under B D, D C is equal to the rectangle under E D, D A: therefore, the difference of the rectangles under B A, A C and B D, D C is equal to the difference of the rectangles under E A, A D, and E D, DA, that is, to the square of A D (I. 31.).

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* This, as is evident from the enunciation, is a property not of the circle, but of a triangle, and belongs as such to I. § 6. The required demonstration has, however, in this and one or two other instances rendered an infringement of the classification unavoidable.

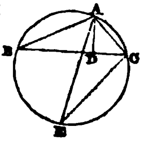
Therefore, &c.

It should be observed in the case of exterior bisection (see the lower figure), that the bisecting line AD must, if produced, cut the circumference in a second point E, in all cases in which it cuts the base B C produced in a point D; that is, in all cases in which the sides A B, AC are unequal. For when AB is equal to A C, the angles A B C, A C B are likewise equal (I. 6.), and therefore (I. 19. and I. ax. 5.) equal to the halves of the exterior angle: therefore, the angle C A D being equal to A C B, A D is parallel to BC (I. 15.), and the same C A D being equal to the angle A B C in the alternate segment, A D touches the circle in A (17. *Cor.*). But, when one of the sides, as A B, is greater than the other, the angle ACB is also greater than ABC (I. 9.); therefore the angle CAD, which (I. 19.) is equal to half the sum of the two ABC, ACB, is less than A C B, and greater than A B C; and because the angle C A D is not equal to ACB, AD is not parallel to BC (I. 15.); and because the same CAD is not equal to ABC, that is to the angle in the alternate segment, A D does not touch the circle in A, but cuts it and meets the circumference in a second point E, as was observed.

PROP. 23. (EUC. vi. C.)

If a triangle be inscribed in a circle, and if a perpendicular be drawn from the vertex to the base; the rectangle under the two sides shall be equal to the rectangle under the perpendicular and the diameter of the circle.

Let A B C be a triangle inscribed in the circle A B C; from A draw A D perpendicular to BC, and AE through the centre of the circle to meet the circumference in E: the rectangle under BA, AC shall be equal to the rectangle under E A, A D.



Join C E. Then, because A C E is a semicircle, (15. *Cor.* 1.) the angle A C E is a right angle: but A D B is likewise a right angle, and the angle A E C is equal to the angle A B D in the same segment (15.); therefore, the triangles E A C, B A D being equiangular, BA : A D :: E A : A C (II. 31.), and (II. 38.) the rectangle under B A, A C is equal to the rectangle under E A, A D.

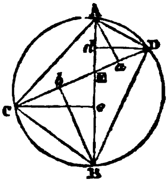
Therefore, &c.

Cor. If two triangles be inscribed in the same, or in equal circles, the rectangle under the two sides of the one, shall be to the rectangle under the two sides of the other, as the perpendicular, which is drawn from the vertex to the base of the one, to the perpendicular which is drawn from the vertex to the base of the other (II. 35.).

PROP. 24.

If a quadrilateral be inscribed in a circle, its diagonals shall be to one another as the sums of the rectangles under the sides adjacent to their extremities.

Let A C B D be a quadrilateral figure, inscribed in the circle A B C and A B, C D its diagonals: A B shall be to C D, as the sum of C A × A D and C B × B D to the sum of A C × C B, and A D × D B.



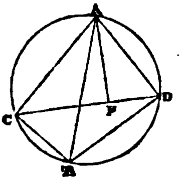
Let A B, and C D cut one another in the point E: and, first, let A B cut CD at right angles. Then, because A C D, B C D, C A B, and D A B are triangles inscribed in the same circle, the perpendiculars A E, B E, C E and D E, are to one another as the rectangles A C × A D, B C × B D, C A × C B, and D A × D B: therefore, (II. 25. *Cor.* 3.) the sum of A E and B E, that is A B, is to the sum of C E and D E, that is C D, as the sum of A C × A D, and B C × B D to the sum of C A × C B and D A × D B. In the next place, let A B cut C D, but not at right angles: and let the perpendiculars $A a$, $B b$, $C c$, and $D d$ be drawn. Then, as before, it may be shown that $A a+B b$ is to $C c+D d$, as A C × A D + B C × B D to C A × C B + D A × D B. But, because the triangles $A E a$, $B E b$, $C E c$, $D E d$ are equiangular, $A a$, $B b$, $C c$, and $D d$ are to one another as A E, B E, C E, and D E (II. 31.). Therefore, $A a + B b$ is to $C c + D d$, as A E + B E to C E + D E, that is, as AB to C D. Therefore, (II. 12.) A B : C D :: A C × A D + B C × B D : C A × C B + D A × D B.

Therefore, &c.

PROP. 25. (EUC. vi. D.

If a quadrilateral be inscribed in a circle, the rectangle under its diagonals *shall be equal to the sum of the rectangles under its opposite sides.*

Let A C B D be a quadrilateral inscribed in the circle A B C; and let A B, CD be its diagonals: the rectangle under A B, C D shall be equal to the sum of the rectangles under AD, B C, and A C, BD.



At the point A make the angle D A F equal to B A C, and let A F meet C D in F.

Then, because the angle A B C is equal to A D F in the same segment (15.), and that B A C was made equal to D A F, the triangles A B C, A D F are equiangular: therefore, (II. 31.) AB : BC :: A D : D F, and (II. 38.) the rectangle under A B, D F is equal to the rectangle under A D, B C.

Again, because the angles B A C, D A F are equal to one another, let the angle B A F be added to each; therefore the whole angle FAC is equal to the whole angle D A B; and the angle F C A is equal to the angle D B A in the same segment (15.); therefore, the triangles A F C, A D B are equiangular. Therefore (II. 31.) A B : B D :: A C : C F, and (II. 38.) the rectangle under A B, C F is equal to the rectangle A C, B D. Therefore, the sum of the rectangles under A B, D F and A B, C F, that is, (I. 30. *Cor.*) the rectangle under A B, C D, is equal to the sum of the rectangles under A D, B C, and A C, B D.

Therefore, &c.

Cor. Hence, a quadrilateral may be constructed, which shall have its sides equal to four given straight lines, in a given order, each to each, and its angular points lying in the circumference of a circle. For, by the 24th proposition, the ratio of the diagonals, and by that which has been just demonstrated, their rectangle is given: therefore, (II. 63.) the diagonals may be found, and (I. 50.) the quadrilateral constructed.

It is only essential to the possibility of the construction that of the four given straight lines, every three be greater than the fourth (I. 10. *Cor.* 2.). It is remarkable that, although the diagonals will be different in different orders of the given sides, the circumscribing circle has the same magnitude whatsoever be their order. (See Sect. 5. Prop. 41. *Scholium.*)

SECTION 4.—*Regular polygons, and approximation to the area of the circle.*

Def. 11. A *regular polygon* is that which has all its sides equal, and likewise all its angles equal.

A figure of five sides is called a *pentagon;* a figure of six sides a *hexagon;* of ten sides a *decagon*; and of fifteen sides a *pente-decagon.* There is so seldom any occasion, however, to specify the number of sides of an irregular figure, as distinct from a multilateral figure in general, that it has become common to appropriate these names with others of similar derivation (as by way of preeminence) to the regular figures—"a hexagon," for instance, is understood to mean a regular figure of six sides, and so of the rest.

It is evident, that regular polygons, which have the same number of sides, are similar figures; for their angles are equal, each to each, because they are contained the same number of times in the same number of right angles (I. 20.); and their sides about the equal angles are to another in the same ratio, viz. the ratio of equality.

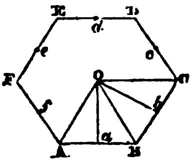
12. The *centre* of a regular polygon is the same with the common centre of the inscribed and circumscribed circles (see *Prop.* 26.): and the perpendicular which is drawn from the centre to any one of the sides is called the *apothem.*

13. *Similar arcs* of circles are those which subtend equal angles at the centre. *Similar sectors* and *segments* are those which are bounded by similar arcs.

PROP. 26. (EUC. iv. 13 and 14.).

If any two adjoining angles of a regular polygon be bisected, the intersection of the bisecting lines shall be the common centre of two circles, the one circumscribing, the other inscribed in, the polygon.

Let A B C D E F be any regular polygon, and let the angles at A and B be bisected by the straight lines AO, B O; which meet in some point O, (I. 15. *Cor.* 4.) because each of the angles F A B, C B A is less than two right angles, and therefore each of their halves OAB, OBA less than a right angle, and the two together less than two right angles The point O shall be the centre



When A C D is a quadrilateral; [illegible] one equal [illegible]
[illegible] A [illegible] B, B C [illegible] C, B D

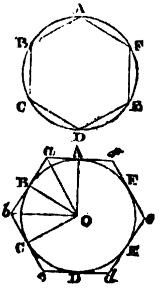
of two circles, one passing through all the points A, B, C, D, E, F, and the other in contact with all the sides A B, B C, C D, D E, E F.

Join O C, O D, O E, O F, and draw the perpendiculars O *a*, O *b*, O *c*, O *d*, O *e*, O *f*. Then, because the triangles O B C, O B A have two sides of the one equal to two sides of the other, each to each, and the included angles O B C, O B A equal to one another, (I. 4.) the base O C is equal to the base O A, and the angle O C B to the angle O A B. But OAB is the half of FAB, and FAB is equal to D C B: therefore O C B is the half of DCB, and the latter angle is bisected by the line O C. By a similar demonstration, therefore, it may be shown that OD is equal to O B, O E to OC, and OF to OD. And, because the angles O A B, O B A, being halves of equal angles, are equal to one another, OB is equal to OA (I. 6.). Therefore the straight lines drawn from O to the angular points of the figure are equal to one another, and O is the centre of a circle passing through those points. And because A B, B C, &c. are equal chords of the same circle, they are at equal distances from the centre O (4. *Cor.*): that is, the perpendiculars O *a*, O *b*, &c. are equal to one another, and O is likewise the centre of a circle described with the apothem O *a* or O *b* for its radius, and (2.) touching the sides in their middle points (3.), *a, b, c, d, e, f*.

Therefore, &c.

Prop. 27.

If the circumference of a circle be divided into any number of equal parts, the chords joining the points of division shall include a regular polygon inscribed in the circle; and the tangents drawn through those points shall include a regular polygon of the same number of sides circumscribed about the circle.



Let the circumference of the circle A C F be divided into any number of equal parts in the points A, B, C, D, E, F. The figure which is included by the straight lines joining those points shall be a regular polygon.

For its sides being the chords of equal arcs of the same circle, are (12. *Cor.* 1.) equal to one another; and its angles standing upon equal arcs, viz. the differences between the whole circumference, and two of the former, are likewise equal (14. *Cor.* 2.).

Next, let *a b c d e f* be the figure which is included by tangents drawn through the points A B C D E F: this shall likewise be a regular polygon.

Let O be the centre of the circle, and join OA, OB, O *a*, O *b*, OC. Then because *a* A, *a* B, are tangents drawn from the same point, they are equal to one another (2. *Cor.* 3.). And because the triangles A O *a*, B O *a* have the three sides of the one equal to the three sides of the other, each to each, the angle B O *a* is equal to A O *a*, that is, to the half of A O B. In like manner, it may be shown that the angle B O *b* is equal to the half of B O C; and A O B is equal to B O C, because the arc A B is equal to the arc B C (12.); therefore the angle B O *a* is equal to B O *b*. Therefore B O *a*, B O *b* are triangles which have two angles of the one equal to two angles of the other, each to each, and the interjacent side O B common to both: consequently, (I. 5.) they are equal in every respect, and B *a* is equal to B *b*; therefore *a b* is bisected in B. In the same manner it may be shown that *a f* is bisected in A; and it has been shown that *a* B, *a* A, are equal to one another; therefore *a b* is equal to *a f*. And by a like demonstration it may be shown that the other sides of the figure are each of them equal to *a b* or *a f*. Therefore, the figure *a b c d e f* has all its sides equal to one another. And because its angles, as *a*, *b*, are supplements (I. 20. *Cor.*) of equal angles, as AOB, B O C, they are likewise all equal to one another. Therefore it is a regular polygon.

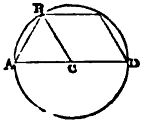
Therefore, &c.

Cor. 1. (Euc. iv. 12.). If any regular polygon be inscribed in a circle, a similar polygon may be circumscribed about the circle by drawing tangents through the angular points of the former; and conversely.

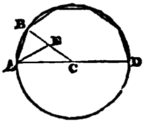
Cor. 2. If, any regular polygon being inscribed in a circle, a tangent be drawn parallel to one of its sides, and be terminated both ways by radii passing through the extremities of that side, such terminated tangent shall be a side of a regular polygon of the same number of sides circumscribed about the circle. For, since the radius drawn to the point of contact bisects the side (3. *Cor.* 1.) at right angles, and therefore also bisects the angle formed by the radii passing through its extremities (I. 6. *Cor.* 3.) it is obvious from I. 5. that the parts of the tangent in question are equal to one another, and to the halves of any side of the regular circumscribed polygon of the same number of sides.

Prop. 28. (Euc. iv. 10. and 15. *Cor.*)

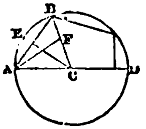
The side of a regular hexagon is equal to the radius of the circle in which it is inscribed; the side of a regular decagon is equal to the greater segment of the radius divided medially; and the side-square of a regular pentagon is greater than the square of the radius by the side-square of a regular decagon inscribed in the same circle.



First, let A B be the side of a hexagon inscribed in the circle ADB, the centre of which is C. Join C A, C B, and let A C produced meet the circumference in D. Then, because the arc A B is contained in the whole circumference six times, it is contained three times in the semi-circumference A B D, and twice in the arc B D. But the angle BAC is measured by half of the arc B D, (14. *Cor.* 1.) and the angle A C B by the arc A B. Therefore the angle B A C is equal to A C B, and the side A B is equal to BC; that is, the side of the regular hexagon is equal to the radius of the circle.



Secondly, let A B be the side of a regular decagon inscribed in the circle A D B the centre of which is C. Join CA, CB, and let AC produced meet the circumference in D. Then, because the arc A B is contained ten times in the whole circumference, it is contained five times in the semi-circumference A B D, and four times in the arc B D. But the angle B A C is measured by half of the arc B D (14. *Cor.* 1.) and the angle A C B is measured by the arc A B. Therefore the angle B A C is equal to twice the angle A C B. Let the angle B A C be bisected by the line A E, and let A E meet C B in E. Then, because the angle EAC is equal to ACE, the side EA is equal to E C; and because A E B is equal to the two E A C, A C E, that is double of A C E, and that A B E being equal (I. 6.) to B A C is likewise double of A C E, A E B is equal to A B E, and AB is equal to AE or E C (I. 6.). But, because the straight line AE bisects the angle B A C, CA : A B :: C E : E B (II. 50.). Therefore CB : CE :: CE : E B, that is the radius C B is medially divided in E; and A B, the side of the decagon, is equal to the greater segment C E.



Lastly, let A B be the side of a regular pentagon incribed in the circle A D B, the centre of which is C. Bisect the angle A C B by the radius C E (I. Post. 4.): then the arcs AE, E B measuring equal angles, are equal to one another (12.). Join A E; take C F equal to A E, and join A F. Then, because the arc A E is the half of A B, it is contained ten times in the whole circumference, and the chord A E is the side of a regular decagon inscribed in the circle A D B. Again, the angle E A C being measured by half the arc E B D, that is, by two-fifths of the semi-circumference AED, is equal to the angle F C A, which is measured by the arc A B, that is likewise by two-fifths of the semi-circumference A E D; therefore, because the triangles E A C, F C A, have two sides of the one equal to two sides of the other, each to each, and the included angles equal to one another, the base AF (I. 4.) is equal to CE or AC. And, because from the vertex A of the isosceles triangle A C F, the straight line AB is drawn to meet the base produced in B, the square of A B is greater than the square of A C (I. 39.) by the rectangle CB, BF. But CF being equal to AE, the side of a regular decagon, the radius C B is medially divided in F, as shown in the second part of the proposition; and therefore the rectangle CB, BF is equal to the square of CF or AE (II. 38. *Cor.* 1.). Therefore the square of A B is greater than the square of AC by the square of AE; that is, the side-square of a regular pentagon is greater than the square of the radius by the side-square of a regular decagon inscribed in the same circle.

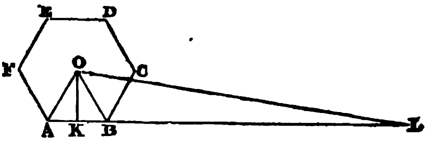
Therefore, &c.

Prop. 29.

The area of any regular polygon is

equal to half the rectangle under its perimeter and apothem.

Let A B C D E F be any regular polygon, O its centre, and O K, which is drawn perpendicular to A B, its apothem: the area of the polygon shall be equal to half the rectangle under the apothem O K and the perimeter A B C D E F.

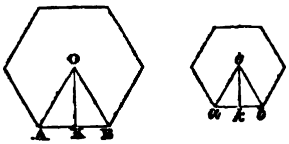


Take A L equal to the sum of the sides or perimeter A B C D E F, and join O L, O B. Then because the sides are equal to one another, the base AL contains the base AB, and therefore the triangle O A L contains the triangle OAB, (I. 27.) as many times as the polygon has sides. But, because the triangles OAB, OBC, &c. having equal bases and altitudes (I. 27.), are equal to one another, the polygon likewise contains the triangle OAB as many times as it has sides. Therefore the polygon is equal to the triangle O A L, (II. ax. 1.) that is, (I. 26. *Cor.*) to half the rectangle under the perimeter AL and apothem OK. Therefore, &c

Prop. 30. (Euc. xii. 1.)

The perimeters of similar regular polygons are as the radii of the inscribed or circumscribed circles, and their areas are as the squares of the radii.

Let O, *o* be the centres of two regular polygons having the same number of sides, A B, *a b*, any two sides, and O K, *o k* lines drawn from the centres

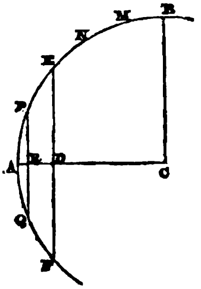


perpendicular to them respectively: then (26.) O A, *o a* are the radii of the circumscribed circles, and O K, *o k* the radii of the inscribed circles. The perimeters of the polygons shall be to one another as OA to *o a*, or OK to *o k* and their areas as OA^2 to oa^2, or OK^2 to ok^2.

Because the polygons have the same number of sides, the angles A O B, *a o b* are contained the same number of times four right angles, and are therefore equal to one another (II. ax. 2.), as also their halves (I. 6. *Cor.* 3.) the angles A O K, *a o k*. Therefore the triangles AO B, *a o b* (II. 32.) as also AOK, *a o k*, (II. 31. *Cor.* 1.) are similar, and AB is to *a b* as OA to *o a*, or as OK to *o k* (II. 31.). But, because the polygons are similar, their perimeters are to one another as AB to *a b*, and their areas as AB^2 to ab^2 (II. 43.). Therefore their perimeters are to one another as OA to *oa*, or as OK to *o k* (II. 12.); and their areas as OA^2 to $o\,a^2$, or as OK^2 to $o\,k^2$. (II. 37. *Cor.* 4.) Therefore, &c.

Prop. 31.

Any circle being given, similar regular polygons may be, the one inscribed, and the other circumscribed, such that the difference of their perimeters or areas shall be less than any given difference.



Let C be the centre and C A the radius of any given circle: and let K be any given straight line; there may be found two regular polygons of the same number of sides, the one inscribed in the circle, the other circumscribed about the circle, which shall have the difference of their perimeters less than the straight line K, or the difference of their areas less than the square of K.

And first of the perimeters. Let L be a straight line equal to the perimeter of some circumscribed polygon; and let the radius AC be divided in the point D in the ratio of K to L (II. 55.). Through D draw the chord E F perpendicular to C A, and draw the radius C B likewise perpendicular to C A: bisect the arc AB in M,* the arc A M in N, the arc A N in P, and so on till the point of bisection fall between A and E; let P be the first point so falling; draw the chord P Q parallel to E F and cutting the radius C A in R. Then, because the double of the arc A P is contained a certain number of times in the quadrant A B, it is contained four times as often in the whole circumference. But because the radius C A cuts the chord P Q at right angles, the arc P A Q is equal to the double of AP (3. and I. 6. *Cor.* 3.). Therefore the arc which it subtends being contained exactly a certain number of times in the whole circumference, the chord P Q is the side of an inscribed polygon (27.). And because the perimeter of this inscribed polygon is to the perimeter of the similar circumscribed polygon as CR to CA (30.), the difference of their perimeters (*invertendo* and *dividendo*) is to the perimeter of the inscribed polygon as A R to R C, that is, in a less ratio than that of A D to DC, or of K to L. But even a magnitude which should have been to the perimeter of the inscribed polygon in the same ratio as that of K to L would have been less than K (II. 18.), because the perimeter of the inscribed polygon is less than L (I. 10. *Cor.* 3.). Much more, then, is the difference of the perimeters of the inscribed and circumscribed polygons less than K, that is, less than the given difference.

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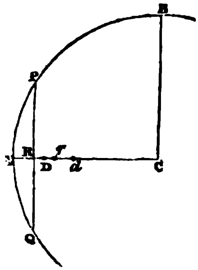
* The necessity of having the lines and letters in the neighbourhood of A clear and distinct has led the engraver to tax the reader's imagination somewhat more than was absolutely necessary in the figures of this proposition. He is requested, therefore, to suppose that the point M bisects the arc AB, the point N the arc A M, and the point P the arc A N. With regard to the operation of bisecting the arc, we should remark that it may be effected by bisecting the angle at the centre (12.).



In the next place, of the areas. Let M be a straight line, such that the square of M is equal to the area of some circumscribed polygon; let AC (II. 55.) be divided in *d* in the ratio of K-square to M-square, and let C D be taken (II. 51.) a mean proportional between C A and C *d*. Then, as before, there may be found an inscribed polygon whose apothem C R is greater than CD. Take C *r* (II. 52.) a third proportional to CA and CR; and, because $CD^2 : CR^2 :: CA \times Cd : CA \times Cr$ (II. 38. *Cor.* 1.), in which proportion the first term is less than the second, the third is also less than the fourth (II. 14.), and therefore C *d* is less than C *r*. And because the area of the inscribed polygon is to the area of the similar polygon circumscribed as CR^2 to CA^2 (30.), that is (II. 35.), as C *r* to C A (*invertendo* and *dividendo*), the difference of their areas is to the area of the inscribed polygon as A *r* to C *r*, or in a less ratio than that of A *d* to C *d*, or of K^2 to M^2. But even a magnitude which should have been to the inscribed polygon as K^2 to M^2 would have been less than K^2, because the inscribed polygon is less than M^2. Much more, then, is the difference between the inscribed and circumscribed polygons less than K square, that is, less than the given difference.

Therefore, &c.

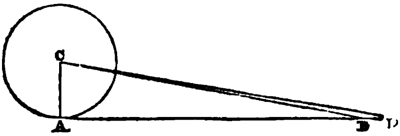
Cor. 1. Any circle being given, a regular polygon may be inscribed (or circumscribed) which shall differ from the circle, in perimeter or in area, by less than any given difference. For the difference between the circle and either of the polygons is less than the difference of the two polygons.

Cor. 2. Any two circles being given, similar regular polygons may be inscribed (or circumscribed), which shall differ from the circles, in perimeter or in area, by less than any the same given difference.

Prop. 32.

The area of a circle is equal to half the rectangle under the radius and circumference.

Let C be the centre, and C A the radius of any circle: from the point A let there be drawn A B perpendicular to C A, and suppose any line AB equal to the circumference of the circle, and join CB: the circle shall be equal to the triangle C A B.



For, if not equal, it must be either greater or less than the triangle. First, let it be supposed greater, and therefore equal to some triangle C A D, the base AD of which is greater than AB. Then, because (31.) there may be circumscribed about the circle a regular polygon, the perimeter of which approaches more nearly to that of the circle (A B) than by any given difference, as B D, a polygon may be circumscribed, the perimeter of which shall be less than A D. But the area of any regular polygon is equal to half the rectangle under its perimeter and apothem (29.). Therefore the area of such circumscribed polygon will be less than the triangle C A D, less that is, than the supposed area of the circle; which is absurd.

Neither can the area of the circle be

less than the triangle CAB; for, as the perimeter of every circumscribed polygon is greater than the circumference AB (I. 10. *Scholium*), and therefore, as before shown, the polygon itself greater than the triangle CAB, to which, however, it may be made to approach within any given difference, because its perimeter may be made (31. *Cor.* 1.) to approach to AB within any given difference; so, because similar polygons may be inscribed and cicumscribed approaching to one another more nearly than by any given difference (31.) a polygon may be inscribed approaching to the triangle C A B within any given difference, that is greater than the circle, if the circle be supposed to be less than the triangle C A B; which is absurd.

Therefore the circle is neither greater nor less than the triangle C A B, that is, it is equal to it.

Therefore, &c.

Cor. Any circular sector is equal to half the rectangle under the radius of the circle, and the arc upon which it stands: for it is less than the circle in the ratio of that arc to the circumference (13.).

Prop. 33. (Euc. xii. 2.)

The circumferences of circles are as the radii, and their areas are as the squares of the radii.

Let R, r, represent the radii of two circles, C, c their circumferences, and A a, their areas: then $C : c :: R : r$, and $A : a :: R^2 : r^2$.

For, in the first place, there may be inscribed (31. *Cor.* 2.) two similar polygons, the perimeters of which approach more nearly to the perimeters C, c of the two circles, than by any the same given difference; and the perimeter of the one polygon (30.) is to the perimeter of the other, always in the same ratio, viz. as R to r: therefore, $C : c :: R : r$ (II. 28.).

And, in the same manner, because there may be inscribed in the circles two similar polygons, the areas of which (31. *Cor.* 2.) approach more nearly to the areas A, a of the circles, than by any the same given difference; and because the area of the one polygon (30.) is always to the area of the other in the same ratio, viz. as R^2 to r^2, $A : a : R^2 : r^2$ (II. 28.).

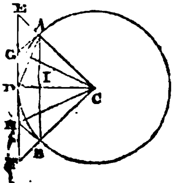
Therefore, &c.

Cor. 1. Hence, similar arcs of circles are as their radii; and similar sectors are as the squares of their radii. For such arcs (or sectors) being to the whole circumferences (or circles) in the same ratio, viz. that of the angle at the centre (13.) to four right angles, are to one another, *alternando*, as the whole circumferences (or circles), that is, by the proposition, as the radii (or the squares of the radii.)

Cor. 2. Similar segments of circles are as the squares of the radii (II. 22.). For they are the differences of similar sectors, and similar triangles, (def. 13. and II. 32.), which sectors (*Cor.* 1.) as also the triangles (II. 42.) are as the squares of the radii.

Prop. 34.

If K *and* L *represent two regular polygons of the same number of sides, the one inscribed in, and the other circumscribed about, the same circle, and if* M *and* N *represent the inscribed and circumscribed polygons of twice the number of sides;* M *shall be a geometrical mean between* K *and* L, *and* N *shall be an harmonical mean between* L *and* M.



Let C be the centre of the circle, A B a side of the inscribed polygon K, C D a radius drawn perpendicular to, and therefore (3.) bisecting A B in the point I. Then, if EF be drawn, touching the circle in D, and terminated by C A and CB produced; EF will (27. *Cor.* 2.) be a side of the circumscribed polygon L of the same number of sides. Also, if AD be joined, and at the points A and B tangents be drawn to meet E F in the points G and H; AD and G H will be sides of the polygons M and N of twice the number of sides (27.).

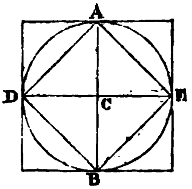
Now, because the triangles C A I, C E D, C A D, C G H are severally contained in the polygons K, L, M, N, the same number of times, viz. as often as the angle ACD, or G C H, is contained in four right angles, the polygons are one to another as those triangles (II. 17.). But the triangle C A I is to the triangle C A D as C I to C D, (II. 39.) that is (because A I, E D are parallels), as CA to C E, that is, again, as the triangle C A D to the triangle C E D (II. 39.). And, because C A I is to CAD as CAD to CED, $K : M :: M : L$ (II. 17. *Cor.* 2.) and M is a mean proportional between K and L.

Again, because the triangle C G H is double of C G D, and, therefore, equal to the quadrilateral C A G D; the triangle A E G is equal to the difference of the triangles C E D, C G H, and the triangle A G D to the difference of the triangles C G H, C A D. But A E G is to AGD as E G to G D, that is, (II. 50.) because the line G C bisects the angle ACD, as EC to CD or CA, that is again, as the triangle C E D to the triangle CAD (II. 39.). And, because CED is to C A D as the difference of C E D, C G H to the difference of C G H, C A D, L : M :: L — N : N — M (II. 17. *Cor.* 2.); that is, N is an harmonical mean between L and M (def. 17.).

Therefore, &c.

Scholium.

The proposition which has been just demonstrated, affords one of the most simple methods of approximating to the area of the circle : to which purpose it may be applied as follows.



Let the diameters A B, D E be drawn at right angles to one another: the straight lines joining their extremities will include an inscribed square, and the tangents drawn through the same a circumscribed square. Now it is plain that the circumscribed square is equal to 4 times the square of the radius, and the inscribed to half the circumscribed, that is, to twice the square of the radius. Therefore, if the radius be assumed for the linear unit, the inscribed square will contain 2, and the circumscribed 4 units of area. But the inscribed figure of eight sides is a mean proportional between them: therefore, the number of units of area which it contains, will be a mean proportional between 2 and 4, $= \sqrt{2 \times 4}, = 2.8284271247$ &c. to the tenth decimal place inclusively. And the number which is an harmonical mean between 4 and 2.8284271247 will, in like manner, be the number of superficial units in the circumscribed figure of 8 sides. Now, to find such a mean x between two numbers m, n, we have this proportion, $m : n :: m - x : x - n$ (II. def. 17); whence, multiplying extremes and means, $m \times (x - n) = n \times (m - x)$; transposing, $mx + nx = 2mn$; and dividing by $m + n$, $x = \frac{2mn}{m+n}$; that is, an [illegible]monical mean between two numbers [illegible]tained by dividing twice their pro[illegible] their sum. Thus we find the circumscribed figure of 8 sides to contain 3.3137084989 &c. superficial units. In the same manner, from these inscribed and circumscribed figures of 8 sides, are to be obtained the inscribed and circumscribed figures of 16 sides; and so on. This process leads us, after 18 times doubling the number of sides,* to the following values of the inscribed and circumscribed polygons of 1,048,576 sides.

3.1415926535 &c.
3.1415926535 &c. values which differ from one another by a quantity which does not appear in the tenth decimal place. But the circle is greater than one, and less than the other of these

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* In the following table of polygonal areas, successively computed as in the text, the letters, A, B, C, D, &c. indicate the regular polygons of 4, 8, 16, 32, &c. sides; and, to show the progress of the approximation, dots are substituted for the figures at the head of their respective columns.

A	in.	2.
	cir.	4.
B	in.	2.8284271247
	cir.	3.3137084989
C	in.	3.0614674589
	cir.	..1825978780
D	in.	..1214451522
	cir.	...517249074
E	in.	...365484905
	cir.	...441183852
F	in.	...403311569
	cir.	22236300
G	in.	12772509
	cir.	17503692
H	in.	5138011
	cir.	6320807
I	in.	5729404
	cir.	6025103
K	in.	5877253
	cir.	5951177
L	in.	914215
	cir.	32696
M	in.	23456
	cir.	28076
N	in.	5766
	cir.	6921
O	in.	6343
	cir.	632
P	in.	487
	cir.	560
Q	in.	523
	cir.	42
R	in.	32
	cir.	37
S	in.	5
	cir.	6
T	in.	5
	cir.	5

Of a calculation which has attracted so much attention, it is not impossible that the student may be curious enough to revise the steps, or even push it to a still greater degree of approximation.

In doing this by the method here given, his labour will be considerably abridged by attending to the following rules.

1°. Annex one more to the decimal places which are required to be exactly ascertained, and with this additional place, use the abbreviated modes of multiplying, dividing, and extracting the square root, viz. by inverting the multiplier, cutting off successively the figures of the divisor, and dividing out when the root is obtained to half the required number of places (See Arith. art, 167. 185.)

2°. When the calculation has proceeded so far that (x being the difference of the two preceding poly-

polygons: therefore, the area of the circle is correctly denoted by 3.1415926535 as far as the tenth decimal place inclusively.

This number is commonly represented by the Greek letter π, being the first letter of the Greek word which signifies *circumference*.* For the same number which represents the area of a circle when the radius is taken for unit, re-

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gons, and b the lesser of the two), the quotient of the fraction $\frac{x^4}{16b^2}$, when expressed in decimals, has no significant figure in the first ten decimal places, the harmonical mean may be found by taking half the sum or arithmetical mean, and subtracting therefrom $\frac{x^2}{4b}$. Since $b = 3.14$, &c., this rule may be used when $2x^4$ does not appear in the last place but three.

3°. And in like manner, when $\frac{x^3}{16b^2}$ is not found in the last decimal place (or which is the same thing nearly, $7x^3$ in the last place but three), the geometrical mean may be obtained by taking the arithmetical mean and subtracting therefrom $\frac{x^2}{8b}$.

4°. When $\frac{x^2}{4b}$ is not found in the last decimal place, (or $8x^2$ in the last but two) neither the harmonical nor the geometrical mean will differ apparently from the arithmetical, which may therefore be taken for them.

Or, when this comes to be the case, instead of computing the intermediate polygonal areas, the area of the circle may be directly found to the required number of places by the following rule.

"Let an inscribed polygon be the last computed; take the difference between its area and that of the preceding circumscribed: divide this difference (considered as a whole number) by that power of 2, say 2^m, which is next less than it; multiply the quotient by $\frac{2^{m-1}-1}{3}$, and add twice the product to the area of the inscribed polygon, placing the units of the product under the last decimal place of the area; the sum shall be the circular area required."

Thus, in the preceding table of areas, the difference between the inscribed polygon L and the circumscribed polygon K is 36962; the power of 2, which is next less, is 32768; the quotient of 36962 divided by 32768 is 1.128; the number by which this is to be multiplied $\frac{1}{3}(\frac{32768}{2}-1)$ or 5461; the product to the nearest unit 6160; and 14215, together with the double of this product, is 26535, which has the remaining digits in question.

The second, third, and fourth of these rules may be established by the assistance of the binomial theorem: the last is derived from the algebraical form of a series of quantities, each of which is an arithmetical mean between the two preceding.

* The letter π is, however, more generally understood to represent the semicircumference of a circle whose radius is unit; this being evidently the same number which represents the circumference when the diameter is assumed for unit.

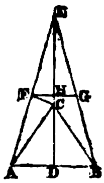
In fact π represents (1), the superficial area of the circle where the unit of superficies is the square of the radius; (2) the linear value of the circumference, where the diameter is the unit of length; and (3) the linear value of the semicircumference, where the radius is the unit of length. The last of these is the meaning most commonly attached to the symbol.

In the method of approximation which is adopted in the text, although the principle is perhaps more obvious, the computation is not so concise as in another method, which may be derived from the following elegant theorem.

If k *and* l *represent the radii of the circles which are inscribed in any regular polygon, and circumscribed about it; and if* m *and* n *represent these radii for a regular polygon which has twice as many sides as the former, and an equal perimeter;* m *shall be an arithmetical mean between* k *and* l, *and* n *a geometrical mean between* l *and* m.

To demonstrate this:

Let AB be a side, and C the centre of any regular polygon; let CD be drawn perpendicular to AB, and join CA, CB: then CD is the radius, *k*, of the inscribed circle, and CA the radius, *l*, of the circumscribed circle. From DC produced cut off CE equal to CA or CB, and join EA, EB: from C draw CF perpendicular to EA, and therefore (I. 6. Cor. 3.) bisecting EA, and through F draw FG parallel to AB, and therefore (I. 14.) perpendicular to ED, which it cuts in the point H.



Then, because the angle AEC is equal to half the angle ACD (I. 19.), the angle AEB, or FEG, is equal to half the angle ACD: also, because EF is equal to the half of EA, FG is equal (II. 30. Cor. 2.) to the half of AB: therefore FG is the side of a regular polygon, which has twice as many sides as the former, E its centre, EH the radius, *m*, of the inscribed circle, and EF the radius, *n*, of the circumscribed circle.

But, because EF is equal to half EA, EH is (II. 30.) equal to half ED, or to half the sum of CD and CA; that is, *m* is an arithmetical mean between *k* and *l*. And, again, because from the right angle F of the triangle EFC, FH is drawn perpendicular to the hypotenuse EC, EF is a mean proportional between EC and EH (II. 34. Cor.); that is, *n* is a mean proportional between *l* and *m*.

Therefore, &c.

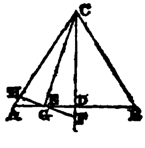
Hence, beginning with the square or hexagon, we may proceed, by alternate *arithmetical* and geometrical means, to determine these radii for a regular polygon, the number of whose sides shall exceed any given number; in which process it is evident that the values of the radii will continually approach to one another, and, therefore, to the intermediate value of the radius of a circle which has the same given perimeter.

There is yet a third theorem, nearly related to the preceding, which may be applied to the purpose of this approximation.

If k *and* l *represent the radii of the circles which are circumscribed about any regular polygon, and inscribed in it, and* m *an arithmetical mean between them; and if* k′ *and* l′ *represent these radii for a regular polygon which has twice as many sides as the former, and an equal area,* k′ *shall be a mean proportional between* k *and* l, *and* l′ *a mean proportional between* l *and* m.

To demonstrate this:

Let AB be a side, and C the centre of any regular polygon; let CD be drawn perpendicular to AB, and join CA, CB: then CA is the radius, *k*, of the circumscribed circle, and CD the radius, *l*, of the inscribed circle. Draw the straight line CE bisecting the angle ACD; in CD produced take CF a mean proportional between CA and CD; from F draw FG perpendicular to CE, and produce it to meet CA in H.



Then, because CG bisects the angle FCH, and FG is perpendicular to CG, the triangle FCH is isosceles (I. 5.); and, because CH×CF is equal to CA×CD, the triangle CFH is equal to the triangle CAD (II. 40. Cor.): therefore FH is the side of a represents also the circumference when the diameter is taken for unit, because the area of a circle, being equal to the rectangle under the semicircumference and radius (32.), bears to the square of the radius the same ratio which the semicircumference bears to the radius, or the circumference to the diameter.

And hence if R be the radius of any circle, its circumference (greater than 2 R in the ratio of $\pi : 1$) is $= 2\,\pi\,R$: and its area (greater than R^2 in the ratio of $\pi : 1$) is $= \pi R^2$.

It remains to observe that the circumference of a circle is incommensurable with its diameter, for which reason their ratio can never be exactly represented by numbers. This was for the first time demonstrated in the year 1761 A. D. by Lambert. During the long period for which it was only matter of conjecture, the quest of the exact numerical ratio (and that by methods not more expeditious than the above) occupied many laborious calculators. Could they have assigned any such, it is evident that they might likewise have assigned the exact value of the area of a circle, whose radius is given, and *vice versâ;* because that area is (32.) equal to half the product of the radius and circumference. But the hope of arriving at a term of the approximation is now demonstrated to have been vain, and accordingly an exact solution of the celebrated problem of squaring the circle, that is, of finding a straight line, the square of which shall be exactly equal to a given circle, impracticable. At the same time, the approximate solution exhibited in the number 3.1415926535 &c. is sufficient for every useful purpose. If the ratio be considered as expressed by the integer and first ten decimal places, the error committed will bear a less proportion to the whole circumference than an inch to the circuit of the earth.

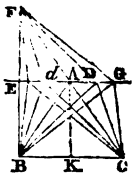
Instead of the number 3.1415 &c. the fractions $\frac{22}{7}$ and $\frac{355}{113}$ may also be conveniently used in cases not requiring a great degree of approximation. The first (discovered by Archimedes) will be found to fail in the third decimal place: the other (due to Metius, and remarkably made up of the odd numbers 1, 3, 5) fails in the seventh decimal place only.

SECTION 5.—*The circle a maximum of area, and a minimum of perimeter.*

In the present section it is proposed to show that of all plane figures having equal perimeters, the circle contains the greatest area; and consequently, of all plane figures containing equal areas, has the least perimeter; in other words, as it is announced in the title of the Section, that the circle is *a maximum of area* and *a minimum of perimeter.*

PROP. 35.

Of equal triangles upon the same base, the isosceles has the least perimeter; and, of the rest, that which has the greater vertical angle has the less perimeter.



Let the triangles A B C, D B C be upon the same base B C, and between the same parallels A D, B C (I. 27.), and let the triangle A B C be isosceles: the triangle ABC shall have a less perimeter than the triangle D B C.

From B draw B E perpendicular to A D, and produce it to F, so that E F may be equal to E B: and join A F, D F. Then, because the triangles B E A, F E A have two sides of the one equal to two sides of the other, each to each, and the included angles B E A, F E A equal to one another, A F is equal to A B (I. 4.) and the angle F A E to the angle B A E, that is, to A B C (I. 15.) or (I. 6.) A C B. But the angles A C B, E A C are together equal to

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gular polygon which has twice as many sides as the former, and an equal area, C its centre, CF the radius, k', of the circumscribed circle, and CG the radius, l', of the inscribed circle

But, by the construction, CF is a mean proportional between CA and CD; that is, k' is a mean proportional between k and l. And, again, because the triangle CGF is similar to CDE, the triangle CGF is to the triangle CDE as CG^2 to CD^2 (II. 42. Cor.); but, because the triangle CGF is equal to half CHF, that is to half CDA, CGF is to CDE as half DA to DE (II. 29.), or, because CE bisects the angle ACD as half CA+CD to CD (II. 50.); therefore (II. 12.) $CG^2 : CD^2 :: \frac{CA+CD}{2} : CD$, and (II. 37. Cor. 2.) CG is a mean proportional between CD and $\frac{CA+CD}{2}$, that is, l' is a mean proportional between l and m.

Therefore, &c.

This theorem is applied in the same manner as the preceding. It is necessary to observe that CG is greater than CE, and not equal to it, as is wrongly represented in the figure: for, if P be taken a third proportional to EC and ED, it may be shown that [illegible] is greater than CE^2 by a square which is to P^2 as [illegible] BD.

two right angles (I. 15); therefore FAE, EAC are likewise equal to two right angles, and (I. 2.) FA, AC are in the same straight line. And because the triangles BED, FED have the two sides BE, ED of the one equal to the two FE, ED of the other, each to each, and the included angles equal to one another (I. 4.) DF is equal to DB; and it was shown that AF was equal to AB. But DF, DC are greater than FC (I. 10.); therefore DB, DC are greater than AB, AC; and, BC being added to each, the perimeter of the triangle DBC is greater than the perimeter of the triangle ABC.

In the next place, let GBC be another triangle upon the same base BC, and between the same parallels, but having the angle BGC less than BDC: the perimeter of the triangle GBC shall be greater than the perimeter of the triangle DBC.

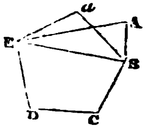
Bisect BC in K, and join AK. Then, because ABC is an isosceles triangle, AK is (I. 6. *Cor.* 3.) at right angles to BC. And, because AK bisects BC at right angles, it passes (3. *Cor.* 2.) through the centre of the circle which is circumscribed about the triangle DBC (5. *Cor.* 2.) Take A*d* equal to AD. Then, because AK passes through the centre of this circle, and bisects the chord BC, it bisects also the chord which passes through the point A parallel to BC (3. *Cor.* 1.); and therefore the point *d* is in the circumference of the circle.

Now, because the angle BGC is less than BDC, the point G must lie without the circle, (15. *Cor.* 3.) that is, G must be some point in the line D*d* produced, and does not lie between the points D, *d*. But if it lie upon the same side of FC with the point D, FG, GC together must be greater (I. 10. *Cor.* 1.) than FD, DC together; and therefore, because FG is equal to BG, and FD to BD, (I. 4.) the perimeter of the triangle GBC must be greater than the perimeter of the triangle DBC. And if it lie upon the other side of FC, FG, GC together will be greater than F*d*, *d*C together. But because the diagonals FC, D*d* bisect one another (I. 22.) the figure FDC*d* is a parallelogram, and (I. 22.) the sides F*d*, *d*C together are equal to the sides FD, DC together. Therefore FG, GC together are greater than FD, DC together, and, as before, the perimeter of the triangle GBC is greater than the perimeter of the triangle DBC.

Therefore, &c.

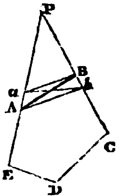
PROP. 36.

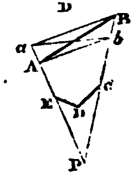
If a rectilineal figure ABCDE *have not all its sides equal and all its angles equal, a figure of equal area may be found, which shall have the same number of sides and a less perimeter.*



For, in the first place, if it have not all its sides equal, there must be at least two adjacent sides which are unequal. Let these be AB, AE, and join BE: and let *a*BE be an isosceles triangle of equal area, and upon the same base BE. Then the whole figure *a*BCDE is equal to the whole ABCDE; and because (35.) *a*B, *a*E together are less than AB, AE together, the figure *a*BCDE has been found of equal area with the figure ABCDE, and having a less perimeter.

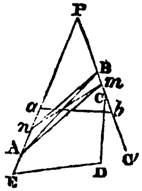
Next, if it have not all its angles equal, there must be two adjacent angles A, B, which are unequal.



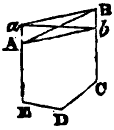


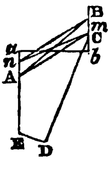
And, first, let the sides AE, BC, meet one another in a point P. Take P*a* a mean proportional (II. 51.) between PA, PB, and make P*b* equal to P*a*. Then, in the first place, if one of these points, as *b*, lie in the corresponding side BC, that is, between B and C, join *ab*: the figure *ab*CDE shall be of equal area with the figure ABCDE, and shall have a less perimeter. For, because PA is to P*a* as P*a* or P*b* to PB, *a*B joined is parallel to A*b* (II. 29.). Therefore (I. 27.) the triangle *a*A*b* is equal to the triangle BA*b*, and the figure *ab*CDE is equal to the figure ABCDE. And because the triangle P*ab* is isosceles, the angle E*ab* is equal to the angle C*ba* (I. 6. or I. 6. *Cor.* 2.); but the two E*ab*, C*ba* are together equal to the two EAB, CBA (I. 19.), of which one, viz. EAB, is the greater; therefore the angle E*ab* is greater than the other CBA: And these latter angles are the vertical angles of the equal triangles *a*A*b*, BA*b*, which stand upon the same base A*b*: therefore (35.) the sides *a*A, *ab* together are less than the sides BA, B*b* together; and the figure *ab*CDE has a less perimeter than the figure ABCDE.

But, in the second place, if neither of the points a, b lie in a side of the figure, but both of them in the sides produced, take any point m in BC, and join mA. Through B draw Bn parallel to mA, and join mn: the figure nmCDE shall be of equal area with the figure ABCDE, and shall have a less perimeter. The figures are of equal area, because, Bn being parallel to mA, the triangles nAm, BAm are (I. 27.) equal to one another. And because the angle mnA is (I. 8. *Cor.* 1.) greater than the angle maA,* much more is it greater than the angle baA; but the latter angle being, as in the preceding case, equal to abC′, is (I. 8. *Cor.* 1.) greater than ABC; therefore much more is the angle mnA greater than the angle ABC. And hence it follows, as before, that nA, nm together (35.) are less than BA, Bm together, and that the perimeter of the figure nmCDE is less than that of the figure ABCDE.



The two cases in which the sides AE, BC are parallel, are easily demonstrated in a manner taken with little variation from the preceding, and readily to be apprehended by aid of the adjoined figures, in which ab is drawn through the middle point of AB perpendicular to AE and BC.





Therefore, in every case, if a figure have not, &c.

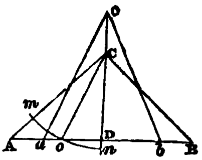
Cor. 1. Of plane rectilineal figures having the same number of sides, and containing the same area, the regular polygon has the least perimeter.

For it is obvious, that a certain area being to be comprehended under a certain number of sides, the perimeter of the containing figure cannot be less than of some certain length depending on the extent of the area; that is, in other words, of figures inclosing the same area, and having the same number of sides, the perimeters cannot, any of them, be less than a certain line, viz. the least possible by which, under the aforesaid condition, the given area can be inclosed. But it is shown in the proposition that, except a figure have all its sides equal and all its angles equal, another may be found inclosing the same area under the same number of sides and with a less perimeter. Therefore, of all the above figures there is one only which is contained by the least possible perimeter, and that one is the regular polygon (*def.* 11.)

Cor. 2. And hence a regular polygon contains a greater area than any other rectilineal figure having the same number of sides and the same perimeter: for a similar polygon which should have the same area with the figure, would have a less perimeter (*Cor.* 1.), and therefore (30.) a less area than the regular polygon which has the same perimeter.

PROP. 37.

Of regular polygons having equal perimeters, that is greatest which has the greatest number of sides.



Let AB, ab be the sides of two regular polygons having equal perimeters, and let the polygon which has the side ab have a greater number of sides than the other. The polygon which has the side ab shall be greater than the other.

Let the sides AB, ab, be placed in the same straight line, and so that their middle points may coincide, as at D. Then, because ab is contained a greater number of times than AB in the common perimeter, ab is less than AB, and the point a lies between A and D. From D draw DC at right angles to AB, and therefore (3. *Cor.* 2. and def. 12.) passing through the centres of both the polygons: and let C be the centre of the polygon which has the side AB, and c the centre of the polygon which has the side ab; it being supposed as yet unknown whether DC or Dc is the greater of the two. Join CA, ca; through C draw Co parallel to ca to meet AD in o; and lastly, with the centre C and radius Co describe the arc mn cutting CA in m, and CD produced in n. Then, because AB is to the common perimeter of the two polygons (II. 17.) as the angle ACB to four right angles, and the common perimeter to ab as four right angles to the angle acb, *ex æquali* (II. 24), AB is to ab as the angle ACB to the angle acb; and hence, by taking the halves, (II. 17. *Cor.* 2.) the line AD is to aD as the angle ACD to acD, or (I. 15.) oCD. Therefore, *dividendo* (II. 20.) Aa is to aD as the angle ACo to oCD, and *invertendo* (II. 15.) Da is to aA as the angle DCo to oCA.

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* The line ma is not drawn in the figure.

Again, because the sector Cno is greater than the triangle CDo, and the sector Com less than the triangle CoA the sector Cno has, on both accounts, (II. 11.) to the sector Com a greater ratio than the triangle CDo to the triangle CoA: but the angles DCo, oCA have the same ratio as the sectors (13.), and the lines Do, oA the same ratio as the triangles (II. 39): therefore the angle DCo has to the angle oCA a greater ratio than the line Do has to the line oA (II. 12. *Cor.* 1 and 2.). And it was shown, that Da has to aA the same ratio which the angle DCo has to the angle oCA: therefore Da has to aA a greater ratio than Do to oA, and Da is greater than Do. And, because Co is parallel to ca, Dc : DC :: Da : Do (II. 29.); therefore Dc is also greater than DC.

But the polygons, being equal, each of them, to half the rectangle under the apothem and perimeter (29.), are to one another as their apothems cD, CD (II. 35.): therefore, the polygon which has the apothem cD and side ab is greater than the other.

Therefore, &c.

Cor. A regular polygon is greater than any other rectilineal figure having the same perimeter, and the same or a less number of sides (36. *Cor.* 2.).

PROP. 38.

A circle is greater than any regular polygon having the same perimeter.

For let a similar polygon be circumscribed about the circle, viz. by dividing the circumference (or conceiving it to be divided) into as many equal parts as the polygon is to have sides, and drawing tangents through the points of division. Then, because the area of the polygon is equal (29.) to half the rectangle under its apothem and perimeter, and the area of the circle to half the rectangle under its radius and perimeter (32.); and that the apothem of the circumscribed polygon is equal to the radius of the circle, the circumscribed polygon is to the circle as its perimeter to that of the circle (II. 35.), that is, as its perimeter to the perimeter of the polygon in question, or (II. 17.) as its side to the side of the latter. Again, because the polygons are similar (II. 43.), they are one to another in the duplicate ratio of their sides. Therefore, the circumscribed polygon has to the other the duplicate ratio of that which it has to the circle, and (II. def. 11.) the circle is a mean proportional between the two polygons.* But the circumscribed polygon is greater than the circle: therefore, the circle (II. 14.) is greater than the polygon of equal perimeter.

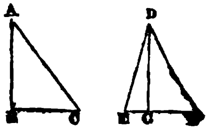
Cor. A circle is greater than any plane rectilineal figure of the same perimeter (37. *Cor.*).

It may be inferred also, from the foregoing propositions, that the circle is *not less* than any *curvilineal* figure of the same perimeter. For there may be inscribed in the latter a rectilineal figure of less perimeter, yet approaching more nearly to it in area than by any supposed excess of the original figure above the circle, so that, were there such an excess, a rectilineal figure might be found greater than the circle, and yet of less perimeter, which is impossible. This method will not, however, carry us any further. In the following propositions, another view is taken of the subject, and by them it will be made to appear, that the circle is *greater* than any other figure, curvilineal or otherwise, which has the same perimeter.

PROP. 39.

If two triangles have two sides of the one equal to two sides of the other, each to each, and the angle contained by the two sides of the first a right angle, but the angle contained by the two sides of the other not a right angle, the first triangle shall be greater than the other.

Let ABC, DEF be two triangles which have the two sides AB, BC of the one



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* It appears from this part of the demonstration, that *if there be two similar polygons, of which one is circumscribed about a circle, and the other has the same perimeter with the circle, the circle shall be a mean proportional between the two polygons*: a proposition which is true, whether the polygons be regular or irregular. For whether the circumscribed polygon be regular or otherwise, it is evident that its area is equal to half the rectangle under its perimeter and the radius of the circle.

equal to the two sides DE, EF of the other, each to each, but the angle ABC a right angle, and the angle DEF greater or less than a right angle; the triangle ABC shall be greater than the triangle DEF.

From the point D draw D G perpendicular to E F, or E F produced. Then, because D G is (I. 12. *Cor.* 3.) less than D E or A B, the rectangle under D G, E F is less than the rectangle under A B, E F, or A B, B C, and therefore (I. 26. *Cor.*) the triangle D E F is less than the triangle A B C.

Therefore, &c.

Cor. Two given finite straight lines with a third indefinite, inclose the greatest possible area, when placed at right angles.

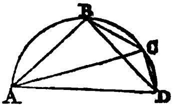
Prop. 40.

If two quadrilaterals have three sides of the one equal to three sides of the other, each to each, and the angles of the first lying in a semi-circumference of which the fourth side is diameter, but the angles of the other not so lying, the first quadrilateral shall be greater than the other.

For if EFGH (see *fig.* 1) be that one of the quadrilaterals which has not the angles lying in a semicircumference, of which the fourth side EH is diameter, and if G be an angle which does not so lie; then, joining EG, the angle EGH will not be equal to a right angle, (15 *Cor.* 3.) and, therefore, if GH′ be drawn perpendicular to EG, and equal to GH, and if EH′ be joined, the triangle EGH′ will be greater than EGH, (39.) and, accordingly, the quadrilateral EFGH′, which has its three sides EF, FG, GH′ equal to the three EF, FG, GH, each to each, greater than the quadrilateral EFGH. Therefore, if a quadrilateral be inclosed by three given sides and a fourth not given, a greater may be found inclosed by the same three given sides and a fourth not given, except when the angles lie in a semicircumference, of which the fourth side is diameter. But, because the fourth side is (I. 10. *Cor.* 2.) necessarily less than the sum of the other three, it is evident that there is some certain area, a greater than which cannot be so inclosed, and therefore some quadrilateral which incloses the greatest possible area. Therefore, the quadrilateral A B C D which has its angles lying in a semicircumference, of which the fourth side is diameter, incloses the greatest possible area, and the quadrilateral ABCD is greater than EFGH.*

Otherwise.

Let A B C D, E F G H be two quadrilaterals which have the sides A B, B C, C D of the one, equal to the sides E F, F G, G H of the other, each to each; and let the angles A, B, C, D of the first lie in the circumference of a circle, of which the side A D is diameter, but the angles E, F, G, H, of the other not lie in the circumference of a circle of which E H is diameter: the quadrilateral A B C D shall be greater than E F G H.



Join E G, F H: and, as most favourable to the figure E F G H in the comparison with ABCD, let one of the angles E F H, E G H, the former for instance, be a right angle; since, F H remaining the same, the triangle E F H, and therefore the whole figure will be greater (39.) upon this than upon the contrary supposition. It will appear in the demonstration, that it is indifferent whether E G H be supposed less or greater than a right angle: we shall set out with supposing it to be less, and, therefore, (15. *Cor.* 3.) the point G to be without the semicircle upon E H.

Fig. 1.

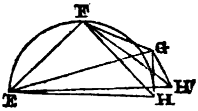


Fig. 2.

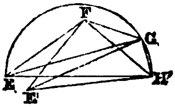
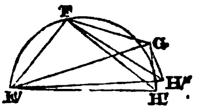


Fig. 3.



Draw G H′ at right angles to E G, and equal to G H (*fig.* 1); and join

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* If two of the given sides as FG, GH should be in the same straight line, E F G H would be a triangle, not a quadrilateral: it may be observed, however, that the demonstration is equally applicable to show that in this case E F G H is less than A B C D. We may add that, by a similar demonstration, it appears that *any number whatever of given finite straight lines with an indefinite inclose the greatest possible area when placed as chords of a semicircumference of which the indefinite is diameter.*

E H′. Then, because E F H is a right angle, E F H′ is greater than a right angle, and therefore (15. *Cor.* 3.) the point F falls within the semicircle drawn upon EH′, as in *fig.* 2. Again, draw F E′ (*fig.* 2) at right angles to F H′ and equal to E F, and join E′ H′. Then, because E G H′ is a right angle, E′ G H′ is less than a right angle, and the point G falls without the semicircle upon E′ H′ (see *fig.* 3), as at first. It appears therefore, that if the process be repeated and continued, we shall obtain in this manner a series of figures (*fig.* 1., *fig.* 2., *fig.* 3., &c.), each of which is greater than the preceding (because one of the triangles remaining the same, the other is made to have a right angle), and in which, one of the angles E F H, E G H, being a right angle, the other is greater than a right angle, and less than a right angle alternately.

Again, of the bases E H, E H′, E′ H′, &c. each is of a magnitude intermediate between the two preceding. For, because the square of E′H′ (*fig.* 2) is equal (I. 36.) to the squares of E′F and F H′, and that FH′ is greater than FH (*fig.* 1), (because the two sides F G, G H′ of the triangle FGH′ are equal to the two sides FG, GH of the triangle FGH, and contain a greater angle (I. 11),) the square of E′ H′ is greater than the squares of E′ F, F H, or of E F, F H; greater, that is, than (I. 36.) the square of E H (*fig.* 1), and therefore E′ H′ (*fig.* 2 *or* 3) is greater than E H (*fig.* 1). But E′H′ (*fig.* 3) is less than E H′ (*fig.* 2), because the two sides E′ F, F H′ of the triangle E′ F H′ are equal to the two E F, F H′ of the triangle E F H′, and contain a less angle, (I. 11.). Therefore, E′ H′ (*fig.* 3) is of intermediate magnitude between E H (*fig.* 1) and E H′ (*fig.* 2). And, in a similar manner, it may be shown that E′ H″ (*fig.* 3. *or* 4) is of intermediate magnitude between E H′ (*fig.* 2) and E′ H′ (*fig.* 3); and so on.

Now A D is greater than E H, E′ H′, &c., and less than E H′, E′ H″, &c., because the chords A B, B C, C D, which together subtend the semi-circumference of which A D is diameter, subtend more than a semi-circumference in the circles of which E H, E′ H′, &c. are diameters, and less than a semi-circumference in those of which E H′, E′ H″, &c. are diameters.

Therefore, every successive base E H being alternately greater and less than A D, and lying between the two preceding, approaches more nearly to A D than did the last in the series which was greater or less than A D; that is, A D is the limit to which, in the foregoing process, the bases E H are made to approach. And it has been shown, besides, that the figure E F G H is increased at every step. Therefore, the figure upon the base AD is greater than any of the figures E F G H.

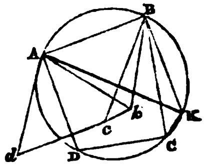
Therefore, &c.

Cor. Three given finite straight lines with a fourth indefinite, inclose the greatest area, when placed as chords of a semi-circumference, of which the fourth side is diameter.

Prop. 41.

If two quadrilaterals have the four sides of the one equal to the four sides of the other, each to each in order, and the angles of the one lying in the circumference of a circle, but the angles of the other not so lying, the first quadrilateral shall be greater than the other.

Let the quadrilaterals ABCD, A B *cd* have the four sides of the one equal to the four sides of the other, each to each in order, and the angles A, B, C, D lying in the circumference of a circle ABD, but the angles A, B, *c*, *d*, not so lying: the quadrilateral A B C D shall be greater than the quadrilateral A B *cd*.



Through A draw the diameter A K, and join B K, K C: and upon B *c*, which is equal to B C, make the triangle B *k c* equal and similar to the triangle B K C, so that the sides B *k*, *k c* may be equal to the sides B K, K C respectively; and join A *k*.

Then, because the straight lines B *c*, B C do not coincide, (for if they did, the figures would coincide altogether by I. 7.) the point *k* does not coincide with the point K: but A B K is a right angle (15. *Cor.* 1.): therefore A B *k* is not a right angle, and (39.) the triangle BAK is greater than the triangle B A *k*. Also, the quadrilateral A D C K, having its three sides A D, D C, C K chords of the semi-circumference upon AK, is greater than any other quadrilateral[*] A *d c k*, having three of its sides equal to AD, DC,

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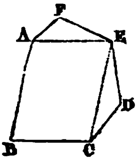
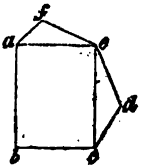
[*] If *d c* and *c k* lie in the same straight line (as is nearly the case in the figure), the figure A *d c k* will be a triangle, not a quadrilateral; but in this case also it is less than A D C K (see note Prop. 40.)

C K, respectively (40.). Therefore, the whole figure A B K C D is greater than the whole figure A B *k c d*; and, taking away the equal triangles B K C, B *k c*, the figure A B C D is greater than the figure A *b c d*.

Therefore, &c.

Cor. 1. If a figure A B C D E F is to be inclosed by any number of given sides, and if these sides be not so disposed that the angles may lie in the circumference of a circle, a greater figure may be inclosed by the same sides. For, if the angle E, for instance, do not lie in the circumference which passes through the points A, B, C, join A E, C E, and let there be constructed the quadrilateral *a b c e*, such that its sides may be equal to those of A B C E, each to each, and its angles in the circumference of a circle (25. *Cor.*): and upon the sides *a e*, *c e*, which are equal to AE, CE, respectively, let there be described the figures *a f e*, *c d e* equal to the figures A F E, CDE, respectively. Then, because, by the proposition, the quadrilateral *a b c e* is greater than A B C E, the whole figure *a b c d e f* is greater than A B C D E F.





Cor. 2. And hence, of all figures contained by the same given sides in the same order, that one contains the greatest possible area which has all its angles in the circumference of a circle. For the area inclosed by the given sides cannot exceed a certain limit depending upon them, which limit is the greatest possible that can be inclosed by the given sides, and is therefore such as by them can be inclosed. But no figure, so inclosed, contains the greatest possible area, of which the angles do not lie in the circumference of a circle. Therefore, the figure which has its angles in the circumference of a circle contains a greater area than any other figure having the same given sides.

Scholium.

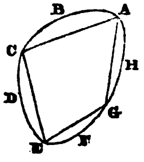
That a circle may be imagined in which any number of given straight lines shall subtend as chords the whole circumference exactly, is evident from this, that a circle may be imagined in which they shall subtend less than the whole circumference, and a second circle in which they shall subtend more than the whole circumference: for the circle required will be of some magnitude between these two.

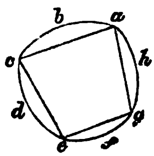
It may be observed, also, that the order of the sides is indifferent as well to the magnitude of the required circle, as to the magnitude of the figure which is to be inscribed in it; for the same chord will subtend an arc of the same magnitude, at whatever part of the circumference it may be placed (12. *Cor.* 1); and therefore the arcs subtended by all the chords will be together equal to the whole circumference, whatsoever may be their order. And, because the same chord always cuts off a segment of the same area, the segments cut off by all the chords will amount to the same area, whatsoever may be their order; and therefore the inclosed area, which is the difference between that amount and the area of the circle, will also be the same. From these considerations it appears that *Prop.* 41. *Cor.* 2. *need not have been qualified by a regard to the order of the sides.*

PROP. 42.

Of all plane figures having the same perimeter, the circle contains the greatest area.

For, if the figure A B C D E F G H be any other than a circle, there must be some four points in the perimeter,





as A, C, E, G, which do not lie in the circumference of a circle. Join these points, and let the quadrilateral *a c e g* be constructed, having its sides equal to those of the quadrilateral ACEG, each to each, and its angles in the circumference of a circle (25. *Cor.*). Then, because (41.) the quadrilateral *a c e g* is greater than A C E G, if upon the sides *a c*, *c e*, *e g*, *g a*, which are equal to A C, C E, E G, G A, respectively, there be constructed the figures *a b c*, *c d e*, *e f g*, *g h a* equal to the figures A B C, C D E, E F G, G H A, each to each, in all respects, the whole figure *a b c d e f g* will be greater than A B C D E F G, and will have the same perimeter.

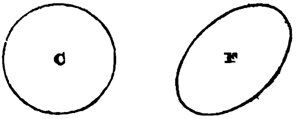
It appears, therefore, that if a plane figure be not a circle, a greater area than is contained by that figure may be inclosed with the same perimeter. But the area inclosed by a given perimeter cannot exceed a certain limit, which limit, being the greatest possible that can be so inclosed, some figure with the given perimeter must be capable of containing. Therefore the circle only contains the greatest area of all figures having the same perimeter.

Cor. In the same manner it may be shown that if a figure is to be inclosed by a given perimeter, of which part is to be a given finite straight line, and if it be not made a circular segment of which the given line is chord, a greater may be inclosed with the same conditions, and therefore that of all figures so inclosed the circular segment is the greatest.

PROP. 43.

Of all plane figures having the same area, the circle has the least perimeter.

Let the circle C have the same area with any other plane figure F: C shall be contained by a less perimeter than F.



Let C′ be a second circle, having the same perimeter with F; then by the last proposition, C′ has a greater area than F has, that is, than C has. But the areas of circles (33.) are as the squares of their radii; therefore the radius of C′ is greater than the radius of C; and the radii of circles (33.) are as their circumferences; therefore the circumference of C′, or perimeter of F, is greater than the circumference of C.

Therefore, &c.

SECTION 6.—*Simple and Plane Loci.*

Def. 14. A *locus* in Plane Geometry is a straight line, circle, or plane curve, every point of which, and none else in that plane, satisfies a certain condition.

The nature and use of loci will be readily apprehended from the following example:—

"Required a point in a certain plane which shall be, first, in a given line in the plane; and, secondly, equidistant from two given points in the same plane."

Here, as far as the first condition only is concerned, any point in the given line, but none else, will answer. The is therefore the locus corresponding to the first condition. But again, the point required must be equidistant from the two given points, that is, it must be in the straight line which bisects the distance of the two given points at right angles; for this, it is easily seen, (I. 6.) is the locus corresponding to this second condition. Therefore, if this straight line be drawn, and intersect the given line, the point of intersection (or any of those points, if there be more than one) will satisfy both conditions, and will be the point required.

If there be no point of intersection, the problem is impossible.

To take another instance—

Let it be required "to find a point in a certain plane, which shall be, first, at a given distance from a given point in the plane; and, secondly, at a second given distance from a second given point in the same plane."

Here it is evident that the locus corresponding to the first condition is the circumference of a circle described about the given point as a centre with the given distance as radius: and again, that the locus corresponding to the second condition is the circumference of a circle described about the second given point as a centre with the second given distance as radius. Therefore the points which are common to the two circumferences, that is, their points of intersection, if there be any, will either of them be the point required.

If the circles do not intersect one another, the problem is impossible.

Such is the use of loci in the solution of problems. We have seen also in the above example, that they serve to determine in what cases the solution is possible or impossible. Thus, in the latter example, it will be impossible, if the distances of the point required from the given points differ by more than the mutual distance of those points, or together fall short of that distance: and in the first example it will be impossible, if the given line, being straight, be perpendicular to the line which passes through the two given points, and does not pass through the point which bisects that line; for if it does so pass, the two conditions proposed are identical, and any point in this line will answer them.

Every locus is the limit between excess and defect. The points upon one side of it fail by defect, and those upon the other side by excess, of possessing the required property which is possessed by every point in the locus. Thus, in the case of the circle, the distance of a point within the circle falls short of the distance of the circumference, while that of a point without exceeds it.

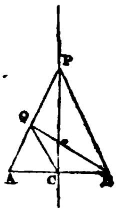
When a locus is merely a straight line, it is called a *simple locus;* when the circumference of a circle, it is called a *plane locus;* when any other curve, it is said to be of *higher dimensions* than the circle.

The following propositions afford examples of the two first only; and, the three concluding propositions excepted, they will be found the same in substance with theorems which have been stated before, and which only reappear in this place under a different form.

It will be observed that they are *investigated*—a species of analytical reasoning which has not hitherto been exemplified either in the demonstration of a theorem or in the solution of a problem, but which is the method commonly pursued in arriving both at the one and the other, and is known under the name of "Geometrical Analysis." The nature of this, as opposed to the ordinary didactic method of solution, commonly called that of *synthesis* or composition, is pointed out by its name, and will be at once apprehended from any of the following examples. We need only observe that rather than two distinct methods of doing the same thing, as might at first appear to be the ease, they are but the different parts of one full and perfect method; that the use of each is essential to a complete solution; the latter (synthesis) always taking the subject up where the other leaves it; the analysis first descending, by geometrical reasoning, from the thing proposed to the minutest particulars of the solution, and the synthesis ascending back through the same steps from these particulars to the thing proposed. It is true, that, because the steps in each are for the most part the same, occurring only in an inverted order, the same principles are developed in each, and, therefore, the detail of either (more especially the synthesis) commonly furnishes a satisfactory view of the question; for which reason, and for the sake of brevity, one is usually given to the exclusion of the other. The student will, however, find himself amply repaid by entering into both, and he is recommended, after following the analysis of the problems of the present section, to supply in each case where it is omitted, the synthesis necessary to complete the solution.

Prop. 44.

Required the locus of all points which are equidistant from two given points A, B.

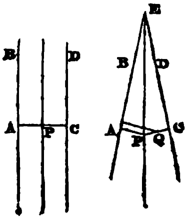


Let P be a point in the locus, and join P A, P B. Then, because PAB is an isosceles triangle, if the base AB be bisected in C, PC joined will be at right angles to AB (I. 6. *Cor.*3). Therefore the point P lies in the straight line which bisects A B at right angles; and it is easily shown, reversely, that every point in this straight line is equidistant from A and B (I. 4.); therefore this straight line is the locus required.

We may observe, that if any point Q lie upon the same side of the locus with A, Q A will be less than Q B; if upon the other side, QA will be greater than QB (I. 11.).

Prop. 45.

Required the locus of all points which are equidistant from two given straight lines A B, C D.



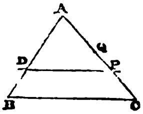
If the given straight lines be parallel, the required locus is evidently a straight line, which is parallel to each, and bisects the distances between them.

But if not, let them meet in E, and let P be a point in the locus. Then if PA be drawn perpendicular to AB, and PC perpendicular to C D, P A will be equal to P C. Join PE: then, because the right-angled triangles P A E, P C E have a common hypotenuse P E, and equal sides P A, P C, they are equal to one another in every respect (I. 13.), and the angle PEA is equal to the angle P E C. Therefore the point P is in the straight line which bisects the angle A E C: and, reversely, it may be shown that every point in this straight line is equidistant from AB and CD (I. 13.): therefore, this straight line is the locus required.

If any point Q lie upon the same side of the locus with the straight line A B, its distance from A B will be less than its distance from C D; if, upon the other side, its distance from A B will be greater than that from C D.

PROP. 46.

Straight lines being drawn from a given point A *to a given straight line* B C, *required the locus of all points dividing them in the same given ratio*

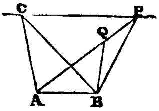


Let A B be any straight line drawn from A to B C, and divided in the given ratio in the point D; and let P be a point in the locus. Then, because A P is to P C in the same ratio as A D to D B, D P joined is parallel to B C (II. 29.). Therefore the point P lies in a straight line drawn through D parallel to B C: and reversely it may be shown that every point in this straight line has the given property (II. 29.); therefore it is the locus required.

If Q be any point on the same side of the locus with A, it will divide the line A C which passes through it, in a less ratio than that of A D to D B: if upon the other side, in a greater ratio.

PROP. 47.

Required the locus of the vertices of all equal triangles, upon the same base A B, *and upon the same side of it.*

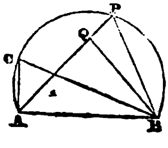


Let A B C be any triangle, upon the given side of the base, and having the given area, and let P be a point in the locus. Then, because the triangle PAB is equal to CAB, PC joined is parallel to A B (I. 27.). Therefore the point P lies in a straight line drawn through C parallel to A B: and, reversely, it may be shown that every point in this straight line has the given property (I. 27.); therefore it is the locus required.

If Q be any point upon the same side of the locus with AB, the triangle QAB will be less than C A B; if upon the other side, greater.

PROP. 48.

Required the locus of the vertices of all triangles having equal vertical angles, upon the same base AB, *and upon the same side of it.*

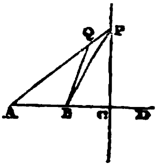


Let A C B be the given vertical angle, and let P be any point in the locus. Then, because the angle at P is equal to the angle at C, the point P is in the circular arc passing through C, and having A B for its chord (15. *Cor.* 3.): and, reversely, it may be shown that every point in this arc has the given property (15.); therefore it is the locus required.

If Q be any point upon the same side of the locus with A B, the angle A Q B will be greater than A C B; if upon the other side, less.

PROP. 49.

Required the locus of the vertices of all triangles upon the same base A B, *having the side terminated in* A *greater than that terminated in* B, *and the difference of the squares of the sides equal to a given square.*



Let P be a point in the locus, and from P draw P C at right angles to A B, or A B produced. Then, because the difference of the squares of A C, B C is equal to the difference of the squares of AP, BP (I. 38.), the difference of the squares of AC, BC is equal to the given square; and the point C may be found (I. 34.) by taking A D (II. 52.) a third proportional to A B and the side of the given square, so that the rectangle under AB, AD may be (II. 38. *Cor.*1.) equal to the given square, and bisecting BD in C.* And it may be shown, reversely, that if from the point C so taken, P C be drawn perpendicular to A C, every point in P C will satisfy the given condition; therefore P C is the locus required.

If Q be any point upon the same side of the locus with the middle point of A B, the difference of the squares of Q A, Q B will be less than the given difference; if upon the other side, greater. For, if a perpendicular Q E be drawn from Q to A B, and E F be taken equal to E B, the difference of the squares of Q A, Q B will be equal to the rectangle under A B, A F, (I. 39.) which is less or greater than the rectangle under A B, A D, according as the position of Q is one or the other of the two just mentioned.

The figure represents the point C in A B produced; if, however, the given square be sufficiently small, the point C may lie between A and B.

PROP. 50.

Required the locus of the vertices of all triangles upon the same base A B,

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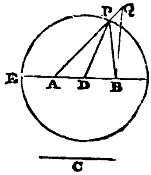
* See also I. 39.

having the sum of the squares of their sides equal to a given square.



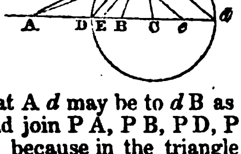
Let the given square be the square of C, and let P be a point in the locus. Bisect AB in D, and join PD. Then, because the base AB of the triangle PAB is bisected in D, the sum of the squares of PA, PB is equal to twice the square of PD, together with twice the square of DA (I. 40.) But it is also equal to the square of C. Therefore twice the square of PD is equal to the difference between the square of C and twice the square of AD, that is, if twice the square of DE be equal to the same given difference, to twice the square of DE; and the point P lies in the circumference of a circle described from the centre D with the radius DE. And it may be shown, reversely, that every point in the circumference of this circle satisfies the given condition (I. 40.); therefore it is the locus required.

If Q be any point without the circle, the sum of the squares of QA, QB will be greater than the given sum; if within it, less. For QD^2 will be greater than PD^2 in the former case, and less in the latter; and therefore the sum of the squares of QA, QB will be (I. 40) greater than twice the sum of the squares of PD, DA, that is than the given sum, in the former case, and less in the latter.

PROP. 51.

Required the locus of the vertices of all triangles upon the same base AB, *having the side terminated in* A *greater than the side terminated in* B, *and their ratio the same with the given ratio of* AD *to* DB.

Let P be a point in the locus. Divide AB produced (II. 55.) in the point



d, so that A*d* may be to *d*B as AD to DB, and join PA, PB, PD, P*d*.

Then, because in the triangle PAB the straight lines PD, P*d* divide the base and the base produced in the ratio of the sides (II. 50.) they bisect the vertical and exterior vertical angles: therefore the angle DP*d* is equal to half the sum of the angles APB, BPX, that is to the half of two right angles (I. 2.) or to one right angle. And because DP*d* is a right angle, the point P lies in the circumference of a circle described upon the diameter D*d*. (15. *Cor.* 3.)

And reversely, if P be any point in the circumference of this circle, PA shall be to PB in the given ratio. For, take C the middle point of D*d*, that is, the centre of the circle, and join CP. Then, because AD is to DB as A*d* to *d*B, the line A*d* is harmonically divided in D and B (II. *def.* 19, page 68); and because the harmonical mean D*d* is bisected in C, (II. 46.) CA, CD and CB are proportionals: also, CD is equal to CP: therefore, in the triangles ACP, PCB, AC is to CP as CP to CB; and consequently (II. 32.) the triangles are equiangular. Therefore (II. 31.) PA is to PB as AC to CP, that is, as AC to CD, or (because CA, CD, and CB are proportionals) as AD to DB (II. 22. *Cor.* 1.).

If any point Q be taken within the locus, QA will be to QB in a greater ratio than that of AD to DB; if without it, in a less ratio. For, if AB be divided in E in the ratio of AQ to QB, and if AB produced be divided in the same ratio in *e*; then, joining QE and Q*e*, the angle EQ*e* will be a right angle, as is above shown. And if one of the points E, *e* lie between D and *d*, the other will also lie between D and *d*; for if AE is to EB in a greater ratio than AD to DB, which is the case when E lies between D and *d*, A*e* will be to *e*B in a greater ratio than A*d* to *d*B, which is the case (as may easily be shown) only when *e* lies between D and *d*: and conversely. Therefore, if the point Q be within the locus, and the angle DQ*d* (by consequence) greater than a right angle (15. *Cor.* 3.), that is, than EQ*e*, the point E cannot lie otherwise than between D and *d*; and consequently the ratio of AE to EB, that is, the ratio of AQ to QB, must be greater than the ratio of AD to DB. In the same manner it may be shown that, if the point Q lie without the locus, AQ will be to QB in a less ratio.

Cor. If there be taken in the same straight line, and in the same direction from a common extremity, three harmonical progressionals, and if upon the mean progressional for a diameter, a circle be described, the distances of any point in the circumference from the other extremities of the first and third shall have to one another always the same ratio, viz. that of the first to the third.

Scholium.

The last proposition may be stated thus: "Required the locus of all points P, the distances of which from two given points A and B, are to one another in a given ratio." And it has been shown that the locus is a circle in every case in which the given ratio is not that of equality; and in that particular case it is (44.) a straight line which bisects AB at right angles. Under this form it readily suggests two other questions of the same kind, which likewise lead to plane loci, and are at the same time so elegant and so nearly related to that we have been discussing, that they claim some notice in this place.

First, then, let it be "required to find the locus of all points P such that the distance PA from a given point A, and the tangent PT drawn to a given circle PCD are to one another in a given ratio."

Take E the centre of the given circle; join AE; and, if PA is to be greater than PT, produce AE to F (II. 55.) so that AF may be to FE in the duplicate of the given ratio (*fig.* 1.); but, if PA is to be less than PT, produce EA to F so that AF may be to FE in the duplicate of the given ratio (*fig.* 2.); take EG (II. 52.) a third proportional to EA and ED, and FK a mean proportional (II. 51.) between FA and FG; and, from the centre F with the radius FK describe a circle KPQ: this circle shall be the locus required.

Fig. 1.

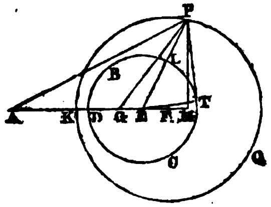
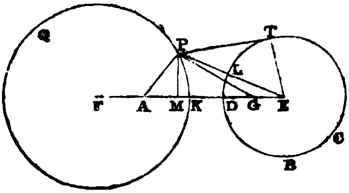


Fig. 2.



For, let P be any point in the circumference of the circle KPQ; join PA, draw the tangent PT to the circle BCD, and join PE, cutting the circumference BCD in L; join also GP, and draw PM perpendicular to AE.

Then, by Prop. 51, because the circle KPQ is described from the centre F with the radius FK, which is a mean proportional between FA and FG, and that P is a point in the circumference KPQ, PA is to PG as AK to KG, or as FA to FK (II. 22. *Cor.* 1.) because FA, FK and FG are proportionals. Therefore also PA^2 is to PG^2 as FA^2 is to FK^2, or as FA to FG (II. 37.). And, because $PA^2 : PG^2 :: FA : FG$,

$PA^2 \times EG : PG^2 \times AE :: FA \times EG : FG \times AE$ (*Rule* 2. *Scholium*, II. [28].) Therefore, *convertendo* $PA^2 \times EG : PG^2 \times AE - PA^2 \times EG :: FA \times EG : FG \times AE - FA \times EG$. (*a*) But, because $PA^2 = PE^2 + AE^2 \pm 2\,AE \times EM$ (I. 37.)

$PA^2 \times EG = PE^2 \times EG + AE^2 \times EG \pm 2AE \times EM \times EG$; and, for the like reason,

$PG^2 \times AE = PE^2 \times AE + EG^2 \times AE \pm 2AE \times EM \times EG$; therefore $PG^2 \times AE - PA^2 \times AG = PE^2 \times AE - PE^2 \times EG, + EG^2 \times AE - AE^2 \times EG$, that is, $= PE^2 \times AG - AG \times AE \times EG$.

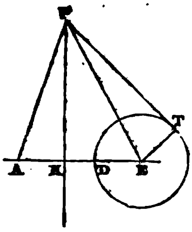
Again, because FG is equal to $FE \pm EG$, $FG \times AE$ is equal to $FE \times AE \pm EG \times AE$, that is, to $FE \times AG + FE \times EG \pm EG \times AG \pm EG^2$, because AE is equal to $AG + EG$; and, in like manner, because FA is equal to $FE \pm EG \pm AG$, $FA \times EG$ is equal to $FE \times EG \pm EG^2 \pm EG \times AG$; therefore, $FG \times AE - FA \times EG$ is equal to $FE \times AG$.

Therefore, substituting these values instead of the second and fourth terms of the proportion, (*a*),

$PA^2 \times EG : PE^2 \times AG - AG \times AE \times EG :: FA \times EG : FE \times AG$, and hence, (*Rule* 2. *Scholium*, II. [28]).

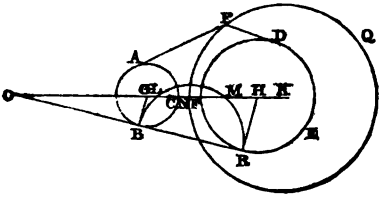
$PA^2 : PE^2 - AE \times EG :: FA : FE$, that is, because $AE \times EG$ is equal to ED^2 (II. 38. *Cor.* 1.), and $PE^2 - ED^2$ is equal to $PE^2 - ET^2$ or PT^2, $PA^2 : PT^2 :: FA : FE$. Therefore PA is to PT in the subduplicate ratio of FA to FE, that is in the given ratio; and the circumference KPQ is the locus required.

If the given ratio be the ratio of equality, the difference of the squares of PA, PE will be equal to the square of ED; and therefore the locus is a straight line (49.) cutting AE at right angles, and may be determined as in Prop. 49.



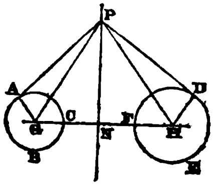
Secondly, let it be "required to find the locus of all points P such that the tangent PA, drawn to a given circle ABC, and the tangent PD drawn to a second given circle DEF, may be to one another in a given ratio.

Take G, H the centres of the two circles; join G H, and produce it on the side of the lesser circle to O so that OG may be to OH as the radius GC to the radius HF (II. 55.): upon OG describe a semicircle cutting the circle ABC in B, so that OB and BG being joined may be perpendicular (15. *Cor.* 1.) to one another, and therefore OB a tangent at B (2.); and from H to OB (produced if necessary) draw the perpendicular H R. Then, because HR is parallel to GB (I. 14.) HR is to GB as OH to OG (II.30.*Cor.*2.), that is, as HF to GC; and, because GB is equal to GC, HR is equal to HF. Therefore R is a point in the circle DEF; and because ORH is a right angle, OB touches the circle DEF in R. Upon BR describe the semicircle BLMR cutting GH in the points L and M; and if PA is to be greater than PD, produce GH to K (II. 55.) so that GK may be to HK in the duplicate of the given ratio (as in the figure); but, if PA is to be less than PD, produce HG to K (II. 55.), so that GK may be to HK in the duplicate of the given ratio; take KN (II. 51.) a mean proportional between KL and KM, and from the centre K with the radius KN describe the circle NPQ; this circle shall be the locus required.



For, if P be any point in the locus, and if the tangents PA and PD be drawn, and PL, PM joined, PA^2 will be to PD^2 in the ratio which is compounded of the ratios of PA^2 to PM^2, PM^2 to PL^2, and PL^2 to PD^2. (II. *def.* 12.) But the circle NPQ stands related to each of the circles ABC, DEF, with the corresponding points M, L, in the same manner in which KPQ is related to BCD in the preceding locus. For, with regard to ABC, because GB is perpendicular to BR, which is the diameter of the semicircle BLMR, GB touches the semicircle (2.) at B, and therefore (21.) GB^2 is equal to GL × GM, that is, GL a third proportional to GM and GB, or GC; and KN was made a mean proportional between KL and KM. And, in the same manner, with respect to the other circle DEF, HM is a third proportional to HL and HF; and KN was made a mean proportional between KL and KM. Therefore, by the last question, PA^2 is to PM^2 as KG to KM; and PL^2 is to PD^2 as KL to KH. Also, because KL, KN and KM are proportionals, PM^2 is to PL^2 as KM to KL. (51.) Therefore the ratio of PA^2 to PD^2 is compounded of ratios which are the same with the ratios of KG to KM, KM to KL, and KL to KH, that is, it is the same with the ratio of KG to KH (II. 27.); and PA is to PD in the subduplicate ratio of KG to KH, that is in the given ratio. Therefore the circle NPQ is the locus required.

If the given ratio be the ratio of equality, the difference of the squares of PG, PH will be equal to the difference of the squares of GC and HF; and therefore the locus is a straight line (49.) cutting GH at right angles, and may be determined as in Prop. 49.

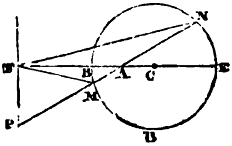


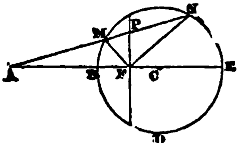
The first of the two loci we have thus discussed is manifestly the same which satisfies the condition that A and E being two given points, and ED^2 a given square, PA^2 shall be to PE^2-ED^2 in a given ratio: and the second, the same which satisfies the condition that G and H being two given points, and GC^2 and HF^2, two given squares, PG^2-GC^2 shall be to PH^2-HF^2 in a given ratio.

Prop. 52.

A point A *being given within or without a circle* B D E, *and to every* *chord* M N *which passes through it, a second point* P *being taken such that the chord produced may be divided by these two points and the circumference harmonically; it is required to find the locus of the points* P.

Let C be the centre of the circle, and let C A, produced if necessary, meet the circle in B: take C F a third proportional to CA, CB, (II. 52.) and join FP, FM, FN. Then, because CF, CB, CA are proportionals, and that CE is equal to CB, the straight lines EA, EB, EF are in harmonical progression (II. 46.). And, because upon the mean EB, the circle EDB is described (51. *Cor.*), AM is to M F as AN to NF. Therefore, *alternando*, (II. 19.) A M : A N : : M F : N F. But by the supposition that P is a point of the required locus, MP : P N :: AM : AN, that is, :: MF : NF: therefore, in the triangle F M N, both the base M N and the base produced are divided in the ratio of the sides.





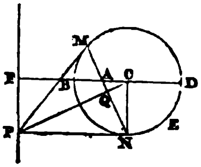
Consequently, as was shown in the like case in the demonstration of the last proposition, the angle AFP is a right angle, and the point P lies in a straight line drawn from the point F perpendicular to C F. It is easy to reverse the reasoning, and to show that every point in this straight line satisfies the given condition. Therefore this straight line is the locus required.

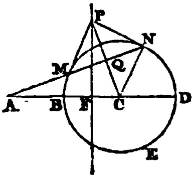
Cor. If the diameter of a circle, and the diameter produced, be divided in the same ratio, or, which is the same thing, (II. 45 *Cor.*) if the diameter produced be divided harmonically, any chord which passes through one point of division shall be divided harmonically by the circumference and the perpendicular to the diameter which is drawn through the other point.

PROP. 53.

A point A *being given within or without a circle* B D E, *and at the extremities of every chord* M N *which passes through it, tangents being drawn intersecting in* P; *it is required to find the locus of the points* P.

Let C be the centre of the circle, and let C A, produced if necessary, meet the circumference in B: take C F a third proportional to C A, C B: join P F, P C, CM, CN, and let P C cut M N in Q.





Then, because P M is equal to P N (2. *Cor.* 3.), and CM to CN, MN is bisected by PC at right angles (3. *Cor.* 3.). And, because C N P is a right angled triangle (2.), and that from the right angle N, a perpendicular N Q is drawn to the hypotenuse, the rectangle under CQ, CP is equal (I. 36. *Cor.* 2.) to the square of CN, that is, to the square of CB, or to the rectangle CA, CF (II. 38. *Cor.* 1.). Therefore (II. 38.) CQ is to CA as CF to CP, and (II. 32.) the triangle CFP is equiangular with the triangle C Q A. Therefore C F P is a right angle, and the point P is in a straight line drawn through the point F perpendicular to C F. It is easy to reverse the reasoning, and to show that every point in this straight line satisfies the given condition. Therefore this straight line is the locus required.

Cor. If the diameter of a circle, and the diameter produced, be divided in the same ratio, or, which is the same thing, (II. 45 Cor.) if the diameter produced be divided harmonically, and if tangents be drawn at the extremities of any chord passing through one of the points of division, they shall intersect one another in the perpendicular to the diameter which is drawn through the other point.

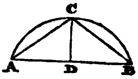
SECTION 7.—*Problems.*

PROP. 54. Prob. 1. (EUC. iii. 30.)

To bisect a given circular arc A C B.

Let C be the required point of bisection: take D the middle point of

AB, and join CA, CB, CD. Then, because the arc CA is equal to the arc CB, the chord CA (12. *Cor.* 1.) is equal to the chord CB: and, because the triangles CDA, CDB have the three sides of the one equal to the three sides of the other, each to each, the angle CDA is equal to CDB: therefore, the line CD bisects AB at right angles. Therefore reversely, bisect the chord AB at right angles by the straight line CD, and the given arc ACB will be bisected in C.



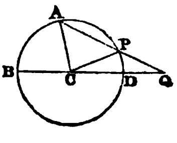
Therefore, &c.

Cor. Hence, a given arc may be divided into 4, 8, 16, &c. equal parts.

Scholium.

The practicability of a geometrical division of a circular arc into any number of equal parts, implies that of the angle at the centre (12.) into the same number of equal parts; and *vice versâ.* It has already been stated that in the cases of 3, 5, &c., equal parts, the division of the angle cannot be effected by a plane construction; and the same is to be understood of the circular arc (I. 46. *Scholium*). We may observe that the problem of *trisecting an arc* has been put under the following form, which gives it an appearance at first of being much easier than upon examination it is found to be.

"From a given point A in the circumference of a given circle ABD to draw a straight line APQ such that the part PQ, which is intercepted between the circumference and a given diameter BD produced, shall be equal to the radius CA."

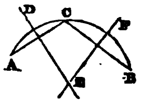


For, if this be done, the arc PD will be found, which is a third of the given arc AB; because, PQ being equal to PC, the angle PCQ is equal (I. 6) to the angle PQC, and therefore, the angle CPA or CAP is equal to twice PCQ (I. 19.), and ACB, which (I. 19.) is equal to CAP and PQC together, is equal to three times PCQ, that is, the arc AB is equal to three times the arc PD (13.)

PROP. 55. Prob. 2. (EUC. iii. 1.)

To find the centre of a given circular arc A C B.

Since the straight line which bisects a chord at right angles passes through the centre of the circle, two such straight lines will cut one another in the centre. Therefore, in the arc A C B take any point C; join A C, C B; and bisect A C, C B at right angles by the straight lines D E, F E: the point E in which they cut one another is the centre of the arc A C B. (See also Prop. 44.)

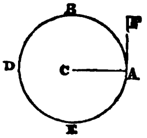


Cor. (Euc. iii. 25.) Hence, any arc of a circle being given, the circumference may be completed of which it is a part.

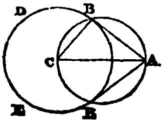
PROP. 56. Prob. 3. (EUC. iii. 17.)

From a given point A, *to draw a tangent to a given circle* B D E.

1. If the point A be in the circumference of the circle, find the centre C (55.), join C A, and from A draw A F perpendicular to C A. Then, because A F is drawn perpendicular to the radius at its extremity, it touches the circle (2.).



2. If A do not lie in the circumference, let the line A B be assumed as the required tangent. Find the centre C, and join C A, C B. Then, because AB is a tangent, the angle CBA is a right angle (2. *Cor.* 1.), and the point B lies in the circumference of a circle of which A C is diameter (15. *Cor.* 3.) Reversely, therefore, upon A C as a diameter describe a circle cutting the given circle in B, and join A B: A B is the tangent required.

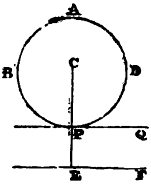


We may observe that in this case there are two points of intersection B, and therefore two tangents. The same may be said, indeed, of the former case; but there the two touch in the same point, and are parts of the same straight line.

PROP. 57. Prob. 4.

To a given circle A B D *to draw a tangent, which shall be parallel to a given straight line* E F.

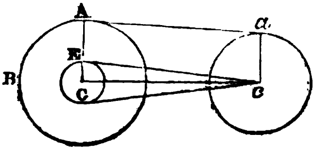
Find the centre C (55.); from C draw CE perpendicular to EF (I. 45.), and let C E, produced if necessary, cut the circumference in P. Then, if PQ be drawn parallel to EF, it will be at right angles to the radius C P (I. 14.); and therefore (2.) will be the tangent required.



Prop. 58. Prob. 5.

To draw a straight line, which shall touch each of two given circles A B D, *a b d.*

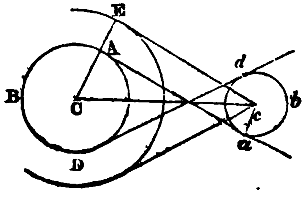
Let A *a* be assumed as the common tangent required: and first, let it touch the circles in the points A, *a* which lie upon the same side of the line joining their centres C, *c.*



Join C A, *c a*; and from *c*, the centre of the lesser circle, draw *c* E parallel to A *a* to meet C A in E. Then, because C A *a* (2. *Cor.* 1.) is a right angle, C E *c* is likewise a right angle (I. 14.), and C E will touch in the point E the circumference of a circle, described from the centre C, with the radius C E, which is equal to the difference of CA, EA, that is (I. 22.) to the difference of the radii C A, *c a.*

Therefore, reversely, from the centre C with a radius equal to the difference of the radii C A, *c a*, describe a circle, and from the point *c* draw a straight line (56.), touching this circle in the point E: join CE, and produce it to meet the circumference ABD in A; draw *c a* parallel to C A, and join A *a*: A *a* is the common tangent required. When the circles are equal, C A and *c a* must be drawn, each of them, perpendicular to C *c*. In this case the common tangent A *a* is evidently parallel to C *c*.

2. If the points of contact are to lie upon opposite sides of the line joining the centres, the like reasoning will lead to a construction which differs from that of the first case in this only, that C E is equal to the sum of the radii.



We may remark that in each of the cases there are two tangents *c* E, and two common tangents parallel to them respectively.

Prop. 59. Prob. 6.

To describe a circle, which shall

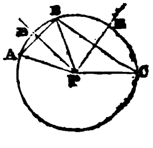
1. (Euc. iv. 5.) *pass through three given points not in the same straight line; or*

2. *pass through two given points, and touch a given straight line; or*

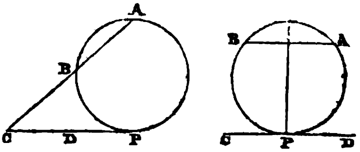
3. *pass through a given point, and touch two given straight lines; or*

4. (Euc. iv. 4.) *touch three given straight lines not parallels.*

1. Let A, B, C be the three given points, and let the point P be assumed for the centre of the required circle. Then, because P is equidistant from A and B, it is in the straight line which bisects A B at right angles (44.); and for a similar reason, it is in the straight line which bisects BC at right angles. Therefore, reversely, to find the point P, bisect AB, BC at right angles (I. 43. *Cor.*) by the straight lines DP, EP, which intersect one another in P: and, from the centre P with the radius PA, describe a circle; it shall pass through the points B, C, and shall be the circle required.



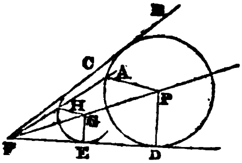
2. Let A, B be the two given points, and C D the given straight line. Suppose the circle to be described, and that it touches C D in P: also, let A B produced meet C D in C. Then, because C P touches, and C A cuts the circle, the square of C P is equal (21.) to the rectangle under C A, C B. Therefore, reversely, take C P a mean proportional between C A, C B, and describe a circle through the points A, B, P; it shall be the circle required.



We may remark that in this case, there may be described two circles, each of which passes through the two given points A, B, and touches the given straight line C D, viz. one for a point P upon the right hand of C, and another for a point P upon the left hand of C.

If A B be parallel to C D, the point P will be in the straight line which bisects A B at right angles, (3. *Cor.* 1.) and being found accordingly, the circle may be described as before.

3. Let A be the given point, and B C, D E the given straight lines. Produce B C, D E to meet one another in F: join F A: draw F G bisecting the angle B F D (I. 46.): in F G take any point G, and from G draw G E (I. 45.) perpendicular to F D: from the centre G with the radius G E describe a circle cutting F A in H: join H G, and through A draw A P parallel to H G to meet F G (produced, if necessary), in P (I. 48.): the circle described from the centre P with the radius P A shall be the circle required.

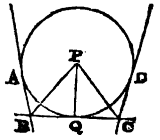


For, if P D be drawn from the point P perpendicular to F D, P D will be parallel to G E (I. 14.), and, therefore, (II. 30. *Cor.* 2.) GE will be to PD as GF to P F: but, because G H is parallel to P A, G F is to P F as G H to P A: therefore (II. 12.) G E is to P D as G H to P A: and in this proportion the first term G E is equal to the third G H; therefore also (II. 18.) P D is equal to P A. Therefore, the circle which is described from the centre P with the radius P A passes through the point D; and it touches the line D E in that point, because P D is at right angles to D E (2.): therefore (45.) it also touches the line B C.*

The case in which B C is parallel to D E differs from the foregoing in this only, that F G must now be drawn parallel to B C or D E, and bisecting the distance between them (See 45.).

In both cases we may observe that there are two circles which satisfy the given conditions, corresponding to the two points in which the circle which is described from the centre G with the radius G E cuts the line F A.

4. Let A B, B C, C D be the three given straight lines, of which not more than two are parallel, and let these two be cut by the third in the points B, C.



Assume the point P for the centre of the circle. Then, because P is equidistant from A B, B C, it is in the straight line B P which bisects the angle ABC (45.); and, for a similar reason, it is in the line P C which bisects the angle B C D. Therefore, reversely, to find P, bisect the angles at B and C by straight lines meeting in P: from P draw P Q perpendicular to A B (I. 45.), and from the centre P with the radius P Q describe a circle: it shall be the circle required.

If A B, C D are parallel, two circles (and two only) can be described, each touching the three given lines: but if no two of the straight lines be parallel, four circles may be described which shall satisfy this condition, viz. one within the triangle included by the given lines, and three others touching the sides of that triangle externally.

Scholium.

The problem of describing a circle about a given triangle (Euc. iv. 5.) belongs to the first case, that of inscribing a circle within a given triangle (Euc. iv. 4.) to the last case of this proposition. The second and third cases are modified by supposing a point and a tangent passing through it to be of the data. Thus, the second becomes "to describe a circle, which shall pass through two given points, and touch a given straight line in one of those points," and the third "to describe a circle which shall touch two given straight lines, and one of them in a given point." The modified solutions corresponding are too simple to detain us here: that of the first occurs in prob. 7.

Instead of touching one, two, or three given straight lines as in the problem we have just considered, it may be required to describe a circle which shall touch one, two, or three given circles, or which shall touch both a straight line and a circle, or two straight lines and a circle, or a straight line and two circles, and at the same time pass through one or two given points, as the other data may happen to admit. The six new problems of contact which are thus suggested are too remarkable to be passed over without further notice; they are accordingly here subjoined.

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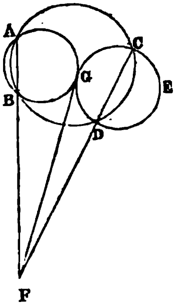
* When the point A is in F G, or in FG produced, the solution here given must be modified by joining H E, drawing A D parallel to H E, and erecting D P perpendicular to F D; which gives the centre P as before. When A is in one of the given lines, as BC, the solution takes a still more simple form.

1. To describe a circle which shall pass through two given points A, B, and touch a given circle C D E.

In the circumference C D E take any point C, and describe (59.) a circle which may pass through the three points A, B, C; then, if this circle meet the circumference C D E in no other point, it is the circle required: but if it do, let that other point be D: join A B, C D, and let them be produced to meet in F: from F draw F G, touching the circle C D E (56.): let G be the point of contact, and describe a circle (59.) passing through the three points A, B, G. Then, because the chord C D of the circle C D E meets the tangent G F in F (21.), the square of G F is equal to the rectangle C F, F D, that is, to the rectangle A F, F B (20.): therefore (21. *Cor.*) G F touches also the circle A B G, and, consequently, the circle A B G touches the circle C D E (2. *Cor.* 2 and 9.), and is the circle required.



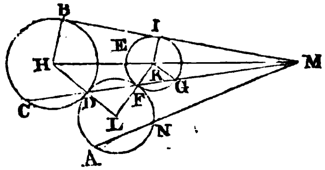
If A B and C D be parallels (which will be the case when the line which bisects A B at right angles passes through the centre of the circle C D E), a tangent F G is to be drawn parallel to A B or C D (57.), and the circle A B G, being then described as before, will be the circle required.

It is evident, in each case, that there are two tangents F G, and two circles A B G corresponding to them, one touching the given circle externally, the other internally.

Cor. The problem which requires a circle to be described which shall touch a given straight line in a given point, and also a given circle, may be solved after a similar manner; viz. by describing a circle which shall touch the given straight line in the given point, and likewise pass through a point assumed in the circumference of the circle, and then proceeding as in the proposition.

2. To describe a circle which shall pass through a given point A, and touch two given circles B C D, E F G.

Suppose that the required circle is described, and that it touches the given circles in the points D, F respectively: join D F, and since the straight line D F cannot touch the given circles in the points D, F, (because then it would touch the required circle A D F in the same points, which (1. *Cor.* 2.) is absurd) let it be produced both ways to meet the circumferences a second time in the points C, G respectively: take H, K, the centres of the circles B C D, E F G, and join H K, H D, K F, K G: then, because the circle A D F touches B C D in D, and E F G in F, the radii H D, K F produced (8. *Cor.* 1.) will meet in its centre, L; and, because K G F, L F D are isosceles triangles, the angle K G F is (I. 6.) equal to K F G, that is (I. 3.) to L F D, that is again (I. 6.) to L D F or (I. 3.) H D C: therefore (I. 15.) K G is parallel to H D, and, consequently, if the circles B C D, E F G be equal, H K, D G will be parallel (I. 21.), or if one of them, as B C D, be greater than the other, H K and D G will meet, if produced in some point M. In the latter case, draw M B touching the circle B C D in B (56.), join H B, and from K draw K I (I. 45.) perpendicular to M B, and therefore (I. 14.) parallel to H B: then, because K I, H B are parallel, K I : H B :: K M : H M (II. 30. *Cor.* 2.) that is, :: K G : H D, because K G, H D are parallel: but H B is equal to H D; therefore (II. 18.) K I is equal to K G, and I is a point in the circumference of the circle E F G, and (2.) M B touches the circle E F G in the point I. Now, M I : M B :: M K : M H (II. 29.), that is, :: M G : M D; therefore, *alternando* (II. 19.), M I : M G :: M B : M D: but, because M I touches the circle E F G, the square of M I is



equal to the rectangle M F × M G (21.), and consequently (II. 38. *Cor.* 1.) M F : M I :: M I : M G: therefore (II. 12.) M F : M I :: M B : M D, and (38.) the rectangle M I × M B is equal to the rectangle M F × M D. Join M A, and let it cut the circumference of the circle A F D a second time in the point N: then the rectangle M A × M N is equal (20.) to the rectangle M F × M D, that is, by what has been just demonstrated, to the rectangle M I × M B.

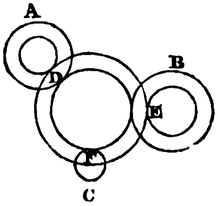
Therefore, reversely, to solve the problem, draw the common tangent B I to the two given circles (58.), and produce it to meet the line H K which joins their centres in M: join M A: take M N a fourth proportional (II. 53.) to M A, M B, M I, and, as in (1), describe the circle A N D, passing through the two points A, N and touching the circle B C D: A N D will be the circle required. For if M D be joined, cutting the circle E F G in F and G, and if K F and K G be joined, the circle described will (20. *Cor.*) pass through F, because (as has been shown above) M D × M F is equal to M B × M I, that is to M A × M N; and (which has been likewise shown) K G will be parallel to H D. Therefore, if in H D produced (which (8. *Cor.* 1.) passes through the centre of the circle A N D, because it passes through the centre H of B C D, which touches A N D in D), there be taken L, the centre of the circle A N D, and L F be joined, the angle L F D will be equal (I. 6.) to L D F, that is (I. 3.) to H D C, that is, again (I. 15.) to K G F, or (I. 6.) K F G; and because D F G is a straight line, K F and F L likewise lie in a straight line (I. 3.), and therefore the circle A N D touches the circle E F G in F (9.).

Should the given circles be equal, or should the point N coincide with the point A, the foregoing solution must be modified accordingly. The latter case is provided for by the corollary of the preceding problem: both, indeed, are comparatively easy, and are therefore left to the student.

It remains to observe, that we have only considered the case in which the required circle touches both of given circles externally: since both, however, may also be touched internally, or one internally and the other externally by the same circle, there are evidently four different circles which satisfy the conditions of the problem. For each of these a different construction is required, to which the reader will be easily conducted by an investigation similar to that which has been already given.

3. To describe a circle which shall touch three given circles A, B, C.

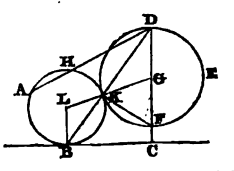
Let C be that one of the three which is not greater than either of the other two. From the centres of A and B describe two new circles with radii less than their own respectively by the radius of C; and, as in the last problem, describe a circle D E F which shall pass through the centre F of the circle C, and touch these two new circles in the points D, E respectively: then, if a circle be described concentric with the circle D E F, with a radius less than that of D E F by the radius of C, it will evidently touch each of the three given circles (10. *Cor.* 2.), and will be the circle required.



In the case here considered, the required circle touches each of the given circles externally: since, however, a circle may be described which shall touch all three internally, or any two externally, and the third internally, or again, any two internally and the third externally, there are no less than eight different circles which satisfy the problem: the construction of each of these is obtained after a similar manner.

4. To describe a circle which shall pass through a given point A, touch a given straight line B C, and also touch a given circle D E F.

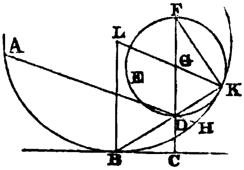
Take G the centre (55.) of the circle D E F, and through G draw (I. 45.) the diameter D F perpendicular to B C to meet B C in C: join A D, and take D H a fourth proportional (II. 53.) to A D, D C and D F, *i. e.* (II. 38.) such that A D × D H may be equal to D C × D F; and describe a circle H A B (59.) passing through the two points A, H,



and touching the straight line B C: this shall be the circle required. For, let it touch B C in B; join B D, and let it cut the circumference D E F in K, and join K F: then, because (15. *Cor.* 1.) D K F is a right angle, the triangles D K F, D C B are equiangular, and (II. 31.) D K : D F :: D C : D B, or (38.) D K × D B is equal to D C × D F, that is, to A D × D H; therefore K is also a point in the circumference of the circle A B H (20. *Cor.*). Now take L the centre of the circle A B H (55.) and join L B, L K, G K: and, because L B is (2. *Cor.* 1.) perpendicular to B C, it is (I. 14.) parallel to D C, and therefore (I. 15.) the angle L B K is equal to G D K; but L K B is equal to L B K, and G K D to G D K, because L B is equal to L K, and G D to G K (I. 6.); therefore the angle L K B is equal to G K D, and (I. 3.) L K G is a straight line: therefore the circle A B H touches (9.) the circle D E F in the point K, and is the circle required.

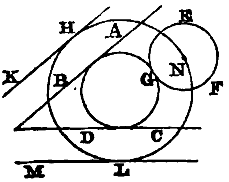
If D H be equal to D A, a circle A B H is to be described touching D A in A, and also touching the straight line B C. (See the beginning of this Scholium.)

In this problem there are two circles satisfying the conditions: the construction of that which touches the given circle internally will readily be understood by applying what has been said to the subjoined figure.



5. To describe a circle which shall touch two given straight lines A B, C D, and also a given circle E F G.

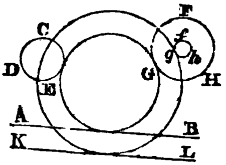
Draw H K parallel to A B, and at a distance from it equal to the radius of E F G: draw also L M parallel to and at the same distance from C D (I. 48.): take N the centre of the circle E F G (55.): and (59.) describe the circle H L N passing through the point N and touching the straight lines H K, L M: then, if a circle be described concentric with H L N, and with a radius less than that of H L N by the radius of E F G, it will evidently touch the straight lines A B, and C D, and the circle E F G (10. *Cor.* 2.) and will therefore be the circle required.



Four different circles may be described to satisfy this problem, viz. two touching the given circle externally, and two internally:[*] the latter two may be found by drawing H K and L M between A B and C D.

6. To describe a circle which shall touch a given straight line A B, and two given circles C D E, F G H.

If the given circles be equal to one another, draw K L parallel to A B, at a distance from it equal to their common radius (I. 48.), and (59.) describe a circle passing through the centres of the two given circles and touching the line K L: then it is evident that a circle concentric with the latter, and having a radius less by the common radius of the given circles, will be the circle required. But, if one of them, as F G H, be greater than the other, describe about its centre a circle *f g h* with a radius less than that of F G H by the radius of the lesser circle C D E: draw K L parallel to A B at a distance from it equal to the radius of C D E; and describe a circle passing through the centre of C D E touching the line K L, and touching also the circle *f g h*, as shown in the last problem but one: then, if a circle be described concentric with this and with a radius less by the radius of C D E, it will evidently be the circle required.



Here, also, there are four different circles satisfying the conditions of the problem; and the same principle leads to the construction of them all.

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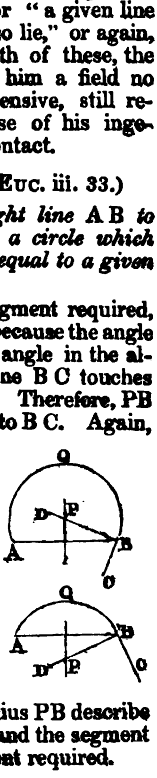
[*] *External* contact is where two circles touch one another, and each of them is without the other; *internal* contact, where two circles touch one another, and one of them is within the other. In the latter case, it is evident that the circumference of the larger circle is without the circumference of the other; and yet each of them is properly said to touch the other *internally*, meaning, with that description of contact which is called internal.

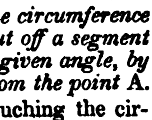
Much as each of the foregoing problems may be varied in appearance by the consideration of internal contact, they admit of being varied yet further by supposing, when a point is given, that it is in the given tangent or circumference: and when it is besides suggested that, in each of the ten problems furnished by the proposition and scholium, one of the *data* be changed for "a given radius," or for "a given line in which the centre is to lie," or again, two of the data for both of these, the student will have before him a field no less pleasing than extensive, still remaining for the exercise of his ingenuity, in problems of contact.

Prop. 60. Prob. 7. (Euc. iii. 33.)

Upon a given straight line A B *to describe a segment of a circle which shall contain an angle equal to a given angle* A B C.

Let A Q B be the segment required, and P its centre. Then, because the angle A B C is equal to the angle in the alternate segment, the line B C touches the circle (17. *Cor.*) at B. Therefore, PB joined is at right angles to B C. Again, because P is the centre of a circle having the chord A B, it is in the straight line which bisects A B at right angles (3 *Cor.* 2). Therefore, reversely, draw B D perpendicular to B C (I. 44.), and bisect A B at right angles (I. 43. *Cor.*) by a straight line, cutting BD in P: with the centre P and radius PB describe the circular arc B Q A: and the segment B Q A will be the segment required.

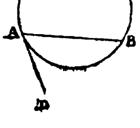




Prop. 61. Prob. 8. (Euc. iii. 34.)

Given a point A *in the circumference of a circle* A B C; *to cut off a segment which shall contain a given angle, by a straight line drawn from the point* A.

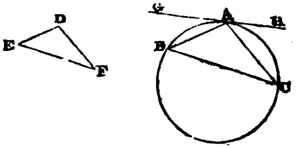
From A draw A D touching the circle, and let A B be the line required: then, because the angle B A D (17.) is equal to the angle in the alternate segment B C A, it is equal to the given angle. Therefore, reversely, make D A B equal to the given angle (I. 47); and A B will be the line required.



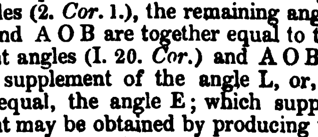
Prop. 62. Prob. 9. (Euc. iv. 2, 3.)

Given a circle A B C, *to* (1) *inscribe in it, or* (2) *describe about it, a triangle similar to a given triangle* D E F.

1. Let A B C be the required inscribed triangle, and through A draw the line G A H touching the circle (56.): then the angle G A B is equal to the angle at C, and the angle H A C to the angle at B (17.) Therefore, reversely, take any point A of the circumference, draw the tangent G A H (56.), and make the angles GAB, HAC equal to the angles at F and E respectively (I. 47.) Then, if A B, A C meet the circumference in the points B, C respectively, and if B C be joined, A B C will be the triangle required; for, the angles at B and C being equal to the angles at E and F, each to each, the third angles at A and D are likewise equal (I. 19. *Cor.* 1.), and therefore the triangles are similar (II. 31. *Cor.* 1.).



2. Let KLM be the required circumscribed triangle, and let its sides touch the circle in the points A, B, C: take O the centre of the circle (55.), and join O A, O B, O C.

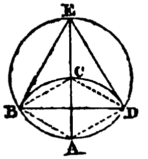


Then, because the angles at A and B of the quadrilateral A L B O are right angles (2. *Cor.* 1.), the remaining angles L and A O B are together equal to two right angles (I. 20. *Cor.*) and A O B is the supplement of the angle L, or, of its equal, the angle E; which supplement may be obtained by producing the side E F. In like manner it may be shown, that the angle B O C is the supplement of the angle M or F. Therefore, reversely, taking any point A in the circumference, at the point O, make (I. 47.) the angle A O B equal to the supplement of E, and the angle BOC equal to the supplement of F: at the points A, B, C draw tangents (56.) meeting one another in K, L, M; and KLM will be the triangle required.

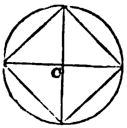
Prop. 63. Prob. 10. (Euc. iv. 6, 7, 11, 12, 15, 16.)

A circle being given; to inscribe in it, or describe about it,
1. *an equilateral triangle; or*
2. *a square; or*
3. *a regular pentagon; or*
4. *a regular hexagon; or*
5. *a regular decagon; or*
6. *a regular pentedecagon.*

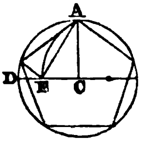
Take C the centre of the circle; and, 1. to inscribe an equilateral triangle: from A, any point of the circumference, with the radius A C, describe a circle cutting the given circle in the points B, D: produce A C to meet the circumference in E, and join B D, D E, E B. Then joining BC, CD, BA, AD, because the triangles ACB, ACD are equilateral, the angles BCA, D C A are each of them a third of two right angles (I. 6. and I. 19.). Therefore the adjacent angles B C E, D C E are each of them two-thirds of the same, as is also the angle B C D; and because the sides of the triangle B D E subtend equal angles at the centre C, they are equal to one another (12. *Cor.* 1.), *i. e.* the triangle B D E is equilateral.



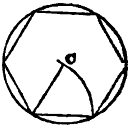
2. To inscribe a square: draw two diameters at right angles to one another, and join their extremities: the included figure will be a square; for its sides are equal, because they subtend equal angles at the centre C (12. *Cor.* 1.), and its angles are right angles, because they are contained in semicircles (15. *Cor.* 1.)



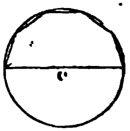
3. To inscribe a regular pentagon: divide the radius C D medially (II. 59) in the point F, so that CF may be the greater segment. Draw the radius C A at right angles to C D, and join A F. Then, because the square of A F is greater than the square of C F by the square of the radius AC, and that CF is the side of the inscribed decagon (28.), AF is the side of the inscribed pentagon. Therefore, a chord, equal to A F, will subtend a fifth part of the circumference, and if the circumference be divided into five parts with chords each equal to AF, a regular pentagon will be inscribed, as required.



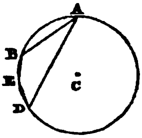
4. To inscribe a regular hexagon: divide the circumference into six parts with chords each equal to the radius (28).



5. To inscribe a regular decagon: divide the radius medially, and divide the circumference into ten parts with chords each equal to the greater segment of the radius so divided (28).



6. To inscribe a regular pentedecagon: take the chord A B equal (as above) to the side of a regular inscribed pentagon, and the chord A D equal (as above) to the side of an inscribed equilateral triangle: bisect the arc BD in E (54.), and join BE. Then, because the arc A B is contained in the whole circumference five times, and A D three times, if the circumference be divided into 5 × 3, or 15 equal parts, A B will contain three and A D five of those parts. Therefore the difference B D contains two of the same parts, and its half B E is one-fifteenth of the whole circumference. Divide the circumference, therefore, into fifteen parts with chords each equal to BE, and a regular pentedecagon will be inscribed, as required.



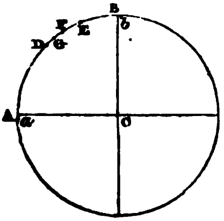
And in every case, if through the angular points of the inscribed figure, or through the bisections of the arcs, (which is sometimes more convenient) there be drawn tangents to the circle (56.), a similar figure will be circumscribed about the circle (27.).

Cor. Hence by the aid of II. 65. any one of the above-mentioned figures may be described upon a given finite straight line.

Scholium.

Besides the figures mentioned in the proposition, it has been discovered that any regular figure which has the number of its sides denoted by $2^n + 1$ and prime, may be inscribed in a circle without any other aid than that of Plane Geometry, that is, by the intersections of the straight line and circle only. And it is evident that by dividing the subtended arcs into two, four, &c. equal parts (54.) a regular figure of twice, four times, &c. the number of sides of any of the above may be inscribed; *e.g.* an *octagon*, (or regular figure of eight sides,) by bisecting the arcs which are subtended by the sides of a square, a *dodecagon* (or regular figure of twelve sides) by bisecting those which are subtended by the sides of a hexagon; and so on. Still there are many regular figures, as the *heptagon*, *enneagon*, *hendecagon*, &c. (figures of 7, 9, 11, &c. sides), for the inscribing of which no exact geometrical rule has ever been discovered. These figures we can only inscribe approximatively; and when it is required to do so with a considerable degree of accuracy, the following method may be adopted.

Let it be required, for instance, to inscribe a regular *heptagon*. Continue the series 4, 8, 16, &c. which represents the numbers of parts into which the circumference may be divided by continued bisections, until a number be found which is greater or less by 1 than a multiple of 7: 64 is such a number, being greater by 1 than 9 × 7.



Now, if the circumference be divided into 64 equal parts, and the arc A G be taken equal to 9 of those parts (which may be done by bisecting the quadrantal arc A B in D, D B in E, D E in F, and D F in G), the arc A G will be less than a seventh part of the circumference by a seventh part of one of them D G. But, the arc D G being small, a seventh part of its chord (which may be found by (I. 49.) may without any considerable error be assumed for the seventh part of the arc itself, being somewhat *less* than the latter; and if the chord of A *a* be taken equal to this approximate seventh part, the error of assuming for it the arc A *a*, which is somewhat *greater* than its chord, will be still less, so that G *a* will be equal, very nearly, to one-seventh of the circumference, and the chord of G *a* very nearly equal to the side of a regular heptagon inscribed in the circle.

For a second example, let us take the *enneagon*, or, as it is sometimes called, *nonagon*: here, again, 64 exceeds 7 × 9 by 1; therefore, the same division of the quadrant being made as in the case of the heptagon, because B G is equal to 7 parts out of the 64, and DG to one part, B G with a ninth of D G will be contained in the circumference 9 times exactly: and if the chord of B *b* be taken equal to the ninth part of the chord of DG, the chord of G *b* will be very nearly equal to the side of a regular enneagon inscribed in the circle.

It is not, however, necessary that we should always proceed with the series till we arrive at a number greater or less than the number of sides by 1. Take, for instance, the *hendecagon*, or, as it is sometimes called, *undecagon*: here 11 × 6 = 66, which exceeds 64 by 2. Now the arc B F contains 6 out of 64 parts D G of the circumference. Therefore, if the circumference be increased by twice DG, BF will be contained in the circumference so increased 11 times, and, consequently, if B F be diminished by two-elevenths of D G, it will be contained in the circumference 11 times exactly; so that the side of the hendecagon will be obtained approximatively by assuming, as before, the chord in place of the arc, and taking from B F two-elevenths of the former instead of two-elevenths of the latter.

In these examples, the real errors, if computed, will be found far more minute than those which the imperfection of our instruments entails upon the most accurate geometrical constructions. Seven times the arc which has been assumed as a seventh of the circumference falls short of the whole circumference by less than the $\frac{1}{150000}$th part, and 9 times the arc which has been assumed as a ninth, by about an equal quantity; while 11 times the arc, which has been assumed as an eleventh, exceeds by only about twice the same quantity. The method may therefore be adopted in these and in similar cases, as practically accurate.

Prop. 64. Prob. 11.

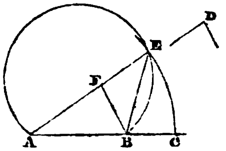
To construct a triangle, any three being assumed out of the four following data, viz. the vertical angle, the base, the sum of the sides, and the area.

This problem comprehends four cases, in which the data are respectively,

1. Vertical angle, base, and sum of sides;
2. Vertical angle, base, and area;

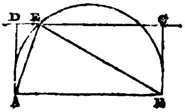
3. Vertical angle, sum of sides and area;

4. Base, sum of sides, and area.

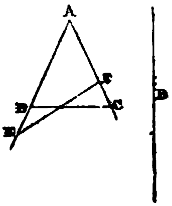


1. Let AB be the given base, AC the given sum of the two sides, and D the given vertical angle. Upon A B (60.) describe a segment A E B, containing an angle equal to the half of D. With the centre A and radius A C describe a circle cutting the arc A E B in E. Join A E, B E, and at the point B make the angle EBF equal to BEA: the triangle F A B shall be the triangle required. For, because the angle F E B is equal to F B E, the side F B is equal to FE (I. 6.), and the two sides AF, FB together are equal to AE, that is, to AC, the given sum of the sides. Again, because the angle A F B (I. 19.) is equal to the sum of the angles at E and B, and that these angles are equal to one another, the angle A F B is equal to twice the angle at E, that is, to the given angle D. And the triangle is described upon the given base A B.

Therefore, &c.



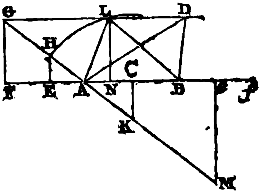
2. Let A B be the given base, upon which let there be described the rectangle A B C D, containing an area equal to twice the given area (I. 57.), and a segment AEB containing an angle equal to the given angle (60.). Then if the arc A E B cut the side C D in E, and EA, E B be joined, E A B will evidently be the triangle required.



3. Let A be the given vertical angle, and let the triangle ABC (II. 69.) contain an area equal to the given area: and let D be the given sum of the two sides. Divide D into two parts, such that their rectangle may be equal to the rectangle under AB, AC (II. 56.). Take AE equal to one, and AF equal to the other of these parts, and join E F. Then, because the triangles ABC, AEF have the common angle A, they are to one another (II. 40. *Cor.*) as the rectangles under the containing sides, that is, they are equal to one another; therefore AEF is the triangle required.

4. Let A B be the given base, and let it be bisected in C. Let twice C E be the given sum of the sides, and let the triangle D A B contain an area equal to the given area.

Take CF (II. 52.) a third proportional to CA, CE, and from the point F draw FG perpendicular to E F (I. 44.): through D draw DG parallel to AB (I. 48.) to meet FG in G, and join G A. Through E and C draw E H and C K parallel to F G, to meet GA and GA produced in the points H, K, so that K G, K H, K A will be proportionals (II. 29.): from the centre K, with the radius KH, describe a circle cutting G D in L, and join L A, L B: L A B shall be the triangle required.



Produce G K to M, so that K M may be equal to K H; and from L draw LN perpendicular to A B. Then, because K A, K H, K G are proportionals, M A, MH, MG are in harmonical progression (II. 46.); but the point L is in the circumference of a circle upon the mean MH; therefore (51. *Cor.*) LA : LG :: AH : HG; but LG is (I. 22.) equal to NF, and AH : HG :: AE : EF (II. 29.), that is, since CA, CE, CF are proportionals (II. 22. *Cor.* 1.) :: CA : CE; therefore LA is to NF as CA to CE, or in the subduplicate ratio of CA to CF; and (II. 38. *Schol. Lem.* 1. *Cor.*) if B *f* be taken equal to A F, L B is to N *f* in the same ratio. Therefore the sum of L A, L B is to the whole line F *f* in the same ratio, or, if C *e* be taken equal to CE, as E *e* to F *f*: therefore, the sum of L A, L B is equal (II. 11. *Cor.* 1.) to E *e*, that is, to the given sum, and L A B is the triangle required.

Therefore, &c.*

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* If the difference of the sides be supposed given instead of the sum in cases 1, 3 and 4, solutions of the same character may be obtained; viz. in case 1, by describing upon the given base AB a segment which shall contain an angle exceeding by a right angle half the given angle D; in case 3, by dividing the *difference* D produced, so that the rectangle under the segments may be equal to A B × A C; and in case 4, by making use of the following corollary to

Prop. 65. Prob. 12.

To find two straight lines, there being assumed any two out of the six following data; viz. their sum, their difference, the sum of their squares, the difference of their squares, their ratio, and their rectangle.

The cases of this proposition are fifteen in number, and may be arranged as follows:—

1. Sum, and difference.
2. Sum of squares, and difference of squares.
3. Sum, and sum of squares.
4. Difference, and sum of squares.
5. Sum, and difference of squares.
6. Difference, and difference of squares.

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7. Ratio, and rectangle (II. 63.)
8. Sum, and ratio (II. 55. fig. 1.)
9. Difference, and ratio (II. 55. fig. 2.)
10. Sum, and rectangle (II. 56. fig. 1.)
11. Difference, and rectangle (II. 56. fig. 2.)

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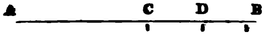
12. Sum of squares, and ratio.
13. Difference of squares, and ratio.
14. Sum of squares, and rectangle.
15. Difference of squares, and rectangle.

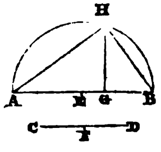
Those of the second division, viz. the 7th, 8th, 9th, 10th, and 11th cases, have in effect been already considered in the propositions referred to at the side. They will accordingly be here omitted. Of the rest the greater part are so obvious, that it will be sufficient to indicate only the construction, leaving the demonstration to the reader.

1. Let AB be the given sum, AC the given difference. Bisect C B in D, and

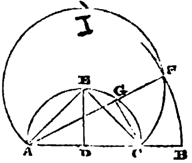


A D, D B will be the straight lines required.

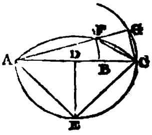
2. Let AB^2 be the given sum of the squares, and CD^2 the given difference of the squares. Bisect AB in E (I. 43.) and C D in F, and take E G a third proportional to E B and C F. From the centre E, with the radius E A, describe a semicircle A H B; and from the point G (I. 44.) draw G H perpendicular to A B, to meet the circumference in H. Join A H, H B: they shall be the straight lines required.



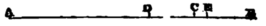
3. Let AB be the given sum, and the square of AC the given sum of the squares. Bisect A C in D, (I. 43.) and from D (I. 44.) draw D E perpendicular to A C. From the centre D with the radius D A or D C describe a circle cutting D E in E; and from the centre E, with the radius E A or E C describe the circle C F A: lastly, from the centre A with the radius A B describe an arc cutting C F A in F. Join A F, let A F cut the circumference A E C in G: and join G C. A G and G C shall be the lines required.



4. Let A B be the given difference, and the square of A C the given sum of the squares. Bisect AC in D (I. 43.), and from D (I. 44.) draw DE perpendicular to AC: from the centre D with the radius DA or DC describe a circle cutting DE in E; and from the centre E with the radius EA or EC describe the circle CFA; lastly, from the centre A with the radius AB describe an arc cutting CFA in F: join AF, and let AF produced cut the circumference AECG in G, and join GC. AG and GC shall be the straight lines required.

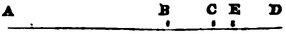


5. Let A B be the given sum, and the square of A C the given difference of the squares: take A D a third pro-



portional to A B, A C (II. 52.), and bisect D B in E (I. 43.); A E, E B shall be the straight lines required.

6. Let A B be the given difference,



and the square of A C the given difference of the squares: take A D a third proportional to A B, A C (II. 52.), and

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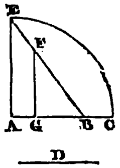
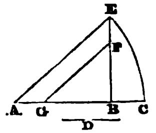
II. 38. Schol. Lem. 2: "If DE be taken to AB as AB∼AC to BC, or, which is the same thing, in a ratio which is the subduplicate of GE to GB, then the other segment DF of the base equally reduced in the opposite direction shall be to the other side AC in the same ratio." The application of these suggestions is left to the student.

bisect B D in E (I. 43.); A E, E B shall be the straight lines required.

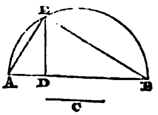
12. Let the given ratio be that of A B to A C, and let the square of D be the given sum of the squares: from A (I. 44.) draw A E at right angles to A B, and from the centre A with the radius A C describe a circle cutting A E in E: join B E: take B F equal to D, and through F (I. 48.) draw F G parallel to E A: G F, G B shall be the straight lines required.



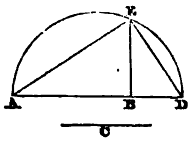
13. Let the given ratio be that of A B to A C, and let the square of D be the given difference of the squares: from B (I. 44.) draw B E at right angles to A B, and from the centre A, with the radius A C, describe a circle cutting B E in E: join A E: take B F equal to D, and through F (I. 48.) draw F G parallel to A E: G F, G B shall be the straight lines required.



14. Let the square of A B be the given sum of the squares, and let the rectangle under A B and C be the given rectangle: divide A B in D (II. 56.), so that the rectangle under AD, DB may be equal to the square of C; from D (II. 56.) draw DE at right angles to AB; and upon AB as a diameter describe a circle cutting DE in E: AE, EB shall be the straight lines required. For it is evident, that the square of ED being, by construction, equal to the rectangle under AD, DB (II. 34. *Cor.*), ED is equal to C; and the rectangle under AE, EB is equal to the rectangle under AB, ED, that is, to the rectangle under AB and C.



15. Let the square of A B be the given difference of the squares, and let the rectangle under A B and C be the given rectangle; produce A B to D (II. 56.), so that the rectangle under A D, D B may be equal to the square of C, from B (I. 44.) draw B E at right angles to A D, and upon A D describe a circle cutting B E in E: A E, E B shall be the straight lines required. For it is evident that the square of ED being (as in the last case) by the construction equal to the rectangle under AD, DB (II. 34. Cor.), ED is equal to C; and, because the triangles ABE, EBD are similar (II. 34.), AB is to AE as EB to ED, and therefore the rectangle under AE, EB is (II. 38.) equal to the rectangle under AB, ED, that is to the rectangle under AB and C.



Scholium.

In the cases of this Proposition we have exhibited geometrical solutions of the following well-known equations.

$$\begin{array}{llll} 1. & x+y=a & 2. & x^2+y^2=a^2 \\ & x-y=b & & x^2-y^2=b^2 \\ 3. & x+y=a & 4. & x-y=a \\ & x^2+y^2=b^2 & & x^2+y^2=b^2 \\ 5. & x+y=a & 6. & x-y=a \\ & x^2-y^2=b^2 & & x^2+y^2=b^2 \end{array}$$

$$\begin{array}{llll} 7. & \frac{x}{y}=\frac{a}{b} & & \\ & xy=c^2 & & \\ 8. & x+y=a & 9. & x-y=a \\ & \frac{x}{y}=\frac{b}{c} & & \frac{x}{y}=\frac{b}{c} \\ 10. & x+y=a & 11. & x-y=a \\ & xy=b^2 & & xy=b^2 \end{array}$$

$$\begin{array}{llll} 12. & x^2+y^2=a^2 & 13. & x^2-y^2=a^2 \\ & \frac{x}{y}=\frac{b}{c} & & \frac{x}{y}=\frac{b}{c} \\ 14. & x^2+y^2=a^2 & 15. & x^2-y^2=a^2 \\ & xy=b^2 & & xy=b^2 \end{array}$$

In the construction of these and other problems of the foregoing Sections, the data have always been supposed such that the problem in question be not impossible. For, as we have already had occasion to observe, many of them are possible, only so long as the mutual relations of the data are confined within certain limits. Thus, if it be required to find two lines, such that their squares may together contain 9 square feet, it is evident that the sum of the lines in question must not be less than 3 feet, nor must their difference exceed 3 feet, (II. 56. N. B.). The solution, therefore, of a problem, which should require the sum of the two to fall short of this quantity, or their difference to exceed it, would be impossible. In this manner does one of the conditions frequently set limits to the other—frequently, but not in every case:—thus, if two lines be required, which shall contain a given rectangle, their ratio may be any whatever, and a problem which should require them to be to one another in any ratio, how great or how small soever, would be possible. The limits of possibility, when there are any, are commonly indicated by the construction, if the problem be solved geometrically, as they are, if algebraically, by the form of the final equation. See the cases of Prop. 64., where the vertex of the triangle sought is determined by the intersection of a straight line and circle, or of two circles: if the data be such that no intersection can take place, the construction fails, and the problem becomes impossible.

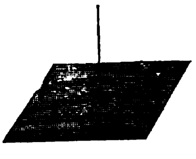
BOOK IV.

§ 1. *Of Lines perpendicular, or inclined, or parallel to planes.*—§ 2. *Of Planes which are parallel, or inclined, or perpendicular to other Planes.*—§ 3. *Of Solids contained by Planes.*—§ 4. *Problems.*

Section 1.—*Of Lines perpendicular, or inclined, or parallel to Planes.*

In the preceding books our attention has been confined to lines which lie in one and the same plane, the intersection of such lines, and the figures contained by them; we are now to consider lines which lie in different planes, planes which intersect one another, and solids which are contained by plane or other surfaces. In other words, we have been hitherto engaged with *Plane* Geometry; we are now to enter upon *Solid* Geometry.

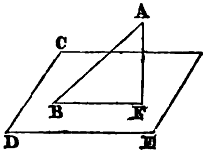
Def. 1. (Euc. xi. def. 3.) A ***straight line*** is said to be *perpendicular* (or *at right angles*) *to a plane*, when it makes right angles with every straight line meeting it in that plane, (see Prop 3.). Also, conversely, in this case the plane is said to be perpendicular to the straight line.



The *foot* of the perpendicular is the point* in which it meets the plane.

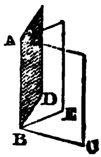
2. (Euc. xi. def. 5.) A ***straight line*** is said to be *inclined to a plane*, when it meets the plane but is not perpendicular to it.

When a straight line A B is inclined to a plane C D E, the angle A B F which it makes with a straight line drawn from the point B in which it meets the plane, through the foot of the perpendicular AF, which is let fall upon the plane from any other point A of the straight line (see Prop. 7.), is called the *angle of inclination.*



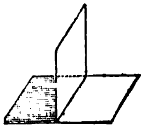
3. A ***straight line*** is said to be *parallel to a plane*, when it cannot meet the plane, to whatever extent both be produced. Also, conversely, in this case the plane is said to be parallel to the straight line.

4. If two planes A B C, A B D intersect one another in a line as A B (see Prop. 2.), they are said to form at that line a *dihedral angle* C A B D.



The magnitude of a dihedral angle does not depend upon the extent of the containing planes, but upon the opening between them. Thus, the dihedral angle C A B D is greater than the dihedral angle E A B D by the dihedral angle C A B E.

5. When one plane standing upon another plane makes the adjacent dihedral angles equal to one another, each of them is called a *right dihedral angle;* and the plane which stands upon the other is said to be *perpendicular* (or *at right angles*) to it.



A dihedral angle is also said to be *acute* or *obtuse*, according as it less or greater than a right angle.

6. (Euc. xi. def. 8.) *Planes*, which do not meet one another, though produced to any extent, are said to be *parallel.*

7. (Euc. xi. def. 9.) If three or more planes pass through a point as A, they are

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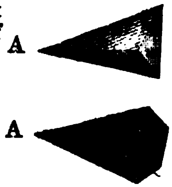
* It is evident that a straight line cannot meet a plane in more than one point, unless it lies altogether in the plane; and in like manner that one plane cannot meet another plane in a portion of surface common to both, unless they coincide altogether. (See Prop. 1.) Therefore a straight line *cuts* a plane in a point; and a plane *cuts* a plane in a line, which line (see Prop. 2.) is a straight line.

said to form at that point a solid angle, of which the intercepted plane angles (see Prop. 2.) are called the *sides* or faces, and the intersections of the planes, *edges*.



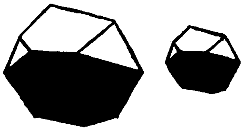
8. A *polyhedron* is a solid figure included by any number of planes, which are called its *faces:* if it have four faces only, which is the least number possible, it is called a *tetrahedron;* if six, a *hexahedron;* if eight, an *octahedron;* if twelve, a *dodecahedron;* if twenty, an *icosahedron;* and so on.*

The intersections of the faces of a polyhedron are called *arrises* or *edges*, and the points of the solid angles *vertices* or angular points. The *diagonals* of a polyhedron are the straight lines which join any two vertices not lying in the same face.

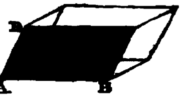
The surfaces of the polyhedrons here treated of are supposed to be convex, that is, such that the same straight line can cut them in two points only.

9. A *polyhedron* is said to be *regular*, when its faces are similar and equal regular polygons, and its solid angles equal to one another. There are only five such figures. (See Prop. 20. *Cor.*)

10. Two *polyhedrons* are said to be *similar*, when they are contained by similar faces similarly situated, and forming equal dihedral and solid angles.†



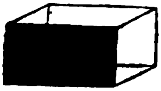
11. (Euc. xi. def. A.) A *parallelopiped* is a solid figure having six faces, of which every opposite two are parallel.



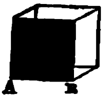
Such a figure may be formed by taking any solid angle A, which is formed by three plane angles, assuming any points B, C, D in the three edges, and passing through those points planes parallel to the planes A C D, A B D, A B C respectively.

The faces of a parallelopiped are sometimes distinguished by naming any two opposite faces the *bases* of the parallelopiped, and the other four the *sides:* in which case the *altitude* of the parallelopiped is the perpendicular distance between the two bases.

12. A *rectangular parallelopiped* is that which has one of its solid angles contained by three right angles, and therefore (see Prop. 17.) every face at right angles to those which are adjoining to it.



13. A *cube* is a rectangular parallelopiped which has the three edges terminated in one of the solid angles equal to one another. The cube of any straight line A B, is the cube of which A B is an edge.



14. (Euc. xi. def. 13.) A *prism* is a solid figure having any number of faces, two of which are similar and equal rectilineal figures, so placed as to have their corresponding sides parallel, and the rest parallelograms.

Such a figure may be formed by

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[*] The Greek word for "seat" being in all cases annexed to the Greek numeral which indicates the number of seats, or faces, on which the figure may be seated. A solid figure may be contained by any number of faces above three, in the same manner as a plane figure may be contained by any number of sides above two, the numbers 4, 6, 8, 12, and 20 being here specified only because the other solids (of 5, 7, &c. faces) are less frequently subjects of consideration.

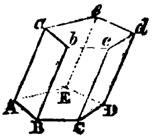
[†] The same observation may be made here as at Book II. def. 14, viz. that in this definition there are some things assumed which have not been as yet demonstrated. These are,

1. If all the plane angles but one which contain two solid angles be equal, each to each, in order, and make with one another equal dihedral angles, the remaining plane angle of the one shall be equal to the remaining plane angle of the other, and the two remaining dihedral angles of the one equal respectively to the two remaining dihedral angles of the other. This may be proved by coincidence.

2. If all the planes but one which form a convex surface be similar and similarly situated to all the planes but one which contain a solid figure, each to each, and if the dihedral angles which every adjoining two of the first make with one another be equal to the dihedral angles which every corresponding two of the latter make with one another, each to each, the remaining edges of the surface (viz. those which are not common to adjoining planes) shall lie in one plane, and shall inclose a rectilineal figure similar to the last face of the solid, and making equal dihedral angles with the corresponding faces adjoining to it. This may be demonstrated by making any two of the equal solid angles coincide. (See Prop. 14. *Cor.*)

It is evident, also, that the definition would be complete without mentioning the equality of the solid angles, for the several plane and dihedral angles of the one being equal and similarly situated with the corresponding plane and dihedral angles of the other, it is evident that any two corresponding solid angles may be said to coincide.

drawing through the several angles of a polygon A B C D E, but not in the same plane with it, the parallels A a, B b, C c, &c., taking A a, B b, C c, &c., each of them equal to any the same straight line, and joining $a\,b$, $b\,c$, &c. (see Prop. 13. and 15.)



The parallels, which join the corresponding angles of the two polygons, are called the *principal edges* of the prism: the polygons are called *bases;* the parallelograms, *sides;* and the surfaces of the parallelograms together constitute what is called the *lateral* or *convex surface* of the prism.

The *altitude* of a prism is the perpendicular distance between its two bases.

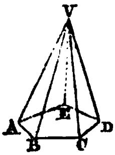
15. A prism is said to be *triangular*, or *quadrilateral*, or *pentagonal*, &c., according as its bases are triangles, or quadrilaterals, or pentagons, &c.

A prism is also said to be *right* or *oblique*, according as the principal edges are perpendicular to the bases, or inclined to them.

16. A *regular prism* is a right prism, which has for its bases two regular polygons; and the straight line which joins their centres is called the *axis* of the prism.

17. (Euc. xi. def. 12.) A *pyramid* is a solid figure, having any number of faces, one of which is a triangle or other rectilineal figure, and the rest triangles which have a common vertex, and for their bases the sides of the first triangle or rectilineal figure.

Such a figure may be formed by drawing straight lines from the angles of any rectilineal figure A B C D E to any point V which is not in the same plane with it.



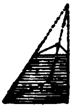
The straight lines V A, V B, &c., which are the sides of the triangles, are called the *principal edges* of the pyramid: the first triangle or rectilineal figure is called the *base*, the other triangles the *sides*, and their common vertex, the *vertex* or *summit:* the surfaces of the latter triangles also constitute what is called the *lateral* or *convex surface* of the pyramid.

The *altitude* of a pyramid is the perpendicular distance of the vertex from the base or the base produced.

18. A pyramid is said to be *triangular*, or *quadrilateral* or *pentagonal*, &c. according as its base is a triangle, or a quadrilateral, or a pentagon, &c.

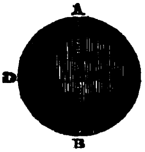
19. A *regular pyramid* is that which has for its base a regular polygon, and the straight line which is drawn from the vertex to the centre of the base perpendicular to the base; and the line so drawn is called the *axis* of the pyramid.

20. If a pyramid be divided into two parts by a plane parallel to its base, the part next the base is called a *frustum* of a pyramid, or sometimes a *truncated pyramid.*



21. (Euc. xi. def. 14.) A *sphere* is a solid figure, every point in the surface of which is at the same distance from a certain point within the figure, which is called the *centre.* The distance from the centre to the surface is called the *radius*, or sometimes the *semidiameter* of the sphere, because it is the half of a straight line which passes through the centre, and is terminated both ways by the surface, which straight line is called a *diameter.*

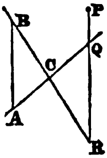
Such a figure may be conceived to be generated by the revolution of a semicircle A D B about its diameter A B;—that is to say, if the semicircle be made to revolve round its diameter AB, its plane will, in the course of the revolution, pass through the whole solid space about the line AB produced, and the semicircular portion A D B will pass through the whole spherical space upon the diameter A B, so that there shall not be a point within that space with which some point or other of the semicircle will not have coincided, but the same cannot be said of any point without the sphere. By the word "generate," it is intended to convey the idea that the parts of the solid start into existence as they are successively traversed by the generating plane.



PROP. 1. (EUC. xi. 2.)

A plane, and one only, may be made to pass through a given straight line and a given point without it, or through three given points which are not in the same straight line.

Let A B be a given straight line, and C a given point without it: a plane, and only one plane, may be made to pass through the straight line A B and the point C.



For a plane may be made to pass through A B, and this plane may be turned about A B until it pass through C. Now, let any other plane be made to pass through the same straight line A B and the same point C; and let P be any point taken in it: P shall likewise be a point in the first plane. For, if in AB any two points A, B be taken, and C A, C B be joined, the straight lines C A, C B will lie in each of the planes (I. def. 7.). And because P is a point in the same plane with CA and CB, through P there may be drawn P Q parallel to A B to meet C A in some point Q and CB in some point R (I. 14. *Cor.* 3). Then, because the lines CA, CB are in the first plane, the points Q, R are likewise in that plane, and therefore the straight line P Q R, which passes through them, and the point P of that straight line, are in the same plane. Therefore, there is no point in either of the two planes which is not also in the other plane, that is, they are one and the same plane.

Again, any plane which passes through the straight line A B and the point C without it, passes also through the three points A, B, C, which are not in the same straight line; and reversely. Therefore, since it has been shown that a plane, and one only, may be made to pass through the straight line A B and the point C, it follows that a plane, and one only, may be made to pass through the three points A, B, C.

Therefore, &c.

Cor. 1. A plane, and one only, may be made to pass through the sides of a given rectilineal angle, or through two given parallels.

Cor. 2. Any number of parallels through which the same straight line passes are in one and the same plane.

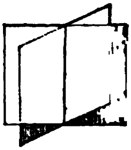
Cor. 3. It follows from the preceding corollary, that a plane may be conceived to be generated by a straight line which moves along a given straight line so as always to continue parallel to another given straight line.

Cor. 4. Any number of planes may be made to pass through the same straight line.

Prop. 2. (Euc. xi. 3.)

If two planes cut one another, their common section shall be a straight line.

For, if any two points be taken in the common section, the straight line which joins them will be in each of the planes (I. def. 7.): but the only line which is in each of the planes is their common section; therefore the common section coincides with the straight line which joins any two of its points, that is, it is a straight line.

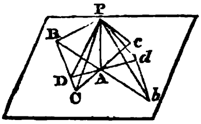


Therefore, &c.

Prop. 3. (Euc. xi. 4.)

If a straight line stand at right angles to each of two other straight lines at their point of intersection, it shall be at right angles to every other straight line which passes through the same point and lies in the same plane with them; that is, (def. 1.) *it shall be at right angles to the plane in which they are.*

Since two different planes may pass through the same straight line A P, and a perpendicular may be drawn to it in each of these planes from the point A, the case supposed in the proposition is evidently possible.



Let the straight line A P, therefore, stand at right angles to each of the straight lines A B, A C, at their point of intersection A, and let A D be any other straight line in the plane A B C, which passes through the same point A: A P shall be at right angles to A D.

In AB, AC, take any points whatever B, C; and in BA, C A produced make A *b* equal to AB, and A*c* equal to AC: join B C, *b c*, and let A D and D A produced cut B C and *b c* respectively in D and *d*: take any point P in A P, and join PB, P C, P D, P *b*, P *c*, P *d*. Then, because in the triangles A B C, A *b c*, the two sides A B, A C are equal to the two A *b*, A *c*, each to each, and the included angles (I. 3.) equal to one another, the base B C is equal to the base *b c*, and the angle A B C to the angle A *b c* (I. 4.); and because in the triangles ABD, Abd the side AB is equal to the side Ab, and the angles A B D, B A D (I. 3.) equal to the angles Abd, bAd, each to each, the side A D is equal to the side Ad (I. 5.), and B D to bd. Again, the triangles P A C, P Ac have the two sides P A, A C of the one equal to the two P A, Ac of the other, each to each, and the included angles right angles; therefore the base PC is equal to the base Pc (I.4.): and for the like reason, PB is equal to Pb; and it was before shown that B C is equal to bc: therefore, the triangles PBC, Pbc have the three sides of the one equal to the three sides of the other, each to each; therefore, also, the angle PBC is equal to the angle Pbc (I. 7.). And, because the triangles PB D, Pbd have two sides PB, BD of the one equal to two sides Pb, bd of the other, each to each, and the included angles P B D, Pbd equal to one another, the bases P D, Pd are likewise equal (I. 4.). Lastly, therefore, because the triangles P A D, P Ad have the three sides of the one equal to the three sides of the other, each to each, the angles P A D, P Ad are equal to one another, and (I def. 6.) P A is at right angles to A D. And because P A is at right angles to every straight line A D, which meets it in the plane BAC, it is at right angles to that plane (def. 1.)

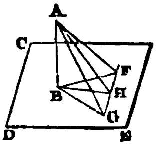
Therefore, &c.

Cor. 1. (Euc. xi. 5.) Any number of straight lines which are drawn at right angles to the same straight line from the same point of it, lie all of them in the plane which is perpendicular to the straight line at that point.

Cor. 2. Hence, if the plane of a right angle be made to revolve about one of its legs, the other leg will describe a plane at right angles to the first leg.

Prop. 4.

If a straight line be perpendicular to a plane, and if from its foot a perpendicular be drawn to a straight line taken in the plane; any straight line, which is drawn from a point in the former perpendicular to meet the foot of the latter perpendicular, shall likewise be perpendicular to the straight line taken in the plane.



Let the straight line AB be perpendicular to the plane C D E at the point B; let F G be a straight line taken in the plane CDE; and from B let B H be drawn perpendicular to F G: take any point A in A B, and join A H: A H shall be perpendicular to F G.

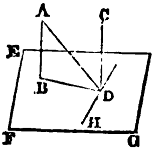
In H F take any point F, make H G equal to H F, and join A F, A G, B F, B G. Then in the triangles B H F, B H G, because the two sides B H, H F are equal to the two BH, HG, each to each, and the included angles right angles, B F is equal to B G (I. 4.). Again, because AB is perpendicular to the plane CDE, the angles ABF, ABG are right angles (def. 1.): and because in the triangles A B F, A B G, the two sides A B, B F are equal to the two AB, BG, each to each, and the included angles right angles, A F is equal to A G (I. 4.). Therefore, lastly, because the triangles A H F, A H G have the three sides of the one equal to the three sides of the other, each to each, the angle A H F is equal to the angle A H G (I. 7.); and they are adjacent angles; therefore, each of them is a right angle (I. def. 10.), and A H is at right angles to F G.

Therefore, &c.

Cor. Hence, also, if a straight line be perpendicular to a plane, and if from any point of it a perpendicular be drawn to a straight line taken in the plane, the straight line which joins the feet of the perpendiculars shall likewise be perpendicular to the straight line taken in the plane.

Prop. 5. (Euc. xi. 6 and 8.)

Straight lines, which are perpendicular to the same plane, are parallel: and, conversely, if there be two parallel straight lines, and if one of them be perpendicular to a plane, the other shall be perpendicular to the same plane.



Let the straight lines A B, C D be each of them perpendicular to the plane E F G: A B shall be parallel to C D.

Join B D; and, in the plane E F G, from D draw D H perpendicular to B D: in A B take any point A, and join A D. Then, because AB is perpendicular to the plane E F G, and BD a perpendicular drawn from its foot to the line D H which is in that plane, A D is likewise perpendicular to D H (4.): and C D is perpendicular to D H, because it is perpendicular to the plane E F G (def. 1.): therefore, DB, DA, and D C lie (3. *Cor.* 1.) in one and the same plane. But BA lies in the plane of D B and D A: therefore, B A and D C lie in the same plane. Again, because AB and C D are perpendicular each of them to the plane E F G, they are perpendicular each of them to the straight line BD (def. 1.). And because they are in the same plane and perpendicular to the same straight line, they are parallels (I. 14.).

Next, let AB, CD be parallels, meeting the plane E F G in the points B, D, respectively, and let A B be perpendicular to the plane E F G: C D shall likewise be perpendicular to it.

Join B D, and draw, as before, in the plane EFG, DH perpendicular to B D; and, taking any point A in A B, join AD. Then, as before, DA is (4.) perpendicular to D H. Therefore, D H is perpendicular to the plane of AD, DB (3.). But C D is in that plane; because it is parallel to AH, and therefore in the same plane with AB and the point D (1. *Cor.* 1.). Therefore C D is likewise perpendicular to DH (3.). Again, because A B, C D are parallel, and that A B (being perpendicular to the plane E F G) is perpendicular to B D, (def. 1.) C D is likewise perpendicular to B D (I. 14.). Therefore, C D is perpendicular to each of the straight lines B D, D H, that is, it is perpendicular to the plane B D H (3.) or EFG.

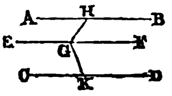
Therefore, &c.

Cor. If from different points in the same straight line perpendiculars be drawn to the same plane, these perpendiculars shall lie in one plane, and their feet in one and the same straight line: for the perpendiculars, being parallel and passing through the same straight line, lie in one plane (1 *Cor.* 2.); and the common section of this plane with the first is a straight line (2.).

Prop. 6. (Euc. xi. 9.)

Straight lines which are parallel to the same straight line, though not both of them in one plane with it, are parallel to one another.

Let the straight lines A B, C D be each of them parallel to the straight line E F, and not in one plane with it: A B shall be parallel to CD.

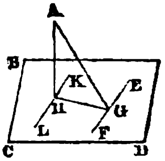


In E F take any point G, and from G, in the plane of A B, E F, draw G H at right angles to E F; and from the same point G, in the plane of C D, E F, draw G K at right angles to E F. Then, because E F is at right angles to each of the lines G H, G K, it is at right angles to the plane HGK (3.). But AB and CD are each of them parallel to EF. Therefore, A B and C D are also at right angles to the plane H G K (5.): and because they are at right angles to the same plane, they are parallel (5.).

Therefore, &c.

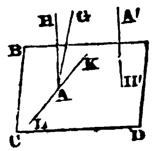
Prop. 7. (Euc. xi. 11, 12, and 13.)

A straight line may be drawn perpendicular to a given plane of indefinite extent from any given point, whether the given point be without or in the plane; but from the same point there cannot be drawn more than one perpendicular to the same plane.



Let A be a point without the plane B C D: a perpendicular, and one only, may be drawn from the point A to the plane B C D.

In the plane BCD draw any line E F, and from the point A draw AG perpendicular to E F: from G draw, in the plane BCD, G H perpendicular to E F; and from A draw AH perpendicular to GH (I. 45.). Through H draw KL parallel to EF (I. 48.), and therefore lying in the same plane with H and E F, that is, in the plane B C D. Then, because E F is at right angles to each of the straight lines A G, G H, it is at right angles to the plane A G H (3.); but K L is parallel to E F; therefore KL is at right angles to the plane A G H, and the angle A H K is a right angle (5.). And because A H is at right angles to each of the straight lines K H, H G, it is at right angles to the plane K H G (3.), that is, to the plane B C D. But from the same point A, there cannot be drawn any other straight line which is at right angles to the plane B C D; for if we suppose AG to be any other straight line drawn from A and meeting the plane BCD in G, and if H G be joined, the angle A H G will be a right angle, and therefore (I. 8.) the angle A G H less than a right angle; so that A G cannot (def. 1.) be at right angles to the plane BCD.

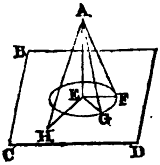


Next, let A be a point in the plane B C D: take any point A′ without the plane, and from A′ draw A′H′ perpendicular to the plane: then, if through A, AH be drawn parallel to A′ H′ (I. 48.), A H will likewise (5.) be perpendicular to the plane. And from the same point A. there cannot be drawn any other straight line perpendicular to the plane B C D: for, if AG be any other straight line drawn from A, and if the plane H A G cut B C D in the line K L (2.), the angle H A K will be a right angle, and therefore G A K will not be a right angle; so that AG cannot be at right angles (def. 1.) to the plane B C D.

Therefore, &c.

Prop. 8.

From a point to a plane the perpendicular is the shortest distance: and of other straight lines which are drawn from the point to the plane, such as are equal to one another cut the plane at equal distances from the foot of the perpendicular; and such as are unequal cut the plane at unequal distances from the foot, the greater being always further from the perpendicular; and conversely.



Let A be a point without the plane BCD, and let there fall from A to the plane, the perpendicular A E, any other straight line A F, the straight line A G which is equal to A F, and the straight line A H which is greater than A F; and join E F, E G, E H: the perpendicular A E shall be less than the straight line AF; the distance E G shall be equal to the distance E F, and the distance E H greater than the distance E F.

For, in the first place, because A E is perpendicular to the plane B C D, the angle A E F is a right angle; wherefore A F E is less than a right angle (I. 8.), and in the triangle A E F (I. 9.) the side A E is less than the side AF. Next, because the angles A E F, A E G are both of them right angles (def. 1.), AEF and A E G are right-angled triangles which have the hypotenuse A F equal to the hypotenuse A G, and the side A E common to both; therefore (I. 13.) the remaining sides E F, E G are equal to one another. Lastly, because A E H is a right angle, the square of A H is equal to the squares of AE, EH (I. 36); and for the like reason the square of AF is equal to the squares of AE, E F: but the square of A H is greater than the square of AF, because AH is greater than A F: therefore, the squares of A E, E H are together greater than the squares of A E, E F; therefore the square of E H is greater than the square of E F, and E H is greater than E F.

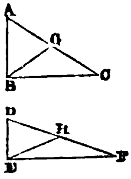
And hence, conversely, if the distances E F, E G be equal to one another, the line AF must be equal to the line A G; for, if not, the distances E F, E G would be unequal: and in like manner, if the distance E H be greater than the distance E F, the line AH must be greater than the line AF.

Therefore, &c.

Cor. If, from the centre A, a sphere be described with a radius less than the perpendicular A E, it will not meet the plane B C D: if with a radius equal to A E, it will meet B C D in one point only, which is the foot of the perpendicular; and, if with a radius greater than AE, its surface will cut the plane in the circumference of a circle which has for its centre the foot of the perpendicular.

Lemma.

If, in two right-angled triangles, the hypotenuse of the one be equal to the hypotenuse of the other, but a side of the first greater than a side of the other, the angle opposite to that side shall be greater than the angle opposite to the side of the other; and conversely.



Let ABC, D E F be two right-angled triangles, which have the hypotenuse AC equal to the hypotenuse D F, but the side A B of the first greater than the side D E of the other: the angle A C B shall likewise be greater than the angle D F E.

Bisect A C and D F in the points G and H respectively (I. 43.): then, if G B and H E be joined, they will be equal respectively to the halves of A C and D F (I. 19. *Cor.* 4.), and therefore (I. ax. 5. equal to one another. And, because in the triangles GAB, HDE, the two sides

AG, GB of the one are equal to the two DH, HE of the other, each to each, but the base AB greater than the base D E, the angle AGB is (I. 11.) greater than the angle D H E. But the angle AGB is double of A C B, for A G B is equal to the sum of GCB and GBC (I. 19.), which are equal to one another, because GB is equal to G C (I. 6.); and for the like reason, the angle D H E is double of D F E. Therefore (I. ax. 8.), the angle A C B is greater than the angle D F E.

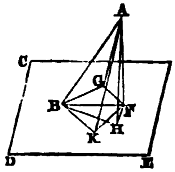
And hence, conversely, if A C B be greater than D F E, AB must also be greater than DE: for it cannot be equal to D E (I. 13.); neither, by what has been just demonstrated, can it be less than D E.

Therefore, &c.

Prop. 9.

If a straight line be inclined to a plane, of all the angles which it makes with straight lines meeting it in that plane, the least shall be the angle of inclination; and, with respect to every other of these angles, a second angle may always be drawn which shall be equal to it, viz. upon the other side of the angle of inclination; but there cannot be drawn in the plane more than two straight lines with which the inclined straight line shall make equal angles, one upon each side of the angle of inclination.

Let the straight line A B be inclined to the plane C D E at the point B: from any point A in A B, draw A F perpendicular to the plane C D E (7.), and join B F, so that A B F may be the angle of inclination (def. 2.); and let BG be any other straight line in the plane C D E passing through the point B: the angle A B F shall be less than the angle A B G.

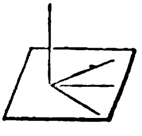


From the point F draw F G perpendicular to B G, and join A G. Then, because A F is perpendicular to the plane in which B G lies, and that F G is drawn from the point F perpendicular to B G, A G is likewise perpendicular to BG (4.). And, because in the right-angled triangles A F B, A G B, the hypotenuse A B is common to both, but the side A F of the first less than the side AG (8.) of the other, the opposite angle A B F is likewise less than the angle A B G, by the foregoing Lemma.

Also, from the point B in the same plane C D E, there may be drawn a straight line BH, making an angle with A B equal to the angle A B G, viz. upon the other side of the angle A B F. For, if the angle F B H be made equal to the angle F B G, then, drawing FH perpendicular to B H, and joining A H, A H will likewise be perpendicular to B H (4.); and, because in the right-angled triangles F G B, F H B, the hypotenuse F B is common to both, and the angle F B G equal to the angle F B H, the opposite side F G is equal to F H (I. 13.): therefore, also, (8.) A G is equal to A H: and, because in the right-angled triangles A G B, A H B the hypotenuse A B is common to both, and the side A G equal to the side A H, the angle A B G is likewise equal to the angle A B H (I. 13.).

But, lastly, there cannot be drawn in the plane CDE more than two straight lines B G, B H, with which the straight line AB shall make equal angles. For, if BK be any other straight line, then, drawing F K perpendicular to B K, and joining A K, A K will likewise be perpendicular to F K (4.): and, because in the right-angled triangles F H B, F K B, the hypotenuse F B is common to both, but the angle F B H not equal to the angle F B K, the side F H is not equal to the side F K (Lemma): therefore (8.) A H is not equal to A K; and, because in the right-angled triangles A H B, A K B, the hypotenuse A B is common to both, but the side A H not equal to the side A K, the angle A B H is not equal to the angle A B K (Lemma).

Therefore, &c.

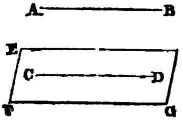


Cor. If three straight lines lie in the same plane and meet in the same point, and if a fourth straight line stand at equal angles to the three at that point; the equal angles shall be right angles, and the fourth straight line shall be at right angles to the plane of the three.

Prop. 10.

If one straight line be parallel to another, it shall likewise be parallel to any plane which passes through that *other;*[*] *and if a straight line be parallel to a plane, it shall be parallel to the line in which any plane passing through it cuts the first plane.*

Let the straight line A B be parallel to C D, and let E F G be any plane passing through C D: the line A B shall be parallel to the plane E F G.



For, since A B is in the plane of A B, C D (I. def. 12.), if it meet the plane EFG at all, it must meet it in the plane of AB, C D, and therefore in some point of the line C D which is the common section of the two planes. But A B cannot meet C D, being parallel to it. Therefore neither can it meet the plane E F G, that is, it is parallel to the plane E F G (def. 3.).

Next, let A B be parallel to the plane E F G, and let C D be the line in which any plane passing through A B cuts the plane E F G: A B shall be parallel to C D. For if not, it must meet it in some point. But in the same point it would meet the plane E F G, to which it is parallel: which is impossible. Therefore A B is parallel to C D.

Therefore, &c.

Cor. 1. If two straight lines be parallel, the common section of any two planes passing through them shall be parallel to either of them. For, by the first part of the proposition, one of the lines is parallel to the plane which passes through the other; and, therefore, by the second part of the proposition, it is parallel to the line in which the plane passing through itself cuts the plane passing through the other.

Cor. 2. If two straight lines, which cut one another, be parallel, each of them, to the same plane; the plane of the two straight lines shall be parallel to that plane. For should the planes meet, their common section would, by the proposition, be parallel to each of the cutting straight lines; which is impossible. (I. 14. *Cor.* 2.)

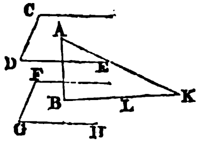
SECTION 2.—*Of Planes which are parallel, or inclined, or perpendicular to other Planes.*

PROP. 11. (EUC. xi. 14.)

Planes, to which the same straight line is perpendicular, are parallel: and, conversely, if two planes be parallel, and if one of them be perpendicular to a straight line, the other shall be perpendicular to the same straight line.

Let the straight line A B be perpendicular to each of the planes C D E, F G H: the plane C D E shall be parallel to the plane F G H.

For, if they are not parallel, let them meet one another, and let K be any point of the common section. Join K A, K B. Then, because A B is perpendicular to the plane C D E, the angle K A B is a right angle (def. 1.); and, because the same A B is perpendicular to the plane F G H, the angle K B A is a right angle: therefore, two angles of the triangle K A B are together equal to two right angles; which (I. 8.) is impossible. Therefore, the planes do not meet one another, to whatever extent they may be produced, that is, (def. 6.) they are parallel to one another.



Next, let the plane C D E be parallel to the plane F G H, and from any point A of the first let A B be drawn perpendicular to the other plane F G H: A B shall likewise be perpendicular to CDE. For, if not, (def. 1.) there must be some line in the plane C D E which meets the line A B, and does not make a right angle with it: let KA be such a line, and let the plane KAB cut the plane FGH in the straight line B L (2.). Then, because AB is at right angles to the plane F G H, the angle A B L is a right angle: but B A K is not a right angle: therefore the straight lines A K and B L will meet, if produced, in some point (I. 15. *Cor.* 4.) which will be common to both the planes; and, because the planes meet one another in this point, they cannot be parallel, which is contrary to the supposition. Therefore the straight line A B makes a right angle with every straight line meeting it in the plane C D E, that is, it is (def. 1.) at right angles to the plane C D E.

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[*] It is possible that a plane may pass through the second straight line, and also through the first, which is supposed to be parallel to it: in this case, it is evident that the latter is not, as is predicated in the proposition, *parallel* to such plane, but lies altogether in it. The enunciation must, therefore, be understood with the exception of this particular case.

Therefore, &c.

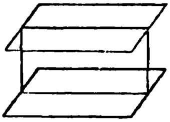
Cor. 1. Through any given point a plane may be drawn, and one only, which shall be parallel to a given plane. For a perpendicular A B may be drawn from the given point A to the given plane F G H (7.); and from the same point A there may be drawn in two different planes straight lines at right angles to this perpendicular; the plane of which straight lines (3.), and evidently none other which passes through the given point, is perpendicular to the straight line A B, and therefore parallel to the given plane F G H.

Cor. 2. Planes, which are parallel to the same plane, are parallel to one another.

PROP. 12. (EUC. xi. 16.)

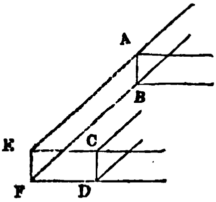
If parallel planes be cut by the same plane, their common sections with it shall be parallels.

For, these common sections lying one of them in one of the planes, and the other in the other, cannot meet one another, unless the planes meet one another; which they do not, because they are parallel: also, the common sections lie in the same (viz the cutting) plane: therefore they are parallels (I. def. 12.).



Therefore, &c.

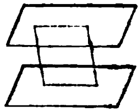
Cor. If two planes which cut one another be parallel to other two which cut one another, each to each, the common sections A B and C D of the first two and second two shall be parallels. For, if one of the first two planes be produced to meet that one of the second two, to which it is not parallel (11. *Cor.* 1.), in the straight line E F (2.), E F will be parallel both to A B and C D; therefore (6.) A B and C D are parallel to one another.



PROP. 13.

If two parallel straight lines be cut by two parallel planes, the parts of the straight lines which are intercepted between the planes shall be equal to one another.

For, if the plane of the parallels be drawn to cut the two parallel planes, the common sections will be parallel (12.), and will therefore include, together with the parts in question, a parallelogram, of which the parts in question are opposite sides, and therefore are equal to one another (I. 22.).



Therefore, &c.

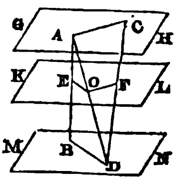
Cor. 1. Parallel planes are every where equidistant (5.).

Cor. 2. If, from any number of points in the same plane, there be drawn without the plane as many equal and parallel straight lines, the other extremities of these straight lines shall lie in a second plane parallel to the first.

PROP. 14. (EUC. xi. 17.)

If any two straight lines be cut by three parallel planes; the parts of the straight lines, which are intercepted by the planes, shall be to one another in the same ratio.

Let the straight lines A B, C D be cut by the parallel planes G H, K L, M N in the points A, E, B, and C, F, D respectively: A E shall be to E B as C F to F D.



Join A D, and let it cut the plane K L in O: and join O E, O F, A C, B D. Then, because the parallel planes K L, M N are cut by the plane of the triangle A B D, the common sections E O, B D are parallel (12.): therefore (II. 29.) A E is to E B as A O to O D. Again, because the parallel planes G H, K L are cut by the plane of the triangle D A C, the common sections A C, O F are parallel; and therefore (II. 29.) C F is to F D as A O to O D. Therefore (I. 12.) A E is to E B as C F to F D.

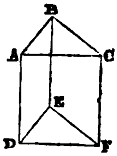
Therefore, &c.

Cor. If straight lines be drawn to a plane from any point without it, and if each of these straight lines, or each of them produced, be divided in the same ratio towards the same parts; the points of division shall all lie in a second plane parallel to the first.

PROP. 15. (EUC. xi. 10 and 15.)

If two straight lines, which meet one another, be parallel respectively to two other straight lines, which meet one another, but are not in the same plane with the first two; the contained angles shall be equal,[*] *and their planes parallel.*

Let the straight lines A B, A C, which meet one another in A, be parallel respectively to the straight lines D E, D F, which meet one another in D. the angle at A shall be equal to the angle at D, and the plane B A C shall be parallel to the plane E D F.



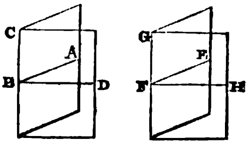
In the straight lines A B, A C take any points B, C: make D E equal to AB and DF equal to AC, and join BC, E F, A D, B E, C F. Then, because the straight lines A B, D E are equal and parallel, B E is equal to A D and parallel to it (I. 21.): and for the like reason C F is equal to A D and parallel to it: therefore also C F, B E are equal and parallel (I. ax. i. and 6), and consequently (I. 21.) B C is equal and parallel to E F. Therefore, because the triangles A B C, D E F have the three sides of the one equal to the three sides of the other, each to each, the angle B A C is equal to the angle E D F (I. 7.). Again, because the straight line A B is parallel to D E, it is parallel to the plane E D F (10.); and for the like reason A C is parallel to the same plane: therefore (10. *Cor.* 2.) the plane B A C is parallel to the plane E D F.

Therefore, &c.

Cor. If the planes, which contain a dihedral angle, be cut by two parallel planes, equal angles shall be intercepted in the latter: for the lines which contain them are parallel (12.).

PROP. 16.

If two planes cut one another, and if other two likewise cut one another, and if the rectilineal angle which is contained by perpendiculars drawn in the two first to their common section from the same point of it, be equal to the rectilineal angle contained by perpendiculars similarly drawn in the other two; the dihedral angles shall be equal to one another.



Let the two planes A B C, D B C cut one another in the straight line B C, and let the two E F G, H F G cut one another in the straight line F G; also, let the angle A B D which is contained by perpendiculars to B C drawn in the two first planes from the point B, be equal to the angle E F H which is contained by perpendiculars to F G drawn from the point F in the two others: the dihedral angle A B C D shall be equal to the dihedral angle E F G H.

For, if the point B be made to coincide with the point F, the straight line B C with F G, and the plane A B C with the plane E F G, the straight line A B will coincide with E F, because the angles A B C, E F G are right angles (I. 1.). Also, because B D is perpendicular to B C, it will be (3 *Cor.* 1) in the plane E F H which (3.) is at right angles to F G, and therefore B D will coincide with F H, because the angle A B D is equal to the angle E F H. Therefore the plane

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[*] It will be observed that this proposition is an extension of I. 18. to the case in which the sides of the one angle are not in the same plane with the sides of the other. In order to exclude the case in which the angles would be supplementary, not equal, to one another, it was required, in the enunciation of I. 18., that the parallel sides should be "in the same order" or direction from one another; and a similar limitation for the same purpose is obviously necessary in the proposition before us. In the former case, a very simple criterion is afforded by the position of the angles relatively to the joining line BE (see the figure of I. 18.); for, in order that the angles may be equal, the sides which are parallel ought to be, each pair, upon the same side, or each pair upon opposite sides, of the joining line BE; for, if one pair lie towards the same parts, and the other towards opposite parts, the angles will be, not equal, but supplementary. And the present case admits of a criterion equally simple; for, in the case of equality, the parallel sides lie, each pair, upon the same side, or each pair upon opposite sides, of any plane which passes through BE; whereas, when the angles are supplementary to one another, one pair of parallel sides lie towards the same parts of any such plane, and the other pair towards opposite parts.

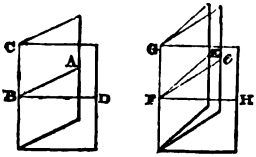
C B D coincides with the plane G F H. And, because the planes A B C, D B C coincide with the planes E F G, H F G respectively, the dihedral angle A B C D coincides with the dihedral angle E F G H, and is equal to it.

Therefore, &c

Prop. 17.

Any two dihedral angles are to one another as the angles contained by perpendiculars drawn as in the last proposition.

Let A B C D, E F G H be any two dihedral angles, and let A B D and E F H be the angles of the perpendiculars A B, B D and E F, F H drawn as in the last proposition: the dihedral angles shall be to one another as the angles A B D, E F H.



For, if the angle E F H be divided into any number of equal angles by the straight lines F *e*, &c., which straight lines, being in the plane of the angle EFH, are all of them perpendicular to the common section FG (3.), the dihedral angle E F G H will be divided into the same number of equal dihedral angles by the planes G F *e*, &c. (16.). And, if the angle E F *e* be contained in A B D any number of times with a remainder less than E F *e*, the dihedral angle E F G *e* will be contained in A B C D the same number of times with a remainder less than E F G *e* (16.). And this will be the case, how great soever be the number of parts into which the angle E F H is divided. Therefore (II. def. 7.), the dihedral angle A B C D is to the dihedral angle E F G H as the angle A B D to the angle E F H.

Therefore, &c.

Cor. If one plane be at right angles to another, the perpendiculars to the common section, which are drawn in the two planes from the same point of the common section, will be at right angles to one another; and conversely.

Scholium.

Hence a dihedral angle is said to be measured by the plane angle of the perpendiculars. And, in thus measuring the dihedral angle, it is indifferent from what point of the common section the perpendiculars are drawn; for, if any two points be taken, the planes of the two pairs of perpendiculars drawn from these points will be (3.) perpendicular to the common section, and therefore parallel (11.); and it has been seen (15.*Cor.*) that, if a dihedral angle be cut by parallel planes, the intercepted angles will be equal to one another.

It has been already stated (17.*Cor.*) that a dihedral right angle is measured by a plane right angle: it follows that a dihedral obtuse angle is measured by a plane obtuse angle, and a dihedral acute angle by a plane acute angle.

Many propositions with regard to cutting planes are readily derived from this mode of measuring dihedral angles. Of these the following only need here be mentioned:

1. If two planes cut one another, the vertical dihedral angles will be equal.
2. If a plane fall upon two parallel planes, it shall make the alternate dihedral angles equal to one another, the exterior dihedral angle equal to the interior and opposite upon the same side, and the two interior dihedral angles upon the same side together equal to two right angles.
3. Of two dihedral angles, if the planes of the one be parallel to the planes of the other, or perpendicular to the planes of the other respectively, the two dihedral angles shall be equal.[*]
4. If two planes cut one another, and if perpendiculars be drawn to them from any the same point, the adjacent angles contained by the perpendiculars shall measure respectively the adjacent dihedral angles contained by the two planes.

Prop. 18. (Euc. xi. 18.)

If one plane be perpendicular to another, any straight line which is drawn in the first plane at right angles to their common section shall be perpendicular to the other plane: and, conversely, if a straight line be perpendicular to a plane, any plane which passes through it shall be perpendicular to the same plane.

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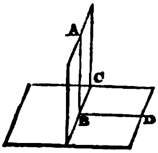
[*] It being provided also, that the parallel planes lie, each pair, upon the same side, or each pair upon opposite sides of the plane which passes through the common sections (A B, C D in the figure of 12. *Cor.*): for, if one pair lie towards the same parts, and the other pair towards opposite parts of that plane, the dihedral angles will be supplementary, not equal, to one another. See the note at Prop. 15.



Let the plane ABC be perpendicular to DBC, and let any straight line AB be drawn in the plane ABC perpendicular to the common section BC: the straight line AB shall be perpendicular to the plane DBC. From the point B, in the plane DBC, let BD be drawn at right angles to BC. Then, because the planes are at right angles to one another, the angle ABD is a right angle (17. *Cor.*): but ABC is likewise a right angle; therefore (3.) AB is perpendicular to the plane DBC.

Next, let the straight line AB be perpendicular to the plane BCD; and let ABC be any plane passing through AB: the plane ABC shall be perpendicular to BCD.

Let BC be the common section of the two planes; and, from the point B, in the plane BCD, draw BD at right angles to BC. Then, because AB meets the line BD drawn in the plane to which AB is perpendicular, the angle ABD is a right angle (def. 1.). But the angle ABD is contained by straight lines drawn in the two planes perpendicular to their common section BC. Therefore (17. *Cor.*) the dihedral angle ABCD is likewise a right dihedral angle, and the plane ABC is perpendicular to BCD.

Therefore, &c.

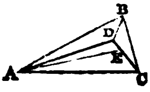
Cor. 1. If through the same point there pass any number of planes perpendicular to the same plane, they shall all of them pass through the same straight line, viz. the perpendicular which is drawn from the point to the plane.

Cor. 2. Euc. xi. 19. If two planes, which cut one another, be each of them perpendicular to a third plane, their common section shall be perpendicular to the same plane.

Prop. 19. (Euc. xi. 20.)

If a solid angle be contained by three plane angles, any two of these shall be together greater than the third.

Let the solid angle at A be contained by the three plane angles BAC, BAD, CAD: any two of them, as BAC, BAD, shall be together greater than the third CAD.



If one of the angles BAC, BAD be greater than CAD, or equal to it, the proposition is evident; for this, together with the other, must be greater than CAD.

But, if the angles BAC, BAD be each of them less than CAD, from the angle CAD cut off the angle CAE equal to BAC; in AB take any point B, make AE equal to AB; through E draw any line CED, not parallel to AC or AD, and therefore cutting AC, AD in the points C, D respectively; and join BC, BD. Then, because the triangles ABC, AEC have two sides of the one equal to two sides of the other, each to each, and the included angles BAC, EAC equal to one another, the base BC is equal to the base EC (I. 4.): but BD and BC are together greater than DC, (I. 10.) that is, than DE and EC: therefore BD is greater than ED. And, because in the triangles ABD, AED the two sides AB, AD of the one are equal to the two sides AE, AD of the other, each to each, but the base BD greater than the base ED, the angle BAD is greater than the angle EAD (I. 11.). Therefore, BAC, BAD together are greater than EAC, EAD together, that is, than CAD.

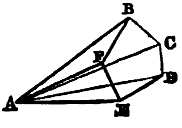
Therefore, &c.

Cor. If three plane angles form a solid angle, any one of them shall be greater than the difference of the other two.

Prop. 20. (Euc. xi. 21.)

If a solid angle be contained by any number of plane angles; these shall be together less than four right angles.

Let the solid angle at A be contained by any number of plane angles BAC, CAD, DAE, EAF, FAB: these shall be together less than four right angles.



Let the edges of the solid angle be cut, by any the same plane, in the points B, C, D, E, F respectively, and therefore the planes which contain it in the straight lines BC, CD, DE, EF, FB (2.), which together contain the rectilineal figure BCDEF. Then, because the triangles ABC, ACD, &c. are as many, in number, as the sides of the figure, the angles of these triangles are (I. 19.) together equal to twice as many right angles as the figure has sides. But the angles of these triangles are those which contain the solid angle at A, together with the pairs of angles at the points B, C, D, &c.; and twice as many right angles as the figure has sides are equal to four right angles together with the angles of the figure (I. 20.). Therefore, the angles at A, together with the pairs of angles at B, C, D, &c. are equal to four right angles together with the angles of the figure B C D E F. But, because the solid angles at B, C, D, &c. are each of them contained by three plane angles, any two of which are (19.) together greater than the third, the pairs of angles at B, C, D, &c. are together greater than the angles of the figure B C D E F. Therefore, the angles at A are together less than four right angles.

Therefore, &c.

Cor. It follows from this proposition that *there cannot be more than five regular solids.* For the solid angles of a regular solid are contained by the plane angles of equilateral triangles, or of squares, or of regular pentagons, &c. (def. 9.). Now, six angles of equilateral triangles are together equal to four right angles (I. 19.): therefore, only three, or four, or five, of such angles, may be taken to form a solid angle; and, accordingly, there cannot be more than three different regular solids, whose faces are equilateral triangles. Again, three angles of squares may be taken to form a solid angle; but four angles of squares are equal to four right angles (I. def. 20.). Lastly, three angles of regular pentagons may be taken to form a solid angle, for these are together less (I. 20.) than four right angles. But, because the angle of a regular pentagon is greater (I. 20.) than that of a square, four angles of a regular pentagon are greater than four right angles, so that not more than three of them can be taken to form a solid angle. With regard to the regular hexagon, and other regular figures that have a still greater number of sides, three angles of a regular hexagon are equal to four right angles, and those of the others still greater (I. 20.). It appears, therefore, that there cannot be more than five regular solids; of which, if there be so many, three will be included by equilateral triangles, one by squares, and one by regular pentagons. It will be seen in the section of problems, that all of these may be constructed, and that they are contained by 4, 8, 20, 6, and 12 faces respectively.

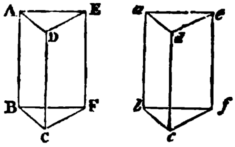
The further consideration of solid angles is deferred to Book vi., where we shall find that the investigation of their properties is greatly facilitated by means of the spherical surface described about the angular point, and that they are, indeed, perfectly analogous to the properties of triangles upon such a surface.

Section 3.—*Of Solids contained by Planes.*

Prop. 21.

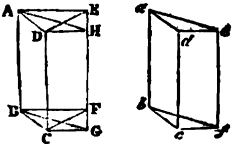
If two triangular prisms have a principal edge of the one equal to a principal edge of the other, and the dihedral angles at those edges equal to one another, and likewise the sides which contain those dihedral angles equal, each to each; the two prisms shall be equal to one another.

Let A B C D E F, *a b c d e f* be two triangular prisms which have the edge A B equal to the edge *a b*, the dihedral angle D A B E equal to the dihedral angle *d a b e*, and the sides A C, A F equal to the sides *a c*, *a f* each to each: the prism A B C D E F shall be equal to the prism *a b c d e f*.



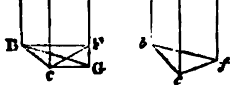
First, let the angle A B C be equal to the angle *a b c*, and the angle A B F to the angle *a b f*, so that the sides A F, A C are similar and similarly placed to the sides *a f*, *a c*, each to each. Then, if the straight line A B be made to coincide with the straight line *a b*, which is equal to it, and the plane A F with the plane *a f*, the plane A C will also coincide with the plane *a c*, because the dihedral angle D A B E is equal to the dihedral angle *d a b e*: also the lines B C, B F will coincide with the lines *b c*, *b f* respectively; and therefore the lines A D, A E, which are parallel and equal to the former, each to each, will coincide respectively with the lines *a d*, *a e*, which are parallel and equal to the latter: therefore the prism A B C D E F will coincide with the prism *a b c d e f*. And because the two prisms may be made to coincide, they are equal to one another (I. ax. 11.).

Next, let the sides A C, *a c* be equal and similar, and the sides A F, *af* equal but dissimilar. Make the angle A B G, in the plane A B F, equal to the angle *abf*; complete the parallelogram A B G H; and join C G, D H. Then because GH is equal and parallel to AB (I. 22.), which is equal and parallel to C D, the three, A B, C D, G H are equal and parallel, and, therefore, (def. 14.) A B C D H G is a triangular prism.



Also, the parallelogram AHGB is equal to the parallelogram A E F B (I. 24.), that is, to the parallelogram *a e f b*; and the parallelogram A D C B is equal to the parallelogram *a d c b*, by the supposition. Therefore, because the sides A C, A G of this new prism are respectively equal, similar, and similarly placed to the sides *a c*, *a f* of the prism *a b c d e f*, it is equal to the prism *a b c d e f*, by the former case. Again, because H G is equal to D C, that is to E F, H E is equal to G F, and may be made to coincide with it; also the dihedral angle A E H D is equal to the dihedral angle B F G C, and may be made to coincide with it; in which case, also, the angles H E D, H E A will coincide with the angles G F C, G F B, which (I. 15.) are equal to them respectively, and the lines E D, E A with the lines F C, F B, which are equal to them respectively (I. 22.). Therefore the solid H E A D, which is bounded by the four triangular planes H D E, H D A, H E A, E A D, may be made to coincide with the solid G F B C, which is bounded by the four triangular planes G C F, G C B, G F B, F B C, and (I. ax. 11.) is equal to it; and, therefore, adding the solid A B C D H F to each, the prism A B C D E F is equal to the prism A B C D H G, that is, to the prism *a b c d e f*.

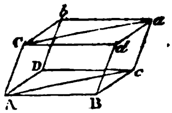
Lastly, let the sides A C, *a c*, as also A F, *a f*, be equal, but dissimilar. Make the parallelogram A G similar to *a f*, and complete the triangular prism A B C D H G, as in the preceding case. Then, it is evident that the prism A B C D H G is equal to each of the prisms A B C D E F, *a b c d e f*; to the former, because the side AC is common to both, and the other sides AG, AF are equal to one another; and to the latter, because the sides A G, *af* are equal and similar (I. 24. and I. ax. 1.), and the other sides AC, *a c* equal to one another. Therefore (I. ax. 1.) the prism A B C D E F is equal to the prism *a b c d e f*.



Therefore, &c.

PROP. 22. (EUC. xi. 24 and 28.)

The opposite faces of a parallelopiped are similar and equal parallelograms: also any two of its opposite edges are parallel to one another; and the plane which passes through them bisects the parallelopiped.



Let A *a* be a parallelopiped, and C D, *cd* any two opposite faces; they shall be similar and equal parallelograms. They are parallelograms, for the opposite sides of each, as A C, D *b*, being sections of parallel planes by the same plane, are parallel (12). Again, their corresponding sides, as A D, B *c*, are equal to one another, because they are opposite sides of a parallelogram A B *c* D (I. 22.); and their corresponding angles, as C A D, *d* B *c* are equal to one another, because they are intercepted upon parallel planes C A D, *d* B *c* by the planes of the same dihedral angle C A B D (15. *Cor.*): therefore the parallelograms C D, *c d* are equal and similar.

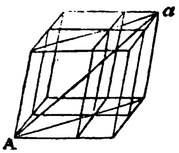
Also, any two of the opposite edges of the parallelopiped, as A C, *a c*, are parallel to one another; for they are the common sections of parallel planes, namely, the opposite faces of the parallelopiped (12. *Cor.*).

Lastly, therefore, let the parallelopiped be divided by the plane of any two opposite edges A C, *a c*, into the two triangular prisms *d* B A C *a c*, *b* D A C *a c*: these prisms shall be equal to one another. For the edges *b* D, *d* B, being each of them equal to C A (I. 22.), are equal to one another; and the dihedral angles at those edges are likewise equal (17. Scholium, 3.) and the sides which contain them are equal, each to each, because they are opposite faces of the parallelopiped. Therefore the prisms are equal to one another (20.).

Therefore, &c.

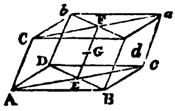
Cor. 1. If a triangular prism be completed into a parallelopiped (which may be done by completing its triangular bases into parallelograms), the prism shall be equal to half theparallelopiped.

Cor. 2. If a parallelopiped be cut by two planes which are drawn parallel respectively to two of its adjoining faces, the line of their intersection will be parallel to the edge made by those faces (12. *Cor.*); and if that line be in the diagonal plane, and the parallelopiped be thus divided into four parallelopipeds, two of which are about the diagonal plane; the other two which, together with the former, make up the whole parallelopiped, shall be equal to one another. For, as in I. 23, the whole parallelopiped, and the two parts which are about the diagonal plane, are bisected by that plane.



Scholium.

It is worthy of remark, that "the four diagonals of a parallelopiped pass through the same point, and that the sum of their squares is equal to the sum of the squares of the twelve edges." For, if the diagonal planes which pass through A C, *a c* and B D, *b d* intersect one another in the line E F, the figures A C *a c* and B D *b d* will be parallelograms, having their opposite sides bisected by the line E F, (I. 22.); wherefore, the diagonals of each (which are the same with the diagonals of the parallelopiped), as, for instance, A *a*, will bisect EF; because the triangles A E G, *a* F G having the angles of the one equal to the angles of the other, each to each (I. 15.), and the side A E equal to the side *a* F, the side E G is likewise equal to the side F G (I. 5.). Therefore, the four diagonals of the parallelopiped pass through the same point, viz. the middle point of EF. And again, because A C *a c* and B D *b d* are parallelograms, the squares of their four diagonals are (I. 41. *Cor.*) together equal to the squares of A C, *a c*, A *c*, *a* C, and B D, *b d*, B *d*, *b* D, that is to say, to the squares of the four edges A C, *a c*, B *d*, *b* D, together with the squares of A *c*, B D, and *a* C, *b d*, which are the diagonals of the parallelograms A B *c* D and *a b* C *d* respectively; that is, to the squares (I. 41. *Cor.*) of the twelve edges, which are made up of the sides of these parallelograms, and the four edges before mentioned.

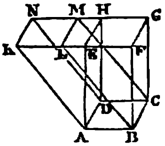


PROP. 23. (EUC. xi. 29 and 30.)

Parallelopipeds upon the same base, and between the same parallel planes, are equal to one another.

Let A B C D be the common base of two parallelopipeds, which have their opposite bases E F G H and K L M N lying in the same plane: the two parallelopipeds shall be equal to one another.

Since E F and K L are both parallel to A B, either they are parts of the same straight line, or they are parallel to one another (6.). And first let them lie in the same straight line. Then, because L M and E H are equal and parallel to B C, and therefore to one another (I. ax. 1. and 6.), M H is parallel to L E or K F (I. 21.), and M N, G H lie in the same straight line with M H and with each other. Therefore, the sides AL, DM, of the one parallelopiped, are in the same planes with the sides A F, D G of the other, each with each. And because the bases K M, E G are equal each to the base A C (22.), they are equal to one another. Therefore, also, the parallelogram K H is equal to the parallelogram L G. And because the triangular prisms K N H E A D, L M G F B C have the edges KN, LM equal to one another, and the dihedral angles at those edges equal (17. *Scholium*, 2.), and the sides containing those angles equal (22.), each to each, they are equal to one another (21.): but, if the first of these prisms be taken from the whole solid A D N K F G C B, there remains the parallelopiped A G; and if the other be taken from the same solid, there remains the parallelopiped A M: therefore the parallelopiped A G is equal to the parallelopiped A M. (I. ax. 3.)



Next, let E F, K L not be in the same straight line. Let the planes A D H E, B C G F be produced to meet the planes A B L K, D C M N, themselves produced if necessary, and let the parallelopiped A P be completed* by producing the plane of the upper bases E F G H, K L M N. Then, by the former case, the parallelopiped A G is equal to A P, because O P, F G are in the same straight line; and, for the like reason, the parallelopiped A M is equal to the same A P. Therefore (I. ax. 1.) the parallelopipeds A G, A M are equal to one another.

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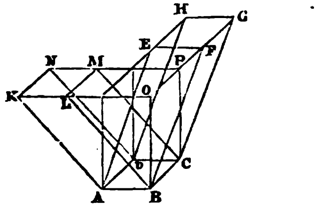
* In this case, the sides of the parallelopiped to be completed are already drawn, viz. by producing the sides of the parallelopipeds A M, A G; and, to complete the parallelopiped, it is only necessary to draw the upper base by producing the plane of EFGH and



Therefore, &c.

Prop. 24. (Euc. xi. 31.)

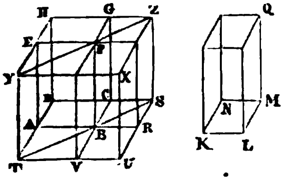
Parallelopipeds upon equal bases, and between the same parallel planes, are equal to one another.

Let A G, K Q be two parallelopipeds upon equal bases A B C D, K L M N, and between the same parallel planes; the parallelopipeds A G, K Q shall be equal to one another.

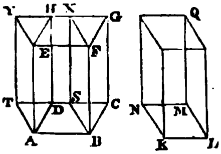
The bases A B C D, K L M N are either similar, or equiangular, or not equiangular.

In the first case, since they are equal as well as similar, they may be made to coincide; and the parallelopipeds will then stand upon a common base, and between the same parallel planes: therefore they are equal to one another.

Secondly, let them be equiangular, and not similar, having the angle at B equal to the angle at L; and therefore, also, their other angles equal, each to each, (I. 15. and I. ax. 3.). Produce A B to R, so that B R may be equal to K L; complete the parallelogram B C S R; join S B, and produce it to meet D A produced in T; through T draw T U parallel to A B, to meet C B and S R produced in the points V and U; complete the parallelopiped D X upon the base D T U S, and with the edge D H; produce the planes A B F E, C B F G, in order to complete the parallelopiped B X, and draw the diagonal plane S T Y Z of the parallelopiped D X. Then the parallelopipeds B H and B X are equal to one another, because, together with the parallelopipeds B Y and B Z about the diagonal plane, they complete the whole parallelopiped D X (22. *Cor.* 2.). Again, because the bases B U and B D are complements of the parallelogram D U, B U is equal to B D (I. 23.), that is, to L N; but B R is equal to K L, and the angle R B V to the angle A B C (I. 3.), that is, to K L M: therefore, also, B V is equal to L M, and the bases B U, L N are both equal and similar. Therefore, by the first case, the parallelopiped K Q is equal to the parallelopiped B X, that is, to B H or A G.



Lastly, let the bases A B C D, K L M N be equal, but not equiangular. Make the angle A B S, in the plane A B C D, equal to the angle K L M, and complete the parallelogram A B S T, and the parallelopiped A X, upon the base A B S T and with the edge B F. Then it is evident that the triangular prisms D H E A T Y, C G F B S X have the edges D H, C G equal to one another (22.), and the dihedral angles at those edges equal (17. *Scholium* 2.), and the sides containing those angles equal, each to each, for the sides A H, B G are opposite faces of the parallelopiped AG (22.), and the sides TH, SG stand upon equal bases (I. ax. 1. and ax. 2. or 3.) TD, SC, and between the same parallels (I. 25.); therefore, the prisms are equal to one another (21.); and, these being taken from the whole solid A E Y T C G F B, there remains the parallelopiped A G, equal to A X. But the parallelopiped K Q is equal to A X by the preceding case; because their bases are equal (I. 24. and



KLMN to cut the sides. It may be observed, however, that a parallelopiped can always be completed when its base A B C D and one of its principal edges AE are given, viz. by drawing the side-planes EAB, EAD, and then through BC, CD the side-planes BG, CH parallel to E A D and E A B respectively, and, lastly, through the point E, the upper base E G parallel to ABCD. This operation is analogous to that by which a parallelogram is completed from two adjoining sides (see note, page 17); and will be sometimes understood when it is directed to *complete* a parallelopiped in future propositions.

ax. 1.) and equiangular. Therefore the parallelopiped KQ is equal to AG. (I. ax. 1.).

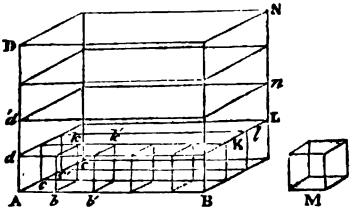
Therefore, &c.

Cor. Every parallelopiped is equal to a rectangular parallelopiped of equal base and the same altitude.

Prop. 25.

If the three conterminous edges of a rectangular parallelopiped contain, each of them, the same straight line a certain number of times exactly; the parallelopiped shall contain the cube of that straight line, as often as is denoted by the product of the three numbers, which severally denote how often the line itself is contained in the three edges.

Let AB, AC, AD be three edges of a rectangular parallelopiped terminated in the same angular point, and let them contain the same straight line M, five, three, and four times respectively: the parallelopiped shall contain the cube of M, $5 \times 3 \times 4$, or 60 times.



In the straight line AB, take the five parts Ab, bb', &c. each equal to M, in AC the three parts Ac, cc', &c. each equal to M, and in AD the four parts Ad, dd', &c., each equal to the same M. Through the points b, c, and d, let planes be drawn parallel to the planes cAd, bAd, and bAc respectively, (11. *Cor.* 1.) and let them meet one another in the point k; then Ak is equal to the cube of M. Let the planes ck, dk be produced to meet the plane BN; and let them be cut by the planes $b'k'$, &c. which are drawn through b', &c. parallel to the plane bk, or the plane CAD. Then, because the lines Ab, bb', &c. are equal to one another, the bases of the parallelopipeds Ak, bk', &c. are equal to one another (I. 25.), and the parallelopipeds have the same altitude; therefore (24.), they are equal to one another, and the whole row AK is equal to five times the cube of M.

Again, through the points c', &c., let the planes $c'l$, &c. be drawn parallel to the plane ck, or (11. *Cor.* 2.) the plane BAD, to cut the plane dk produced. Then, because the lines Ac, cc', &c. are equal to one another, the bases of the parallelopipeds AK, cl, &c. are equal to one another (I. 25.); and the parallelopipeds have the same altitude; therefore (24.) they are equal to one another, and the whole tier AL is equal to three times AK; that is, to 3×5, or fifteen times the cube of M.

Lastly, through the points d', &c., let planes be drawn parallel to the plane dk, or (11. *Cor.* 2.) the plane BAC. Then, as before, because Ad, dd', &c. are equal, the parallelopipeds AL, dn, &c. are likewise equal to one another; and, therefore, the whole parallelopiped AN is equal to four times AL, that is to $4 \times 3 \times 5$, or sixty times the cube of M.

And, it is evident that a similar demonstration may be applied, whatever other numbers be taken in place of the numbers 5, 3, and 4.

Therefore, &c.

Scholium.

Hence the *solid content* of a rectangular parallelopiped is said to be equal to the *product of its three dimensions*, that is to $AB \times AC \times AD$, if AB, AC, AD are the three edges; this expression being interpreted in the same sense with the product of the two dimensions or sides, which is said to constitute the area of a rectangle, viz., that the number of cubical units in the parallelopiped is equal to the product of the numbers which denote how often the corresponding linear unit is contained in the three edges. Thus, if the linear unit be a foot, and the edges 3, 4, and 5 feet respectively, the solid content will be $3 \times 4 \times 5$, or 60 cubic feet.

It is likewise said, in a similar sense, that the solid content of a rectangular parallelopiped is equal to the *product of its base and altitude:* thus, in the example just stated, the number of square feet in the base is 4×5, or 20; and this, being multiplied by 3, the number of linear feet in the altitude, gives 60 for the number of cubic feet in the parallelopiped, as before.

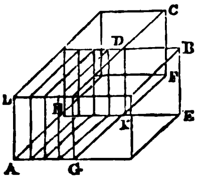
The cube is also the unit in the mensuration of all other solids; their contents being the same with the contents of rectangular parallelopipeds equal to them. Thus, since every parallelopiped (24. *Cor.*) is equal to a rectangular parallelopiped of equal base and altitude, *the solid content of every pa-*

rallelopiped is equal to the product of its base and altitude. The prism and the pyramid will presently furnish new examples.

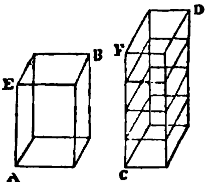
Prop. 26. (Euc. xi. 32.)

Parallelopipeds, which have equal altitudes, are to one another as their bases; and parallelopipeds, which have equal bases, are to one another as their altitudes; also, any two parallelopipeds are to one another in the ratio, which is compounded of the ratios of their bases and altitudes.

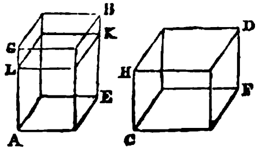
First, let A B, A C be two *rectangular* parallelopipeds, having equal altitudes, and let A E, A F be their bases; the parallelopipeds being so placed that a solid angle of the one may coincide, as at A, with a solid angle of the other. Then, the parallelopipeds will have a common part A D, which is likewise a rectangular parallelopiped, having the same altitude A L with them, and the two adjoining edges A G, A H common to it with A B, A C respectively. Let the base A K of this parallelopiped be divided into any number of equal rectangles by lines parallel to A H; and, therefore, the parallelopiped into the same number of equal parallelopipeds, by planes passing through these lines parallel to the planes L A H (24.) Then, if one of the first parts be contained in the base A E any number of times, exactly, or with a remainder; one of the others will be contained in the parallelopiped A B the same number of times, exactly, or with a corresponding remainder (24.). Therefore (II. def. 7.) the parallelopiped A B is to A D as the base A E to the base A K. And in the same manner it may be shewn that the parallelopiped A D is to A C as the base A K to the base A F. Therefore, *ex æquali* (II. 24.), the parallelopiped A B is to the parallelopiped A C as the base A E to the base A F.



Next, let A B, C D be two *rectangular* parallelopipeds, having equal bases; and let A E, C F be their altitudes. Then, if the altitude C F be divided into any number of equal parts; it may easily be shewn that the parallelopiped C D will be divided into the same number of equal parallelopipeds, by planes drawn through the points of division parallel to the base (I. 25. and 24.). And, if one of the first parts be contained in the altitude AE a certain number of times exactly or with a remainder, one of the others will be contained in the parallelopiped A B the same number of times exactly, or with a corresponding remainder (I. 25. and 24.). Therefore (II. def. 7.) the parallelopiped A B is to the parallelopiped C D as the altitude A E to the altitude C F.



Lastly, let A B, C D be any two *rectangular* parallelopipeds, having the bases A E, C F, and the altitudes A G, C H. Take A L equal to C H, and let A K be a third parallelopiped having the base A E and the altitude A L. Then, because the parallelopiped A B has to C D the ratio which is compounded of the ratios of A B to A K, and A K to C D (II. def. 12.); which ratios, by what has been just demonstrated, are the same with those of AG to AL or CH, and AE to C F; the parallelopiped A B has to the parallelopiped C D, the ratio which is compounded of the ratios of the altitude A G to the altitude G H, and of the base A E to the base C F.



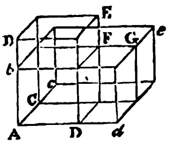
And what has here been demonstrated of rectangular parallelopipeds is true also with regard to any two parallelopipeds whatever, because these (24. *Cor.*) are equal to rectangular parallelopipeds having equal bases and altitudes with them.

Therefore, &c.

Prop. 27.

Any two rectangular parallelopipeds are to one another in the ratio, which is compounded of the ratios of their edges.

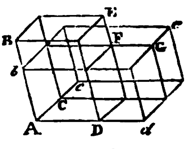
Let A E, A *e* be two rectangular parallelopipeds, the first having the edges A B, A C, A D, the other the edges A *b*, A *c*, A *d*; the parallelopiped A E shall be to the parallelopiped A B in the ratio which is compounded of the



ratios of A B to A *b*, A C to A *c*, and A D to A *d*.

For, the parallelopipeds being placed so that a solid angle of the one may coincide with a solid angle of the other as at A, they will have a common part A F, which is likewise a rectangular parallelopiped; and the face of this parallelopiped, which is opposite to D *b*, being produced to meet the plane *d e*, will cut off from the parallelopiped A *e*, the part A G, which is also a rectangular parallelopiped. Now, the parallelopipeds A E, A F have a common base D C; therefore (26.) they are to one another as their altitudes A B, A *b*; and, for the like reason, the parallelopipeds A F, A G are to one another as their altitudes A D, A *d*, and the parallelopipeds A G, A *e* are to one another as their altitudes A C, A *c*. But the parallelopiped A E has to the parallelopiped A *e* the ratio which is compounded of the ratios of A E to A F, A F to A G, and A G to A *e*. Therefore, the parallelopiped A E has to the parallelopiped A *e*, the ratio which is compounded of the ratios of A B to A *b*, A D to A *d*, and A C to A *c*; or, which is the same thing, which is compounded of the ratios of A B to A *b*, A C to A *c*, and A D to A *d*. (II. 27.)*

Therefore, &c.



Cor. 1. (Euc. xi. D.) The same may be said of any two parallelopipeds A E, A *e*, in which a solid angle of the one may be made to coincide with a solid angle of the other; that is, such parallelopipeds are to one another in the ratio, which is compounded of the ratios of the edges about the equal angles. For, the parallelopiped A G being completed as in the proposition, A E is to A F as the base B C to the base *b* C (26.), that is, as A B to A *b* (II. 35. *Cor.*); and in like manner A F is to A G as A D to A *d*, and A G to A *e* as A C to A *c*; therefore, as in the proposition, the parallelopiped A E is to the parallelopiped A *e*, in the ratio which is compounded of the ratios of A B to A *b*, A D to A *d*, and A C to A *c*; that is, in a ratio which is compounded of the ratio of the edges about the common angle A.

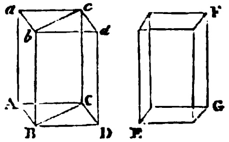
Cor. 2. Cubes are to one another in the triplicate ratio of their edges (II. 27. *Cor.* 2.); therefore, the triplicate ratio of two straight lines A B, *a b* is the same with the ratio of their cubes.

Cor. 3. If one straight line be to another as a third to a fourth, the cube of the first shall be to the cube of the second as the cube of the third to the cube of the fourth (II. 27.).

PROP. 28.

Every triangular prism is equal to a rectangular parallelopiped, which has an equal base and the same altitude.

Let A *b c* be a triangular prism, and let E F be a rectangular parallelopiped, having its base E G equal to the base A B C of the prism, and its altitude F G the same with the altitude of the prism: the prism A *b c* shall be equal to the parallelopiped E F.



Let the bases A B C, *a b c* of the prism be completed into the parallelograms A D, *a d*, and let D *d* be joined. Then it is evident that the solid A *d* is a parallelopiped (def. 11.). But the prism A *b c* is equal to half the parallelopiped A *d* (21.); and the parallelopiped E F is also equal to half the parallelopiped A *d*, because the base E G is equal to A B C, that is, to half of the base A D, and the altitude F G is the same with the altitude A *a* (26.). Therefore (I. ax. 5.) the prism A *b c* is equal to the parallelopiped E F

Therefore, &c.

PROP. 29.

Every prism is equal to a rectangular parallelopiped, which has an equal base and the same altitude.

Let A *d* C be a prism, having the

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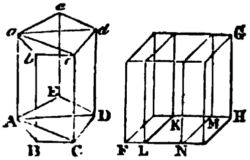
* This demonstration is analogous to that of II. 36. The proposition may, however, be otherwise, and more concisely established by the aid of Prop. 26.; for, since the parallelopipeds are to one another in the ratio which is compounded of the ratios of their bases and altitudes, and that the bases, which are rectangles under two of the conterminous edges of each, are to one another in the ratio which is compounded of the ratios of those edges (II. 36.), and the altitudes equal to the third edges, it follows that the parallelopipeds are to one another in the ratio which is compounded of the ratios of their edges. And the first corollary (analogous to II. 36. *Cor.* 1.) admits of a demonstration little different.

polygonal bases A B C D E, *a b c d e*, and let F G be a rectangular parallelopiped, having its base F H equal to the base A B C D E, and its altitude G H the same with that of the prism: the prism A *d* C shall be equal to the parallelopiped F G. Join A C, A D, *a c*, *a d*;



then, because A *a* and C *c* are parallel, they lie in the same plane (I. def. 12.) and the solid A *b c* is a triangular prism upon the base A B C (def. 14.). In the same manner it may be shewn, that A *c d*, A *d e*, are triangular prisms upon the bases A C D, A D E. Now, because the rectangle F H is equal to the polygon A B C D, it may be divided (I. 57.) into rectangles F K, L M, N H, severally equal to the triangles A B C, A C D, A D E, which together make up the polygon; and the parallelopiped F H may be divided into as many rectangular parallelopipeds, having the same altitude G H, and these rectangles for their bases. And, because these bases are severally equal to the bases of the triangular prisms A *b c*, A *c d*, A *d e*, and that their common altitude G H is the same with the common altitude of the prisms, the parallelopipeds, which stand upon them, are severally equal to the prisms (28.). Therefore their sum is equal to the sum of the prisms; that is, the parallelopiped F G is equal to the prism A *d* C.

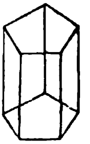
Therefore, &c.

Cor. 1. The solid content of every prism is equal to the product of its base and altitude (25. *Scholium*).

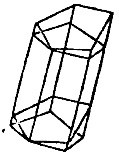
Cor. 2. (Euc. xii. 7. *Cor.* 2.) Prisms which have equal altitudes are to one another as their bases; and prisms which have equal bases are to one another as their altitudes: also, any two prisms are to one another in the ratio which is compounded of the ratios of their bases and altitudes. (26.).

Scholium.

With regard to the lateral surface of a prism, if it be a right prism, it is measured by the product of the principal edge, and the perimeter of the base; if oblique, by the product of the principal edge and the perimeter of a section, which is made by a plane perpendicular to the principal edge. For, in the case of the right prism, the sides are rectangles, which have a common altitude, and for their bases the sides of the base of the prism; therefore (I. 30.) their sum is equal to a rectangle of the same altitude, and having its base equal to the sum of those sides. And, in the case of the oblique prism, the sides of the perpendicular section, being perpendicular to the principal edges (5.), are the altitudes respectively of the parallelograms, which are the sides of the prism and have for their bases each a principal edge of the prism; therefore the sum of those parallelograms is equal to a rectangle, having likewise for its base a principal edge of the prism, and for its altitude the sum of their altitudes, that is, the perimeter of the perpendicular section.



In the latter case it is easy also to perceive that the content of the oblique prism is measured by the product of the principal edge, and the area of the perpendicular section above mentioned: for, if the lateral surface be produced, and cut by two such perpendicular planes passing through the extremities of any principal edge, as in the figure, the solids included between these planes and the bases of the oblique prism, may be made to coincide, and are therefore equal to one another: therefore the whole oblique prism is equal to the right prism, which has for its bases the two perpendicular sections.



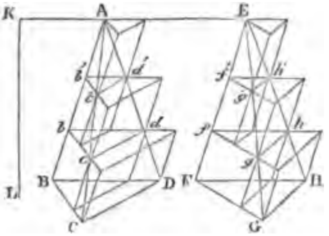
The convex surface of a prism has this obvious but remarkable property, that the sections made by any two parallel planes are similar and equal figures (12. and 15. *Cor.*): the convex surface, also, of a pyramid has its parallel sections similar, but they are not equal to one another.

PROP. 30.

Triangular pyramids, which have equal bases in the same plane, and their vertices in a straight line parallel to that plane, are equal to one another.

Let A B C D, E F G H be two triangular pyramids, having equal bases B C D, F G H in the same plane, and the straight line A E, which joins their vertices parallel to that plane: the pyramid A B C D is equal to the pyramid E F G H.

Since A E is parallel to the plane of the bases, the pyramids have a common



altitude K L. Let K L be divided into any number of equal parts; and let the pyramids be cut by planes, parallel to the plane of the bases, passing through the points of division. Let bcd and fgh, $b'c'd'$ and $f'g'h'$, &c. be the sections made by these planes. Then, because the triangles bcd, B C D lie in parallel planes, which cut the dihedral angles at the edges A B, A C, and A D of the pyramid A B C D, the angles b, c, d of the one are (15. *Cor.*) severally equal to the angles B, C, D of the other, and therefore (II. 31. *Cor.* 1.) the two triangles are similar. Therefore the triangle bcd is to the triangle B C D as bc^2 to BC^2 (II. 42. *Cor.*), that is, because bc is (12.) parallel to B C, as Ab^2 to AB^2 (II. 30. *Cor.* 2. and II. 37. *Cor.* 4.) And, in the same manner it may be shewn, that the triangle fgh is to the triangle F G H as Ef^2 to EF^2, that is (14.) as Ab^2 to AB^2. Therefore the triangle bcd is to the triangle B C D as the triangle fgh to the triangle F G H: and, because B C D is equal to F G H (II. 12.), bcd is equal to fgh (II. 18. *Cor.*). In the same manner it may be shewn, that $b'c'd'$ is equal to $f'g'h'$; and so of the rest.

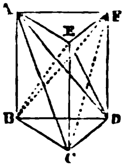
Now, if the triangular prisms with the bases bcd, $b'c'd'$, &c., and the edges bB, $b'b$, &c., be completed, and also the triangular prisms with the bases B C D, bcd, &c., and the edges Bb, bb', &c.; the former will, together, constitute a series of prisms inscribed in the pyramid A B C D, and the latter a series of prisms circumscribed about the same pyramid, which, excepting that upon the base B C D, are equal to the former, each to each, because they have equal altitudes towards opposite parts of the same base (29. *Cor.* 2.). And, because the pyramid exceeds the inscribed series, but is less than the circumscribed; it differs from the former, by less than the difference of the two series, that is, by less than the prism upon the base B C D. But, by increasing the number of parts into which the altitude is divided, the altitude of this prism may be diminished without limit, and so the prism itself be made less than any given solid. Therefore, the series of inscribed prisms may be made to approach to the pyramid A B C D, by less than any given difference. And in the same manner it may be shewn, that if prisms be similarly inscribed in the other pyramid E F G H, this series will approach to the pyramid E F G H, by less than the prism upon the base F G H, which is equal to that before-mentioned upon the equal base B C D; by less, that is, than the same given difference. But of the prisms inscribed in the two pyramids, those upon the bases bcd, and fgh, $b'c'd'$ and $f'g'h'$, &c. are equal to one another, because their bases are equal to one another, and they have the same altitude (29. *Cor.* 2.). Therefore, the series inscribed in the pyramid A B C D is equal to the series inscribed in the pyramid E F G H (I. ax. 2.). And because this equality always subsists, and that the two series may be made to approach to the two pyramids respectively within any the same given difference, the pyramid A B C D is equal to the pyramid E F G H. (II. 28.).

Therefore, &c.

Prop. 31. (Euc. xii. 7. *Cor.* 1.)

Every triangular pyramid is equal to the third part of a prism, having the same base and the same altitude with the pyramid.

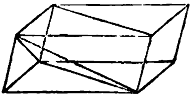
Let A B C D be a triangular pyramid, having the base B C D; let the sides A B C, A B D be completed into the parallelograms A B C E, A B D F, and join E F; then it is evident that the triangle A E F is the upper base of a prism B E F, standing upon the same base B C D with the pyramid, and having the same altitude: the pyramid A B C D shall be equal to a third part of the prism B E F.



Join D E. Then the prism B E F is made up of three pyramids, viz. A B C D, A C D E, and A D E F. The first of these is the original pyramid; the second is equal (30.) to a pyramid B C D E, upon the same base C D E, and having its vertex B in the same line A B parallel (10.) to the base; and the third is equal to a pyramid B D E F, upon the same base D E F, and having its vertex B in the same line A B parallel to the base (30.), that is, to a pyramid B C D F, upon the same base B D F with the last, and having its vertex C in the same line C E, parallel to the base. Therefore the prism B E F is equal to the three pyramids A B C D, B C D E, and B C D F. But of these, the two last are each of them equal to the first A B C D, because they stand upon the same base B C D with it, and their vertices are in the same lines with it, viz., A E, F A, parallel to the base (30.). Therefore, the prism B E F is equal to three times the pyramid A B C D; and the pyramid A B C D is equal to a third part of the prism B E F.

Therefore, &c.

Cor. If a parallelopiped be described, having three of its edges the same with three conterminous edges of a triangular pyramid, the pyramid shall be equal to a sixth part of the parallelopiped; for the latter is divided by its diagonal plane into two equal prisms (22.), one of which stands upon the same base, and is of the same altitude with the pyramid, and is, therefore, by the proposition, equal to three times the pyramid.



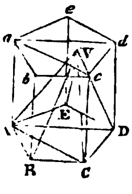
Scholium.

This proposition might have been demonstrated somewhat more concisely. It will be found, however, that our demonstration applies equally to the following more general theorem:—"If the upper part of a triangular prism (see the figure in the proposition) be cut away by a plane A E F, whether parallel to the base B C D, or otherwise, the remaining solid will be equal to the sum of three pyramids standing upon the same base B C D with the prism, and having for their vertices the upper extremities A, E, F of the diminished edges."

Prop. 32.

Every pyramid is equal to the third part of a prism, having the same base and the same altitude.

Let V A B C D E be a pyramid, and let *a b c d e* be the upper base of a prism standing upon the same base A B C D E, and having the same altitude with the pyramid: the pyramid shall be equal to the third part of the prism. Join A C, A D, *a c*, *a d*. Then the pyramid V A B C D E is divided by the planes V A C, V A D into triangular pyramids, having the common vertex V; and the prism, into as many triangular prisms, by the planes A C *c a*, A D *d a*. And because the former have the same altitude, and stand upon the same bases with the latter respectively, they are equal to their third parts, each of each. Therefore, also, the sum of the former, that is, the whole pyramid V A B C D E, is equal to a third part of the sum of the latter, that is, to a third part of the whole prism A *d c*.



Therefore, &c.

Cor. 1. The solid content of every pyramid is equal to one-third of the product of its base and altitude (29. *Cor.* 1.).

Cor. 2. (Euc. xii. 5, 6.) Pyramids which have equal altitudes are to one another as their bases: and pyramids which have equal bases are to one another as their altitudes: also, any two pyramids are to one another in the ratio compounded of the ratios of their bases and altitudes. (29. *Cor.* 2.).

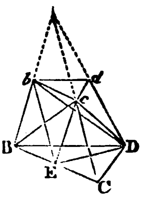
Scholium.

It was observed in the *Scholium* at i. 29. that the measurement of any plane rectilineal figure in general depends upon that of the triangle, because every such figure may be divided into triangles. In like manner, we may here remark, that the solid content of the pyramid leads to that of every other polyhedron; and that the latter may be found by adding together the contents of all the pyramids which constitute the polyhedron. These may be assumed with a common vertex, which may be either within the solid, in which case the several faces will be the bases of so many pyramids together making up the polyhedron, or at one of the angular points, in which case the adjoining faces will be divided into sides, and the others will be bases of the component pyramids.

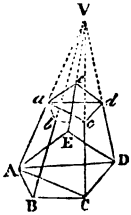
Prop. 33.

If the upper part of a pyramid be cut away by a plane parallel to the base; the remaining frustum shall be equal to the sum of three pyramids, having the same altitude with the frustum, and for their bases the bases of the frustum, and a mean proportional between them.

First, let B C D, *b c d* be the two bases of the frustum of a triangular pyramid: draw *c* E parallel to *b* B, to



meet B C in E, and join D E. Then, because the triangles B C D, *b c d* are equiangular (12. and 15.), and therefore similar, B C D is to *b c d* in the duplicate ratio (II. 42.) of B C to *b c*, or (I. 22.) BE, that is, in the duplicate ratio of the same B C D to B E D (II. 39.). Therefore the triangle B E D is a mean proportional between B C D and *b c d* (II. def. 11.) Now, the frustum is made up of three pyramids, namely, *c* B C D, D *b c d*, and *c b* B D. And, of these the first has the same altitude with the frustum, and stands upon the base B C D ; the second has likewise the same altitude with the frustum, and stands upon the base *b c d*; and the third, because *c* E is parallel (10.) to the plane B *b d*, is equal (30.) to the pyramid E *b* B D, which has the same altitude, and stands upon the base B E D, that is, upon a base which is a mean proportional between B C D and *b c d*.



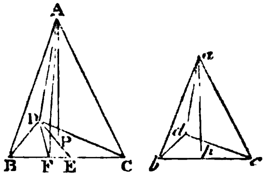
Next, let A B C D E, *a b c d e* be the two bases of the frustum of any pyramid, having the vertex V. Then, if the planes V A C, V A D be drawn cutting the bases in the straight lines A C, *a c*, A D, *a d*, the frustum in question will be divided into frustums of triangular pyramids. And each of these, by what has been already demonstrated, is equal to the sum of three pyramids, having the same altitude with the frustum, and for their bases respectively, the two bases of the frustum and a mean proportional between them. Therefore, their sum, that is, the frustum in question, is equal to three pyramids having the same altitude, and for their bases the sum of the lower bases, the sum of the upper bases, and the sum of the means respectively. Now, the two former sums are the bases A B C D E, *a b c d e*. And the sum of the means is a mean between the bases A B C D E, *a b c d e*; for, the triangles ABC, ACD, &c. having to the triangles *a b c*, *a c d*, &c. the same ratio, each to each, viz. that of VA^2 to va^2 (II. 42. and II. 32.) must have the same ratio also to their respective means (II. 27. *Cor.* 3. and note[*]); wherefore the sum of the former, that is, the base A B C D E, has to the sum of the means the same ratio (II. 23. *Cor.* 1.), viz. the subduplicate ratio of A B C to *a b c*, or of A B C D E to *a b c d e*; and accordingly (II. def. 11.) the sum of the means is a mean proportional between A B C D E and *a b c d e*. Therefore, in this case also, the frustum in question is equal to the sum of three pyramids, having the same altitude with it, and for their bases its two bases, and a mean proportional between them.

Therefore, &c.

PROP. 34. (EUC. xii. 8. and 8 Cor.)

Similar pyramids are to one another in the triplicate ratio of their homologous edges.

In the first place, let A B C D and *a b c d* be two similar *triangular* pyramids, and let the edges B C, *b c* be homologous: the pyramid A B C D shall have to the pyramid *a b c d* the triplicate ratio of that which B C has to *b c*.



Because the faces of the two pyramids are similar triangles, it is evident that their corresponding edges are to one another in the same ratio, so that A B is to *a b* as B C to *b c*, and as B D to *b d*. And, because the plane and dihedral angles of the solid angle B are severally equal in order to the plane and dihedral angles of the solid angle *b*, these solid angles may be made to coincide. Therefore (27 *Cor.* 1.) the parallelopipeds, which have the edges B A, B C, B D, and *b a*, *b c*, *b d*, are to one another in the ratio which is compounded of the ratios of B A to *b a*, B C to *b c*, and B D to *b d*, that is, since these ratios are equal to one another, the triplicate ratio (II. 27. *Cor.* 2.) of that which B C has to *b c*. But the pyramids A B C D, *a b c d* are (31. *Cor.*) the sixth parts of these parallelopipeds, each of each, and therefore have to one another (II. 17.) the same ratio with the parallelopipeds. Therefore, the pyramid A B C D has to the pyramid *a b c d* the triplicate ratio of that which B C has to *b c*.

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[*] If a magnitude A be the first of three proportionals A, B, C, and if the same A be the first also of three other proportionals A, *b*, *c*, then if B be greater than *b*, C shall likewise be greater than *c*. For C is to *c* in the ratio which is compounded of the ratios of C to B, B to *b*, and *b* to *c*, that is, (because the first of these ratios is the same with that of B to A, and the last the same with that of A to *b*,) in the ratio which is compounded of the ratios of B to A, A to *b*, and B to *b*, or in the duplicate ratio of B to *b*. Wherefore because B is greater than *b*, C is likewise greater than *c*.

Hence it follows, that if A, B, C be proportionals, and if A', B', C' be likewise proportionals, and if A' has a greater ratio to B' than A to B, A' shall also have a greater ratio to C' than A to C. For, if *b* be taken such that A' may have to *b* the same ratio as A to B, *b* will be greater than B' (II. 11. *Cor.* 3.); wherefore, if *c* be taken such that A', *b*, *c* may be proportionals, *c* will likewise be greater than C'; and therefore A' will have to C' a greater ratio than it has to *c*. But A' has to *c* the same ratio as A to C (II. 24.). Therefore A' has to C' a greater ratio than A to C.

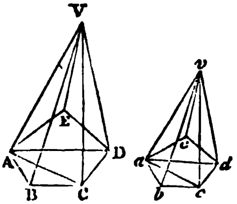
And hence it is evident, that *if any three magnitudes* A, B, C *be proportionals, and likewise any other three*, A', B', C', *and if* A *is to* C *as* A' *to* C', A *is to* B *as* A' *to* B'; for if A has not to B the same ratio as A' to B', then neither can A have to C the same ratio as A' to C'. This last is the theorem demanded in the text; and is evidently the converse of II. 27. *Cor.* 3., from which alone, however, it may be considered as sufficiently apparent.

Otherwise:

Take B E (II. 52.), a third proportional to B C, *b c*; and, again, B F a third proportional to *b c*, B E; so that B C may have to B F the triplicate ratio of that which it has to *b c* (II. def. 11.): join D E, D F, A F; and, from A, *a*, let A P, *a p* be drawn perpendicular to the bases B C D, *b c d* of the pyramids, each to each. Then, because the solid angles B, *b* may be made to coincide, and that in this case (5. *Cor.*) the triangles A B P,* *a b p* will be in the same plane, and on account of the parallelism of A P, *a p* (5.) will be equiangular (I. 15.), A P is to *a p* (II. 31.) as A B to *a b*, or as B C to *b c*. Again, because, in the triangles D B C, *d b c*, D B is to *d b* as B C to *b c*; that is, as *b c* to B E; the triangles D B E, *d b c* are equal to one another (II. 41.) But D B E is to D B F (II. 39.) as B E to B F; that is, as B C to *b c*. Therefore, *d b c* is to D B F as B C to *b c*. But B C is to *b c* as A P to *a p*; therefore, *d b c* is to D B F as A P to *a p*, and consequently (II. 11. *Cor.* 2.) the ratio, which is compounded of the ratios of *d b c* to D B F, and *a p* to A P is a ratio of equality. But the pyramid *a b c d* is to the pyramid A B F D in a ratio which is compounded of the ratios of *d b c* to D B F, and *a p* to A P (32. *Cor.* 2.); therefore, the pyramid A B F D is equal to the pyramid *a b c d*. Now, the pyramid A B C D is to the pyramid A B F D, as the base D B C to the base D B F (32. *Cor.* 2.); that is, as B C to B F (II. 39.), or in the triplicate ratio of B C to *b c*. Therefore, the pyramid A B C D is to the pyramid *a b c d* in the triplicate ratio of that which B C has to *b c*.*

The proposition is therefore demonstrated with regard to triangular pyramids.

Next, let V A B C D E, *v a b c d e* be any two similar pyramids, and let B C, *b c* be homologous edges: the pyramid V A B C D E shall be to the pyramid *v a b c d e* in the triplicate ratio of that which B C has to *b c*.



Because the plane and dihedral angles of the solid angle V are severally equal to the plane and dihedral angles of the solid angle *v*, these solid angles may be made to coincide; and, if this be done, the bases A B C D E, *a b c d e* will be parallel (15.), because the faces V A B and *v a b*, V B C and *v b c*, &c. being similar, their sides A B and *a b*, B C and *b c*, &c. will be parallel (I. 15.). Hence it appears that the planes V A C and *v a c*, V A D and *v a d* make equal angles with the adjoining faces of the pyramids; and that the triangles V A C and *v a c*, as also V A D and *v a d*, are similar. And it is easy to show that the triangles A B C, A C D, A D E are respectively similar to the triangles *a b c*, *a c d*, *a d e* (II. 32.). Therefore, the pyramids in question are divided by those planes into similar triangular pyramids V A B C and *v a b c*, V A C D and *v a c d*, VADE and *v a d e* (def. 10.). And, since the homologous edges of these pyramids, B C and *b c*, C D and *c d*, D E and *d e*, are to one another in the same ratio; and, by what has been already demonstrated, the pyramids to one another in the triplicate ratios of their homologous edges, that is likewise (II. 27. *Cor.* 3.) in the same ratio; the whole pyramid V A B C D E is (II. 23. *Cor.* 1.) to the whole pyramid *v a b c d e* in the latter ratio, *i. e.* in the triplicate ratio of that which B C has to *b c.*

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* The lines B P, *b p* are omitted in the figure.

* These demonstrations are respectively analogous to those given of II. 42. in page 66. The second, it is obvious, admits of considerable abridgment, by reference to 32. *Cor.* 2.; for, since the pyramids are to one another in the ratio which is compounded of the ratios of their bases and altitudes; and that the bases are to one another in the duplicate ratio of two homologous edges (II. 42.), and the altitudes have to one another the same ratio with those edges, it follows that the pyramids are to one another in the triplicate ratio of their homologous edges. And in this form the demonstration applies equally to the general case in which the bases are similar polygons.

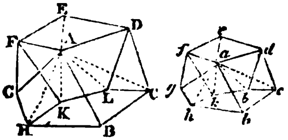
Therefore, &c.

Cor. Since the triplicate ratio of two straight lines is the same with the ratio of their cubes (27. *Cor.* 2.), similar pyramids are to one another as the cubes of their homologous edges.

Prop. 35.

Similar polyhedrons are to one another in the triplicate ratio of their homologous edges.

Let A C G, *a c g* be two similar polyhedrons, and let B C, *b c* be any two



homologous edges: the polyhedron A C G shall be to the polyhedron *a c g* in the triplicate ratio of that which B C has to *b c.*

In the first place, it is evident that the corresponding edges of the polyhedrons are to one another each to each in the same ratio, because they are homologous sides of the similar faces of the polyhedrons. Let the polyhedrons be so placed as to have two corresponding edges A B, *a b* parallel to one another, and two corresponding faces A B C D, *a b c d* adjoining to those edges likewise parallel. Then, because the angles A B C, *a b c* are equal to one another, B C is likewise parallel to *b c* (15.); and for the like reason C D, D A are parallel to *cd, da* respectively. Again, since A B is parallel to *a b*, a plane may be made to pass through A B parallel to the plane *a b h g* (15.), and that plane is no other than A B H G, because the dihedral angles at A B and *a b* are equal of one another (17. *Scholium*, 3. and 11. *Cor.* 1.); and for the like reason the other similar faces of the two polyhedrons lie in parallel planes. Therefore, also, any two corresponding edges being the common sections of such planes, are parallel (12. *Cor.*).

Join A C and *a c*, A H and *a h*, A K and *a k*, A L and *a l*; that is, join the points A, *a*, with all the angular points of their respective figures, to which they are not already joined by the edges terminating in A, *a*. Then, because the triangles A B C, *a b c*, are similar (II. 32.) the angles BAC, *b a c* are equal to one another; but the planes B A C, *b a c* are parallel, and A B is parallel to *a b*; therefore, also (15.), AC is parallel to *a c*: and, for the like reason, A H is parallel to *a h*. Again, because A F and F K are parallel respectively, and in the same ratio, to *af* and *fk*, the triangles AFK, *afk* are similar (15. and II. 32.), and their planes parallel; wherefore, as before, AK is parallel to *a k*; and for the like reason the triangles A D L, *a d l* are similar and in parallel planes, and A L is parallel to *a l*. And because the triangles A C L, *a c l* have the three sides of the one parallel to the three sides of the other, each to each, they are equiangular (15.), and lie in parallel planes: the same also may be said of the triangles A K L, *a k l*, and A H K, *a h k.*

Therefore, the pyramids into which the solid A C G is divided by the planes ACL, ADL, AFK, AHK, AKL are similar to the pyramids into which the solid *a c g* is divided by the planes *a c l*, *a d l*, *a f k*, *a h k*, *a k l*, each to each: for it has been shown that the faces in these planes are similar, each to each, and their other faces are similar, because they are similar faces, or similar parts of similar faces, of the polyhedrons; also the dihedral angles made by any two similar and adjoining faces are equal (17. *Scholium*, 3.), because their planes are parallel. And the former pyramids are to the latter (34.) in the triplicate ratios of their homologous edges, each to each; that is, in the triplicate ratio of B C to *b c*, because their edges have to one another the common ratio of B C to *b c*. Therefore, also, the sum of the former pyramids, that is, the polyhedron A C G, is (II. 23. *Cor.* 1.) to the sum of the latter, that is, to the polyhedron *a c g*, in the same ratio, viz., the triplicate ratio of B C to *b c.*

Therefore, &c.

Cor. Similar polyhedrons are to one another as the cubes of their homologous edges (27. *Cor.* 2.).

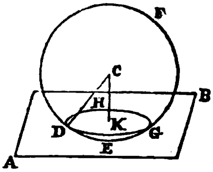
Section 4.—*Problems.*

In the following problems it is assumed that a sphere may be described from any centre, and at any distance from that centre: also, a plane is considered to be drawn, when two straight lines are drawn which lie in that plane.

Prop. 36. Prob 1.

To draw a straight line perpendicular to a given plane A B, *from a given point* C *without it.*

From C, to the plane, draw any straight line C D; from the centre C, with the radius C D, describe a sphere D E F. If this sphere touch the plane in D, C D is the perpendicular required (8. *Cor.*); if not, take K, the centre of the circle D G H in which it cuts the plane, and join C K; C K will be the perpendicular required (8. *Cor.*).



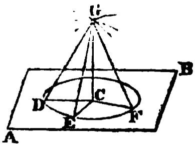
Therefore, &c.

Another method has been seen in prop. 7.

Prop. 37. Prob. 2.

To draw a straight line perpendicular to a given plane A B, *from a given point* C *in the same.*

From the centre C, with any radius C D, describe, in the plane A B, the circle D E F. In the circumference of this circle, take any three points D, E, F; and join C D, C E, C F.



Now, let C G be the perpendicular required. From the centre D, with any radius greater than D C, describe a sphere, and let its surface cut the perpendicular C G in G; and join G D, G E, G F. Then, because G E and G F are each of them equal to G D (8.), if spheres be described from the points E and F, each with a radius equal to D G, their surfaces will likewise each of them pass through the point G. And G is the only point on that side of the plane A B, which is found at the same time in the surface of each of the three spheres. For, every point, as G, which is at equal distances from the three points D, E, F, must lie in the perpendicular which passes through the centre of the circle D E F, because (I. 7.) the angles G C D, G C E, G C F are equal to one another, and are therefore (9. *Cor.*) right angles; and every other point of the perpendicular on the same side of A B is at a greater or a less distance from D than G is (I. 12. *Cor.* 2.). Reversely, therefore, from the centres D, E, F, with any the same radius greater than D C, describe three spheres, and let G be the point common to the surfaces of these spheres; join C G; C G will be the perpendicular required.

Therefore, &c.

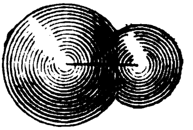
Another method has been seen in prop. 7.

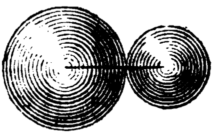
Cor. It appears from the demonstration, that, if a point be equally distant from the three angles of a triangle, it must lie in a perpendicular to the plane of the triangle, which passes through the centre of the circumscribing circle.

Scholium.

Had the intersection of the spherical surfaces, in this problem, been considered at a greater length, the analogy of the construction to that employed in the first method of Book i. prop. 44, might have been lost sight of in the detail. The subject is, however, of sufficient interest to merit the attention of the student; and, as it has not hitherto found a place in this treatise, the present is, perhaps, the most proper for its consideration.

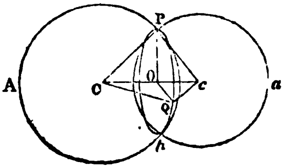
1. Two spheres will, 1°, cut one another, or 2°, touch one another, or 3°, one of them fall wholly without the other, according as the distance between their centres is, 1°, less than the sum, and greater than the difference of their radii, or, 2°, equal to the sum, or to the difference of their radii, or, 3°, greater than the sum, or less than the difference of their radii. This is easily inferred from what has been already demonstrated with regard to the intersection of circles; for it is evident that the sections of two spheres, made by a plane passing through their centre and through any





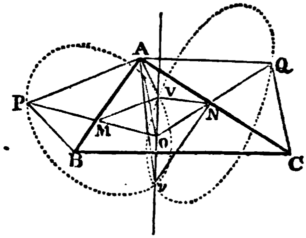
point which is supposed to be common to the two surfaces, will be circles having the same radii and centres with the spheres respectively.

2. If two spheres cut one another, they shall cut one another in a circle, the plane of which is perpendicular to the line joining their centres, and its centre in that line.



Let C, *c* be the centres of two spheres, and P, Q any two points common to the surfaces of both; join C P, P *c*, C Q, Q *c*, and C *c*; draw P O perpendicular to C *c*; and join O Q. Then, because the triangles C P *c*, C Q *c* have the three sides of the one equal to the three sides of the other, each to each, the angle P C O is equal (I. 7.) to the angle Q C O. And because the triangles P C O, Q C O have two sides PC, CO of the one equal to two sides Q C, C O of the other, each to each, and the included angles equal to one another; the remaining sides P O, Q O are equal to one another, and the angle Q O C is equal to P O C, that is, to a right angle. Therefore, the point Q is in the circumference of a circle which is described from the centre O, with the radius O P, in a plane (3. *Cor.* 1.) perpendicular to C O or C *c*. And the same may be shewn of every other point Q in the common section of the two spheres. Therefore that common section is a circle, the plane of which is perpendicular to the line C *c*, and its centre in that line, as above stated. In fact, if the plane P C *c* cut the spheres in the circles A P *p* and *a p* P, and if it be made to revolve about the line C *c*, the circles A P *p*, *a p* P will generate the two spheres (def. 21.), and the line O P (3. *Cor.* 2.) the circle of intersection.

3. If three spheres be described, whose centres A, B, C do not lie in the same straight line, the surfaces of the three cannot have more than two points in common, which points (if there be any such) lie in a straight line perpendicular to the plane of the centres, and at equal distances on either side of that plane.



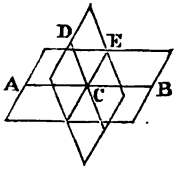
From the points A, B, with the radii of the spheres described from A, B, respectively, in the plane A B C, describe two circles, and let them cut one another in P; and from P draw P M perpendicular to A B: then M is the centre, and M P the radius, of the circle (dotted in the figure) in which the spheres, whose centres are A and B, cut one another. In like manner draw the line Q N, so that N may be the centre, and N Q the radius, of the circle (dotted in the figure) in which the spheres, whose centres are A and C, cut one another. Then, it is evident that the only points, common to the surfaces of the three spheres, are those which are common to the circumferences of these two circles. Let P M, Q N be produced to meet one another in the point O; and let the planes of the circles cut one another in the line V *v*, which passes through O, and is perpendicular to the plane A B C (18. *Cor.* 2.), because the planes of the circles, being perpendicular to the lines AB, AC respectively, are perpendicular, each, to the plane A B C, which passes through each of those lines (18.). Then, if M P be greater than M O, the circle, which has the centre M and the radius M P, will cut the line V *v* in two points V, *v*, which are equally distant from the point O (I. 12.). Join V N, *v* N. Then, because A N is at right angles to the plane Q N V, the angles A N V, A N *v* are right angles. And, because V, *v* are points in the surface of the sphere which has the centre A and radius A P or A Q, A V and A *v* are, each of them, equal to A Q. Therefore, because the right-angled (3.) triangles A N V, A N *v* have their hypotenuses A V, A *v*, and sides A N equal respectively to the hypotenuse A Q and side A N of the right-angled triangle A N Q, their other sides N V, N *v* are equal each to N Q (I. 13.); and, consequently, the circle which has the centre N and the radius N Q also passes through the same two points V, *v*. But, the circumferences of the two circles do not meet one another in any other point; for they can meet only in the line V *v*, in which their planes cut one another. Therefore the points V, v, which are in a perpendicular to the plane A B C, and at equal distances on either side of that plane, and none else, are common to the surfaces of the three spheres; as above stated.

If the point V coincide with O, the point v will likewise coincide with it; and in this case the three spherical surfaces will have only one point common to them, viz. the point O, which is in the plane of the three centres A, B, C.

If the centres of three spheres lie in the same straight line, their circles of intersection (if there are any) cannot meet one another, because their planes are perpendicular to this straight line, and therefore (11.) parallel; and accordingly the surfaces can have no point in common; unless each of them passes through the same point of the straight line in which the centres lie, for then each of them will touch the other two in that point.

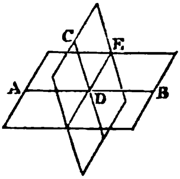
Prop. 38. Prob. 3.

Through a given point C, *to draw a plane, which shall be perpendicular to a given straight line* A B.



If the point C lies in the straight line A B, from C draw any straight line C D perpendicular to A B (I. 44.): through A B draw any plane not coinciding with the plane A C D, and draw C E in that plane perpendicular to A B (I. 44.): then the plane D C E (3.) is the plane required.

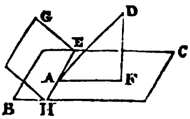
But if C do not lie in A B, draw C D perpendicular to A B (I. 45.): through A B draw any plane not coinciding with the plane A D C, and draw D E in that plane perpendicular to A B (I. 44.): then the plane C D E is the plane required.



Therefore, &c.

Prop. 39. Prob. 4.

Through a given point A *in a given plane* B C, *to draw a straight line, which shall lie in the plane* B C, *and be at right angles to a given straight line* A D, *which cuts the plane in that point.*



Through A draw a plane G H (38.) perpendicular to A D, and let it cut the given plane B C in the line A E: A E will be the straight line required. For, A D, being perpendicular to the plane G H, is perpendicular to the line A E, which meets it in that plane (def. 1.).

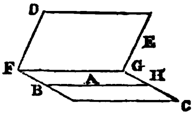
Otherwise:

From any point D in A D draw D F perpendicular to the plane B C (36.): join A F; and, in the plane B C, draw A E perpendicular to A F; then A E (4.) will be the line required.

Therefore, &c.

Prop. 40. Prob. 5.

Through a given point A *in a given plane* B C, *to draw a straight line, which shall lie in that plane, and be parallel to another given plane* D E.



If the planes B C, D E are parallel; then any straight line which is drawn through the point A in the plane B C will be parallel to the plane D E (def. 6. and def. 3.).

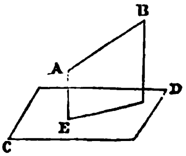
If the planes are not parallel, let them be produced to cut one another in the straight line F G, and through A draw A H parallel to F G (I. 48.): A H will be the line required (10.).

Therefore, &c.

Cor. In the same manner a straight line may be drawn through a given point, which shall be parallel to each of two given planes; viz., by producing the planes to cut one another, and drawing the required straight line parallel to their common section.

Prop. 41. Prob. 6.

Through a given straight line A B, *to draw a plane, which shall be perpendicular to a given plane* C D.



If the straight line A B is perpendicular to the plane C D, then any plane which passes through A B will be perpendicular to the plane C D (18.).

But, if A B be not perpendicular to C D, from any point A in A B draw A E perpendicular to C D (36. or 37.); and the plane B A E will be the plane required.

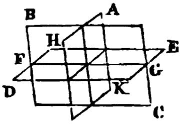
Therefore, &c.

Prop. 42. Prob. 7.

Through a given point A *to draw a plane which shall be perpendicular to each of two given planes* B C, D E.

If the given planes are parallel, from A draw (36.) a straight line perpendicular to either of them; and any plane passing through this straight line will be perpendicular to each of the given planes (11. and 18.)

If the planes are not parallel, let them be produced to meet one another in the straight line F G, and through A draw the plane H K perpendicular to F G (38.): H K will be the plane required (18.). Or, in this case, draw from the point A perpendiculars to the two planes (36.); and the plane of these perpendiculars will be the plane required (18.).

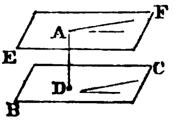


Therefore, &c.

Prop. 43. Prob. 8.

Through a given point A, *to draw a plane, which shall be parallel to a given plane* B C.

From A draw the straight line A D perpendicular to the plane B C (36.); and through A draw the plane E F (38.) perpendicular to the straight line A D: the plane E F is the plane required (11.).



Otherwise:

In the given plane B C, draw any two straight lines cutting one another; and, through the given point A, draw parallels to them respectively: the plane of these parallels will be the plane required (15.).

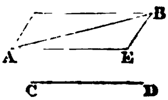
Therefore, &c.

Prop. 44. Prob. 9.

Through a given straight line A B, *to draw a plane, which shall be parallel to a second given straight line* C D.

If A B is parallel to C D, then any plane (10.) which passes through A B, and does not pass through C D, is parallel to C D.

But, if A B is not parallel to C D, through any point A of A B draw the straight line A E parallel to C D (I. 48.): the plane B A E is the plane required (10.).

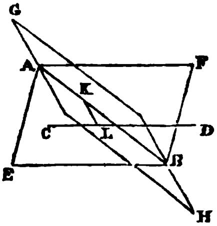


Therefore, &c.

Prop. 45. Prob. 10.

To find the shortest distance between two given straight lines, A B *and* C D, *which do not lie in the same plane.*

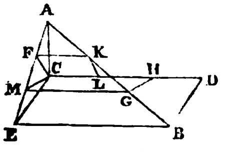
Through A B draw the plane E F, parallel to the line C D (44.); and through the same A B draw the plane G H, perpendicular to the plane E F (41.). Then, because the straight line C D is parallel to the plane E F, which cuts the plane GH in the line A B, it cuts, or may be produced to cut, the plane G H in some point L. From L, draw L K perpendicular to A B: K L shall be the shortest distance required.



For, the plane G H, being perpendicular to the plane E F, the straight line K L, which is drawn in the former plane, at right angles to the common section A B, must be perpendicular to the plane E F (18.), and therefore (10. and I. 14.) to the line C D, which is parallel to the plane E F. Now, because the line A B lies in the plane E F, C D cannot be nearer to A B than it is to the plane E F, to which it is parallel. But because K L is perpendicular to the plane E F, K L is the shortest distance of C D from the plane E F. Therefore K L is the shortest distance between the straight lines A B and C D.

Otherwise:

From any point A in A B, draw A C perpendicular to C D (I. 45.): from C draw C E perpendicular to the plane A C D (37.), and let A B cut the plane D C E in B: complete the rectangle B D C E, and join A E: from C draw C F perpendicular to A E (I. 45.): through F draw F K (I. 48.), parallel to B E or CD, to meet AB in K: take C L equal to F K, and join K L: K L shall be the shortest distance required.



For, let G be any other point in A B,

and G H its perpendicular distance from C D: through G draw G M, parallel to B E or D C (I. 48), to meet A E in M, and join C M. Then, because G M and H C are in the same plane (I. def. 12.), G H and M C are likewise in the same plane; and, because C D is perpendicular both to C A and C E, it is perpendicular also to CM and CF (3.): but, CD is likewise perpendicular to G H; therefore (I. 14.) C M is parallel to H G, and the quadrilateral C G is a rectangle. Again, because F K, C L are parallel and equal, and that the angle F C L is a right angle, C K is a parallelogram (I. 21.) and a rectangle. Therefore (I. 22.) G H is equal to C M, and K L to C F. But C F (I. 12. *Cor.* 3.) is less than C M, because C F is perpendicular to A E. Therefore, also, K L is less than G H. And in the same manner it may be shown that K L is less than any other distance between the lines A B, CD. If AB do not meet the plane DCE, that is, if it be parallel to that plane, AC is the shortest distance required.

Therefore, &c.

Cor. 1. The shortest distance between two straight lines is a straight line which is at right angles to each of them. This is evident from each of the constructions above given. Indeed, if such were not the case, it could not be the shortest distance; for a perpendicular to the line which it does not meet at right angles, drawn from the point in which it meets the other, would be shorter (I. 12. *Cor.* 3.).

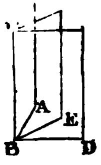
The problem shows, therefore, how to draw a straight line, which shall be at right angles to each of two given straight lines not lying in the same plane.

Cor. 2. If, in the second construction, the perpendiculars A C and B D be given, the length of the shortest distance K L may be found; for it is equal to the perpendicular C F, which is drawn from the right angle to the hypotenuse of a right-angled triangle having those perpendiculars for the two sides.

PROP. 46. Prob. 11.

To bisect a given dihedral angle A B C D.

Draw from any the same point B in the edge B C, BA and B D perpendicular to it (I. 44.) in the planes C B A and C B D respectively, and bisect the angle ABD by the straight line B E (I. 46.): the plane E B C will bisect the given dihedral angle A B C D (17.).



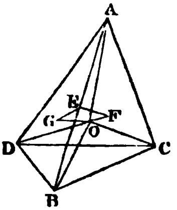
Therefore, &c.

Cor. Any point in the bisecting plane is at equal distances from the planes of the dihedral angle A B C D (I. 13); for it may easily be shown that these distances are the sides of right-angled triangles, which have a common hypotenuse and the angles opposite to them equal to one another.

PROP. 47. Prob. 12.

To circumscribe a sphere about a given tetrahedron or triangular pyramid A B C D.

Bisect any one of the edges, A B, in E (I. 43.): from E, in the adjoining faces A B C, A B D, draw E F, E G perpendicular to AB (I. 44.): take F, G, the centres of the circles circumscribing the triangles A B C, A B D (III. 59.); and therefore in the straight lines E F, E G respectively: from the points F, G, in the plane F E G, draw F O, G O, perpendicular to F E, G E (I. 44.) respectively, and meeting one another in O, and join OA: the sphere, which is described from the centre O, with the radius O A, shall pass through the points B, C, D, and circumscribe the tetrahedron.



Join O B, O C, O D. Then, because E F and E G are each of them at right angles to E A, the plane F E G is perpendicular to E A (3.), and therefore (18.) to each of the planes A B C, A B D, which pass through E A. But O F and O G are perpendicular to the common sections F E and G E respectively. Therefore (18.) O F and O G are perpendicular to the planes A B C and A B D respectively. And because the points A, B, C, are equidistant from the foot of the perpendicular O F, they are likewise equidistant from the point O in the perpendicular (8.); therefore O B and O C are each of them equal to O A. And for the like reason O B (as before) and O D are each of them equal to O A. Therefore, the four distances, O A, O B, O C, O D, are equal to one another; and the sphere, which is described from the centre O, with the radius O A, is circumscribed about the tetrahedron A B C D.

Otherwise:

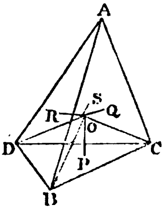
Bisect three of the edges at right angles by planes meeting one another in the point O (38.): O is evidently the centre of a sphere, passing through the points A, B, C, D, and circumscribing the pyramid.

Therefore, &c.

Prop. 48. Prob. 13.

To inscribe a sphere in a given tetrahedron or triangular pyramid A B C D.

Bisect the dihedral angles at the edges BC, CD, DB of any the same face,(46.) and let the bisecting planes meet one another in the point O: from O draw OP, OQ, O R, O S, perpendicular to the faces B C D, ABC, ABD, ACD, respectively (36.): the sphere, which is described from the centre O, with the radius O P shall pass through the points Q, R, S, and be inscribed in the tetrahedron.



For, the point O being in the plane which bisects the dihedral angle at B C, is equidistant (46. *Cor.*) from the planes A B C, B C D, that is, O Q is equal to O P; and, for the like reason, O R and O S are each of them equal to O P; therefore, the four distances, O P, O Q, O R, O S, are equal to one another: and the sphere, which is described from the centre O, with the radius O P, is inscribed in the tetrahedron A B C D.

Therefore, &c.

Cor. If the faces and altitude A L of the tetrahedron be given, the length of the radius O P may be found. For, since the pyramid, which has the altitude A L and the base B C D, is equal to the sum of four pyramids having the common altitude O P, and for their bases the four faces of the first pyramid; or, which is the same thing, to a pyramid, which has the altitude O P, and a base equal to the sum of those faces (30.); it may easily be shown (32. *Cor.* 2. and II. 10. *Cor.*) that O P is to A L as the base B C D to the sum of the four faces.

Scholium.

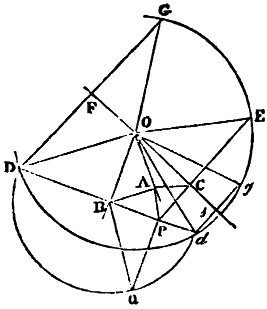
In these problems, 12. and 13., it is assumed that, three planes being given, a point may be found which is common to the three, or that, any *three planes* being drawn, there is some point through which each of them passes, and which may be called their *point of intersection.* It is evident, however, that if the common section of two of the planes be parallel to the third plane (in which case, also, by Prop. 10, the common sections are all parallel to one another), there can be no such point; for, since all the points which are common to the two first planes are found in their common section, if this common section does not cut the third plane (that is, in other words, if there is no point common to this common section and the third plane), there can be no point which is common to the three planes. When, on the contrary, the common section of two of the planes is not parallel to the third, it may be produced to cut that plane, and the point in which it cuts it is the point which is common to the three planes.

When a point is to be determined, therefore, by the intersection of three planes, in order to be satisfied with regard to the meeting of the planes, it is only necessary to consider the common section of any two of them, and examine whether it is or is not parallel to the third plane. Thus, in Problem 13, the question occurs, whether O C is, or not, parallel to the plane which is made to pass through DB; and it is easily perceived that it is not parallel; for, if the plane D B A be supposed to be turned round D B, in order to arrive at the position in which it will be parallel to O C, before it arrives at that position it must evidently become parallel to A C, and DBO is necessarily more distant from the position of parallelism than D A B when parallel to AC, for it passes between the face A D B of the pyramid, and the base C D B. In the second solution of Prob. 12, the three planes, which are supposed to meet in O, are at right angles to the edges, and therefore the common section of any two of them, as, for instance, the two which are at right angles to A B, A C, is perpendicular to the face ABC of the pyramid (18. and 18. *Cor.* 2.), for which reason any plane which is parallel to it must be perpendicular to the same face, which is evidently not the case with the plane which is at right angles to the edge AD.

Prop. 49. Prob. 14.

Three plane angles A O B, B O C, C O A, *which form a solid angle* O, *being given; to find by a plane construction the angle contained by two of the planes, viz.* A O B *and* B O C.

At the point B, in the plane B O C, and upon either side of the angle B O C, make the angles B O D, C O E equal to the angles BOA, COA, each to each



(I. 47.): in O D take any point D, and make O E equal to O D; from the points D, E, draw D B, E C perpendicular (I. 45.) to A B, A C respectively, and let them be produced to meet one another in P: from P (I. 44.) draw Q P perpendicular to P B, and from the centre B, with the radius B D, describe the circle D Q d cutting P Q in Q, and join B Q; the angle Q B P shall measure the angle of inclination contained by the two planes A O B, B O C.

Take OA equal to OD or OE, and join AB, AC, AP. Then, because the triangles OBA, OBD have the two sides OB, OA of the one equal to the two sides OB, OD of the other, each to each, and the included angles equal to one another, B A is equal to B D (I. 4.), and the angle O B A to the angle O B D, that is to a right angle. And, because O B is at right angles both to B A and B P, it is at right angles to the plane A B P (3.), and therefore the plane B O C, which passes through O B, is at right angles to the same plane A B P (18.), or, which is the same thing, the plane A B P is at right angles to the plane O B C. And in the same manner it may be shown that the plane A C P is likewise at right angles to the plane O B C. Therefore the straight line A P, which is the common section of the planes A B P, A C P, is at right angles to the plane O B C (18. *Cor.* 2.), and the angle A P B is a right angle. And, because in the right-angled triangles Q B P, A B P, the hypotenuse Q B is equal to the hypotenuse A B, and the side B P common to both the triangles, the angle QBP is equal to the angle ABP (I. 13.). But ABP measures the inclination of the planes AOB, B O C, because AB and BP are each of them perpendicular to the common section OB (17. *Schol.*) Therefore, the angle Q B P measures the same inclination.

Therefore, &c.

It will be found that the above construction becomes impossible, when the third angle C O E is greater than the sum or less than the difference of the other two. For, in order that it may be applied, it is necessary that B P be less than B D. Now, if D F be drawn perpendicular to O C, and produced to meet the circle described from the centre O with the radius O D in G, it may easily be shown that the angle F O G is equal to the angle F O D (III. 3. and I. 7.); and, therefore, the angle C O G to C O D, that is, to the sum of B O C and B O D. And in like manner, if df be drawn perpendicular to O C, and produced to meet the circumference of the same circle in g, the angle C O g will be equal to the angle C O d, that is, (because B O d, by III. 3. and I. 7., is equal to B O D) to the difference of B O C and B O D. But if C O E be greater than C O G, or less than C O g, in either case the perpendicular E C, which is drawn to O C from a point E in the circumference of the circle D C g, will not cut D d between the points D and d; and, therefore, B P will be greater than B D. The limits of possibility are, therefore, those above stated. The inferences from this are evidently the same with 19. and 19. *Cor.*

Cor. 1. The same construction, slightly modified, may be used to find the third angle C O A from the two A O B, B O C, and their angle of inclination. For, if P B Q be made equal to the angle of inclination, and if B Q be taken equal to B D, the point P will be determined by drawing Q P perpendicular to B P; and thence the angle C O E or C O A, by drawing P C perpendicular to O C, and producing it to meet the circumference D G g in E.

Cor. 2. (Euc. xi. A.) It follows, from the solution of this problem, that if two solid angles be contained each of them by three plane angles, and if the plane angles which contain one of them be equal to those which contain the other, each to each, the dihedral angles of the one will likewise be equal to the corresponding dihedral angles of the other, each to each.

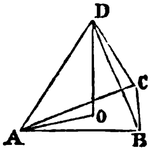
Cor. 3. And in like manner it follows, from *Cor.* 1, that if two solid angles be contained each of them by three plane angles, and if two of the plane angles which contain one of them be equal to two of those which contain the other, each to each, and equally inclined to one another, the third angle of the one shall be equal to the third angle of the other, and its plane inclined at equal angles to the adjacent planes.

PROP. 50. Prob. 15.

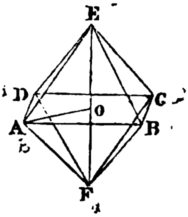
An edge A B *of a regular polyhedron being given; to construct the polyhedron.*

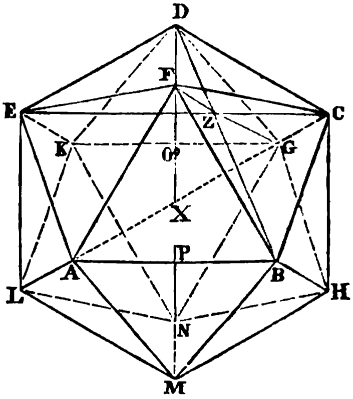
It has been already seen (20. *Cor.*) that the number of regular polyhedrons cannot exceed five, three of which (if there be so many) will be contained by equilateral triangles, one by squares, and one by pentagons. In the present problem it will appear that each of these five solids may be constructed.

1. Let the solid angle of the polyhedron in question be contained by *three* angles of *equilateral triangles.* Upon the given edge A B, describe (I. 42.) an equilateral triangle ABC: take the centre O (III. 26.) of the triangle A B C, and from O draw OD perpendicular to the plane A B C (37.): in O D take the point D such that A D may be equal to A B,[*] and join D A, D B, D C. Then, because O B and O C are each of them equal to O A, D B and D C are each of them equal to D A or A B (8.), and, therefore, the faces D A B, D A C, D B C of the solid D A B C are equilateral triangles: and because its solid angles are each of them contained by three angles of equilateral triangles, they are equal to one another (49. *Cor.* 2.). Therefore, D A B C is a regular solid; and since it has four faces, it is a *regular tetrahedron.*



2. Let the solid angle be contained by *four* angles of *equilateral triangles.* Upon the given edge A B describe a square A B C D (I. 52.); take the centre O (III. 26.), and from O draw O E (37.) perpendicular to the plane A B C D; take O E equal to O A, and, therefore, (I. 22. *Cor.* 2. and I. 36.) such that A E shall be equal to A B; and produce E O to F, so that O F may be equal to O E; join E A, E B, E C, E D, and F A, F B, F C, F D. Then, because the point O is equally distant from the points A, B, C, D, E B, E C, E D are each of them equal to E A or A B (8.): and for the like reason F B, FC, FD are each of them equal to FA; that is, because O F is equal to O E, to E A, (I. 4.) or A B. Therefore, the eight faces of the solid E A B C D F are equilateral triangles. Again, each of the quadrilaterals E A F C, E B F D is a square, because its diagonals are equal, and (I. 22. *Cor.* 2.) bisect one another at right angles: therefore, the planes of any two adjoining faces are inclined to one another at the same angle as the planes of any other two, (49. *Cor.* 2.) viz., the angle which is contained by two angles of equilateral triangles forming with the angle of a square a solid angle. Therefore, any two of the solid angles, as E and F, may be made to coincide, and are equal to one another. Therefore, E A B C D F is a regular solid; and, since it has eight faces, it is a *regular octahedron.*





3. Let the solid angle be contained by *five* angles of *equilateral triangles.* Upon the given edge A B describe a regular pentagon A B C D E (III. 63. *Cor.*): take the centre O (III. 26.), and from O draw O F perpendicular to the plane A B C D E (37.): take O F such that F A may be equal to A B (which may be done by joining O A, and taking (I. 59.) O F such that the square of OF may be equal to the difference of the squares of O A, A B), and join FA, FB, FC, FD, FE. Then, because the point O is equally distant from the points A, B, C, D, E, FB, FC, FD, FE are equal each of them to F A or A B (8.), and therefore F A B, F B C, F C D, F D E, F E A are equilateral triangles. Join BD; then because the triangles B F D, B C D have the three sides of the one equal to the three sides of the other, each to each, the angle B F D is equal to the angle B C D (I. 7.), that is, to the angle of a regular pentagon; and the same is evidently true of the plane angle formed by any two of the principal edges of the

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[*] That is, take OD^2 equal to $AB^2 - AO^2$ (I. 59.)

pyramid F A B C D E, which are not adjacent to one another. Complete the regular pentagon B F D G H, and join C G, C H. Then, because the point C is equally distant from the three points B, F, D, it lies in that perpendicular to the plane B F D, which passes through the centre of the circle circumscribing the triangle B F D (37. *Cor.*), that is, through the centre of the pentagon B F D G H; therefore C G, C H are each of them equal to C B (8.) or AB, and C D G, C G H, C H B are equilateral triangles. Join CE, and let it cut BD in the point Z. Then, because the diagonal BD is common to the two pentagons A B C D E, H B F D G, and that another diagonal C E of the former cuts it in Z, it may easily be shown that the diagonal F G of the latter cuts it in the same point Z.[*] Therefore EF, FC, and CG lie in one plane; and because the angles EFC, FCG, as shown in the case of the pyramid first constructed, are equal to the angles of a regular pentagon, let the regular pentagon E F C G K be completed, and join DK. Then, as above, it may be shown that D K is equal to D E or AB, and that DEK, DKG are equilateral triangles. Complete in like manner the pentagon A F D K L, and join L E; and in like manner the pentagon B F E L M, and join MA: then, as before, E L is equal to EA or AB, and EAL, E L K are equilateral triangles; and A M is equal to A B, and A B M, A M L are equilateral triangles; and because F A, A M, F C, C H may be shown in the same way as above to be in one plane, and that they make with one another angles of a regular pentagon, the line MH will complete the regular pentagon F A M H C, and B M H will be an equilateral triangle.

Again, because G K is parallel to the diagonal C E of the pentagon F C G K E, and K L to the diagonal D A of the pentagon FDKLA, and so on, the lines GK, K L, L M, M H, H G lie all in the same plane parallel to the plane A B C D E (15.); and the angle contained by every adjacent two is equal to the angle of a pentagon, because K G H is equal to E Z B (15.), that is, to E A B (I. 22), and so on. Therefore G K L M H is a regular pentagon. From the centre P of this pentagon draw (37.) a perpendicular PN and take P N such (I. 59.) that N G may be equal to G K, and join N G, N K, N L, N M, N H: then N G K, N K L, N L M, N M H, N H G are equilateral triangles. Therefore all the faces of the solid F N are equilateral triangles. And it is evident that any two adjoining faces on the same side of the plane GKLMH are inclined to one another at the same angle as any other two (49. *Cor.* 2.), viz., the angle which is contained by two angles of equilateral triangles forming a solid angle with the angle of a pentagon. Also, that any two upon opposite sides of that plane, as A L M, N L M, are inclined to one another at the same angle, may be shown by comparing it with the angle contained by any two G D C, F D C, upon opposite sides of the plane A B C D E: for, since a solid angle is formed at C by the three plane angles B C D, B C F, D C F, of which the first is an angle of a regular pentagon, and the two others angles of equilateral triangles, and in the same manner a solid angle at L by the three plane angles K L M, K L N, M L N, of which the first is an angle of a regular pentagon, and the two others angles of equilateral triangles, the dihedral angle, contained at the edge C D by the planes B C D, F C D, is equal to the dihedral angle contained at the edge L M by the planes K L M, N L M (49. *Cor.* 2.): and again, since a solid angle is formed at C by the three plane angles B C D, B C G, D C G, of which the two first are angles of regular pentagons, and the other an angle of an equilateral triangle, and in the same manner a solid angle at L by the three angles K L M, K L A, M L A, of which the two first are angles of regular pentagons, and the other an angle of an equilateral triangle, the dihedral angle contained at the edge C D by the planes B C D, G C D, is equal to the dihedral angle contained at the edge L M by the planes K L M, A L M (49. *Cor.* 2.); therefore, the whole dihedral angle contained at the edge C D by the planes F C D, G C D, is equal to the whole dihedral angle contained at the edge L M by the planes N L M, A L M (I. ax. 2.). Therefore, all the plane and likewise all the dihedral angles of the solid F N are equal to one another; and, consequently, any two of its solid angles, as at A and K, may be made to coincide, and are equal to one another. Therefore, F N is a regular solid (def. 9.); and, since it has twenty faces (viz. five, forming the pyramid which has the vertex F and base A B C D; five more, forming the pyramid which has the vertex N and the base G K L M H; and ten more between the two bases;) it is a *regular icosahedron.*

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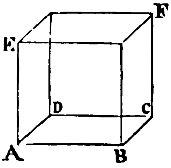
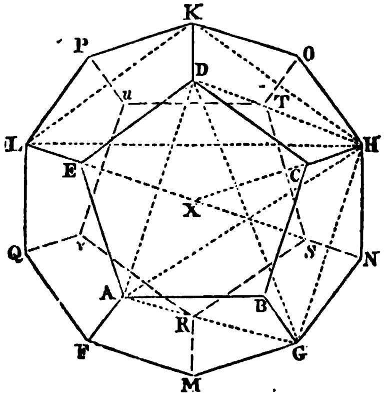
[*] Because the diagonals of a regular pentagon divide one another always in the same (viz. the medial) ratio. In fact, the diagonals BD, EC of the pentagon ABCDE being parallel to the sides AE, AB respectively (III. 26. III. 12. *Cor.* 1. and III. 18.), it follows (I. 22.) that EZ is equal to AB or DC; and the triangles ECD, DCZ (I. 6.) being isosceles and similar, EC is to CD (or EZ) as CD (or EZ) to CZ (II. 31.).

4. Let the solid angle be contained by three angles of *squares*, that is, by three right angles. Upon the given edge A B describe a square ABCD: from A draw A E at right angles to the plane A B C D (37.), and therefore at right angles both to A B and AD; take A E equal to AB, and complete the parallelopiped E C. Then it is evident that the six faces of the parallelopiped E C are squares (I. def. 20. and IV. 22.); and because each of its solid angles is contained by three right angles, any two of them may be made to coincide, and are equal to one another. Therefore E C is a regular solid, and since it has six faces, is a *regular hexahedron.* This figure is the same as the cube (def. 13.).



5. Let the solid angle be contained by three angles of *pentagons.* Upon the given edge A B describe a pentagon A B C D E: find the angle, I, at which two angles of regular pentagons must be mutually inclined, in order that with a third angle, which is likewise an angle of a regular pentagon, they may contain a solid angle (49.); and through the sides A B, B C, &c. of the pentagon, already described, draw planes, each making with it that angle of inclination, and intersecting one another in the lines AF, BG, CH, DK, FL: then the angles at the points A, B, C, D, E, will all of them e angles of regular pentagons. Comlete the regular pentagons A B G M F, B C H N G, &c.: then, since the angles B G M, B G N are angles of pentagons, and so inclined, that with a third angle of a pentagon they may form a solid angle, the third angle M G N is an angle of a pentagon; and for the like reason the third angles at H, K, L, F are likewise angles of pentagons. Now, it is evident from the figure thus far constructed, that, if at adjacent angles F A B, G B A of a pentagon, other angles F A E, G B C of pentagons be placed at the inclination I, the edges A E, A B, B C will lie in one plane.[*] If, therefore, the pentagon G M R S N be completed, not only will H N S be an angle of a pentagon, for the reason before stated, but, for the reason just mentioned, O H, H N, N S will lie in one plane, because at the adjacent angles C H N, G N H, other angles C H O, G N S of pentagons are placed at the inclination I: and for the like reasons F M R is an angle of a pentagon, and QF, FM, MR lie in one plane. In like manner, if the pentagon H N S T O be completed, PK, K O, O T will be in one plane; and if K O T U P be completed, Q L, L P, P U will lie in one plane; and if L P U V Q be completed, M F, F Q, Q V will lie in one plane, viz., the plane Q F M, or Q F M R, and therefore M R being joined will complete the pentagon F Q V R M. Lastly, also, by the same rule, the lines R S, S T, T U, U V, V R lie in one plane, and make with one another angles of pentagons, and therefore R S T U V is a pentagon completing the solid A T. And because A T has all its faces regular pentagons, and all its solid angles (49. *Cor.* 2.) equal (for each of them is contained by three angles of pentagons), it is a regular solid; and, since the faces are twelve in number, it is a *regular dodecahedron.*



And, in every case the regular solid is described with the given edge A B.

Therefore, &c.

As it is possible that the dotted or occult lines, which have been necessarily introduced in the foregoing constructions, may prevent the reader's acquiring from them a clear notion of the solids intended, we have here added shaded representations of the five regular solids, each in two different positions, in which only so much of the convex surface is exhibited as would present itself to the eye if they were opaque bodies.

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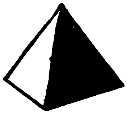
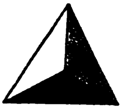
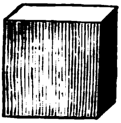
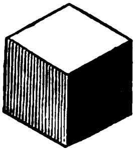
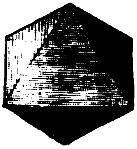
[*] The same will evidently be the case if FAB, GBA are any other equal angles, F A E, G B C any other two likewise equal to one another, and the common inclination I any whatever.

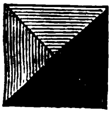


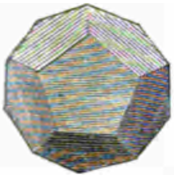


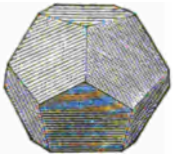


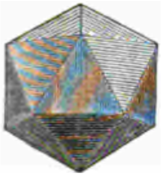


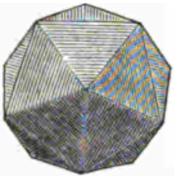








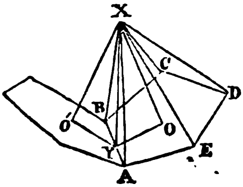




Cor. 1. Regular solids of the same name are similar (def. 10.).

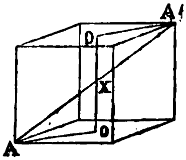
Cor. 2. Any regular solid being given, a point may be found within it, which is the common centre of two spheres, one inscribed in the solid, and touching all its faces, the other circumscribed about it, and passing through all its solid angles.

Take O the centre of any face having the edge A B for a side, and draw O X perpendicular to the face (37.): let the dihedral angle at A B be bisected by a plane cutting O X in X: X shall be the point in question.



Take O′ the centre of the adjoining face, and Y the middle point of A B, and join O Y, O′ Y, X O′, X Y. Then, because O Y is perpendicular to A B (III. 3.) X Y is likewise perpendicular to A B (4.); also O′ Y is perpendicular to the same A B: therefore O Y, X Y and O′ Y lie in one plane (3. *Cor.* 1.). And because the plane X Y A bisects the dihedral angle O Y A O′, the angle X Y O is equal to the angle X Y O′ (17.): also Y O is equal to Y O′; therefore X O′ is equal to X O (I. 4.) and the angle X O′ Y to the angle X O Y, that is, to a right angle. But the plane X O′ Y is perpendicular to the face which has the centre O′ (18.), because it is perpendicular to the line A B, through which that face passes. Therefore (18.) X O′ is perpendicular to the face which has the centre O′. Now, because X lies in the perpendicular passing through the centre O, it may easily be shown (4. and I. 4.) that the planes X B C, X C D, &c. make dihedral angles with the plane O A B, equal each to the dihedral angle O Y A X; also the dihedral angles of the solid are equal to one another; therefore those planes bisect the dihedral angles of the solid at B C, C D, &c. Hence, as above, it may be shown, that the straight lines drawn from X to the centres of each of the adjoining faces are perpendicular to those faces, and equal each to X O. And because the dihedral angles at A B, B C, &c. are bisected by planes meeting the perpendiculars from the centres of those faces in X, the same may be said of the faces adjoining to them, and so on. Therefore the straight lines drawn from X to the centres of all the faces are perpendicular to them respectively, and equal each to X O. And hence again, because the centres of the faces are equidistant from their several angles, the point X is likewise equidistant from the several angles of the solid (8.). Therefore X is the common centre of two spheres, one inscribed, and the other circumscribed, as before said. The point X is called the *centre* of the solid.

Cor. 3. Each of the regular solids of six, eight, twelve, and twenty faces has for every face a face opposite and parallel to it, and the opposite edges of those faces likewise parallel; and the straight line which joins two opposite angles passes through the centre of the solid. That the opposite faces and edges are parallel, is evident from the construction of the solid; and hence it is evident (11. and 7.) that the lines X O, X O′ drawn to their centres from the centre of the solid, are in one and the same straight line: therefore, again, because the opposite edges are parallel, it may easily be shown that the lines X A, X A′ which are drawn from the centre to the opposite angles A, A′, lie in the same straight line (15. and I. 4.)



Scholium.

Upon examining the number of the solid angles in each of these figures, it will appear, that the tetrahedron has four solid angles, which is also the number of its faces; the cube eight, which is the number of faces of the octahedron; the octahedron six, which is the number of faces of the cube; the dodecahedron twenty, which is the number of faces of the icosahedron; and the icosahedron twelve, which is the number of faces of the dodecahedron. Hence it is easily inferred, that *if the centres of the faces of a regular solid be taken, they will be the vertices of another regular solid inscribed in the first.* In this manner a tetrahedron may be inscribed in a tetrahedron; an octahedron in a cube, and a cube in an octahedron; an icosahedron in a dodecahedron, and a dodecahedron in an icosahedron.*

This mutual relation of the regular solids is very striking. We may observe that if lines are drawn from the centre of the circumscribed solid to its different angular points, these lines will be perpendicular respectively to the faces of the inscribed solid: hence, if we cleave or cut away the solid angles of the circumscribed figure by planes perpendicular to these lines; and if we continue the process until we arrive at the centres of the several faces, we shall obtain the regular solid which is inscribed, and which forms as it were the nucleus of the other. There are two stages of this process, which geometers have marked by bestowing names upon the figures which the derived solids are made to assume on arriving at them. The first is when the solid angles are so far cut away that the remaining portions of the faces are regular polygons, which have twice as many sides as the original faces. The derived solids at this stage are called the *ex-tetrahedron, ex-cube, ex-octahedron, ex-dodecahedron,* and *ex-icosahedron.* They are obtained from the regular solids by inscribing in each of the faces a regular figure, having twice as many sides as the face, and then cutting away the small pyramids which have for their vertices the several solid angles of the regular solid. Thus, in the face of a regular tetrahedron a hexagon may be inscribed by inscribing a circle in the face, joining the centre with the angles of the face, and drawing tangents to the circle at the points where the circumference is cut by the joining lines: and in a similar manner an octagon may be inscribed in a square, and a decagon in a pentagon. The number and character of the faces of any of these derived solids may be readily obtained from the number and character of the faces and solid angles of the regular solid from which it is derived. Thus the faces of the ex-cube are six octagons and eight equilateral triangles.

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* With the aid of this relation it will be found, also, that a regular solid being given, any one of the regular solids which have a less number of faces, may be inscribed in it by taking for the vertices certain of the vertices of the former, or else of the centres of its faces, or of the middle points of its edges.

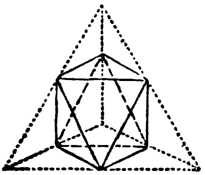
Thus, in the cube AF, the vertices B, D, E, F are the vertices also of an inscribed tetrahedron.

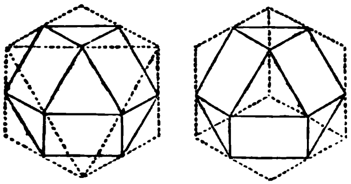
In the octahedron E F, the centres of the several faces are the vertices of an inscribed cube; and the centres of the faces E A B, E D C, F A D, F B C the vertices of an inscribed tetrahedron.

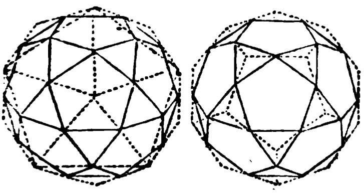
In the dodecahedron A T, the vertices H, G, A, D, P, Q, R, T, are the vertices of an inscribed cube; for AD and GH being equal, and also, because they are parallel to BC, parallel to one another (6), the figure ADHG is a parallelogram (I. 21.); but the side A D is equal to the side D H, and the diagonal A H may be shown to be equal to the diagonal D G; therefore A D H G is a rhombus, which has its two diagonals equal to one another, that is, a square; and, since the same may be shown of the other figures A D P Q, &c., the inscribed solid H G A D P Q R T is a cube: hence, also, the vertices A, H, P, R are the vertices of an inscribed tetrahedron; and the middle points of B C, U V, E L, S N, K O, M F, the vertices of an inscribed octahedron.

In the icosahedron A G, the centres of the several faces are the vertices of an inscribed dodecahedron; the centres of the faces F B A, A L M, M N H, H B C, C G D, D F E, E L K, K G N, the vertices of an inscribed cube; the centres of the faces F B A, M N H, C G D, E L K, the vertices of an inscribed tetrahedron; and the middle points of the edges B C, K L, E F, H N, A M, D G, the vertices of an inscribed octahedron.

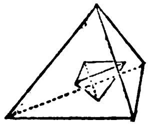
The second stage occurs when, the solid angles being still further cut away, the planes of cleavage meet at the middle points of the edges, thus reducing the original faces to regular polygons which have the same number of sides with the faces, and are inscribed in them by joining the middle points of the edges. In fact, if the edges of a regular solid be bisected, and the points of bisection joined, there will be inscribed in each of its faces a figure similar to that face, that is, an equilateral triangle, if the face be an equilateral triangle; a square, if a square; and a pentagon, if a pentagon. Here the forms of the derived solids apprise us at once of the mutual relations of their originals; the two derived from the cube and octahedron being precisely similar, as are likewise those derived from the dodecahedron and icosahedron; from which circumstance the new figures with which we are thus presented have received the names of the *cuboctahedron* and the *icosadodecahedron*. From the tetrahedron treated in this manner we obtain the octahedron.

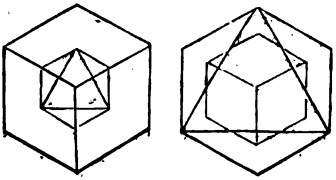


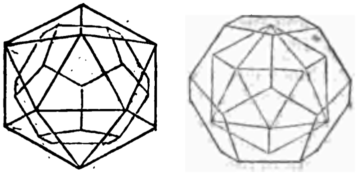




Finally, the cleavage being continued till we arrive at the centres of the faces, we obtain the inscribed regular solids.





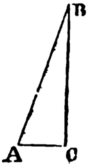


Prop. 51. Prob. 16.

To find the inclination of two adjoining planes of a given regular solid.

1. If the given solid be a regular tetrahedron, the required inclination is that of two angles of equilateral triangles, which, together with a third, form a solid angle, and therefore may be found by the construction given in Prop. 49.

Fig. 1.

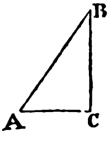


Or thus: describe the right-angled triangle A C B, having the hypotenuse A B equal to three times the side A C; and the angle B A C will be the angle of inclination required.

2. If the solid be a cube, the angle of inclination will be a right angle (17. *Cor.*)

3. If an octahedron, the required inclination is that of two angles of equilateral triangles, which, together with the angle of a square, form a solid angle, and may be found as in Prop. 49.

Fig. 2.

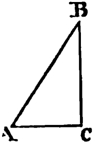


Or thus: describe the right-angled triangle ACB, having its two sides A C, C B equal, respectively, to the side and diagonal of a square, and twice the angle B A C will be the angle of inclination required.

4. If a dodecahedron, the required inclination is that of two angles of re-

gular pentagons which, together with a third angle of a pentagon, form a solid angle, and therefore may be found as in Prop. 49.

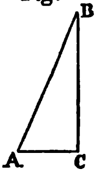
Fig. 3.



Or thus: describe the right-angled triangle A C B, having its sides A C, C B to one another in the medial ratio, and A C the lesser of the two; and twice the angle B A C will be the angle required.

5. If an icosahedron, the required inclination is that of two angles of equilateral triangles, which, together with the angle of a regular pentagon, form a solid angle, and therefore may be found as in Prop. 49.

Fig. 4.



Or thus: describe the right-angled triangle A C B, having its sides A C, C B to one another in a ratio which is the duplicate of the medial ratio; and twice the angle B A C will be the angle required.

It will be sufficient to notice briefly the steps which lead to the foregoing constructions.

With regard to the tetrahedron; if a perpendicular be drawn from the centre of an equilateral triangle to one of the sides, such perpendicular will be a third of the whole perpendicular which is drawn to the same side from the angle opposite to it (see the method of inscribing an equilateral triangle in a circle at III. 63.). Now, in the tetrahedron the faces are equilateral triangles, and the line which joins any of its solid angles with the centre of the opposite face is perpendicular to that face (37. *Cor.*); whence, by the aid of Prop. 4., the first construction.

In the octahedron, the square which divides the figure (see the construction in Prop. 50) bisects the angles made by the adjoining faces upon either side of it: and the line which joins the centre of this square with either of the two solid angles above and below it, is equal to half its diagonal, while the perpendicular drawn from the centre of the square to any of its sides is equal to half the side; whence the construction in this case.

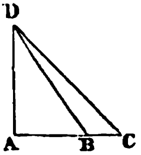
The cases of the dodecahedron and icosahedron admit of an easy demonstration by help of the mutual relation of the dodecahedron and icosahedron mentioned in the last *Scholium*. For, if X be the centre of the icosahedron A G (see the figure of Prop. 50.), X F and X G will be perpendicular to two adjoining planes of the inscribed dodecahedron, and therefore, A X G being a straight line (50. *Cor.* 3.), the angle A X F will measure the inclination of those planes (17. *Schol.* 4.): now because X F is equal to X A or X G (50. *Cor.* 2.), the angle A F G is a right angle (I. 19. *Cor.* 4.), and the angle A X F is double of the angle A G F (I. 6. and I. 19.); also, F G is the diagonal of a regular pentagon, whose side is equal to A F, and therefore F G is to A F in the medial ratio (see note p. 159). Hence the construction given for the inclination of the faces of a dodecahedron.

And that given for the icosahedron is similarly derived, from considering it as inscribed in a dodecahedron. For if X be taken, the centre of the dodecahedron L N (see the figure of Prop. 50.), X N, and X H will be perpendicular to two adjoining planes of the inscribed icosahedron, and therefore, L X N being a straight line (50. *Cor.* 3.) the angle L X H will measure the inclination of those planes (17. *Schol.* 4.): now, if L K, K H, L H, be joined, the angle L K H will be equal to the angle E D C of a pentagon (15.), because L K and K H are parallel to E D and D C respectively; therefore, the triangle L K H is similar to K O H (II. 32.), and O H is to H K as H K to L H; and since O H is to H K in the medial ratio, O H or H N is to L H in a ratio which is the duplicate of that ratio (II. def. 11.): and, because X H is equal to X L or X N, the angle L H N is a right angle (I. 19. *Cor.* 4.), and L X H is equal to twice L N H (I. 6. and I. 19.), that is, the angle of inclination is equal to twice the greater acute angle of a right-angled triangle, whose sides are to one another in a ratio which is the duplicate of the medial ratio.

Therefore, &c.

Prop. 52. Prob. 17.

The edge of any regular solid being given, to find the radii of the inscribed and circumscribed spheres.



Find AB and AC, the radii of the circles inscribing and circumscribing a face of the given solid (III. 26.); from A draw AD perpendicular to AB or AC; at the point B make the angle ABD equal to half the angle which measures the inclination of two adjoining faces (51.); let B D meet A D in D, and join C D. Then it is evident, from the construction of 50. *Cor.* 2, that D A is the radius of the inscribed sphere, and D C that of the circumscribed sphere.

Or the radius D C of the circumscribed sphere may be determined in the several cases, by the following constructions; and then D A from the triangle D A C, described with the hypotenuse D C and side A C.

1. If the given solid be a tetrahedron, describe the right-angled triangle ACB (see 51. *fig.* 2.), having the sides A C, C B, equal respectively to the side and diagonal of a square; and the diameter of the circumscribed sphere will be to the edge of the tetrahedron as A B to B C.

2. If a cube, the diameter of the circumscribed sphere will be to the edge of the cube as A B to A C in the above triangle (51. *fig.* 2.); and that of the inscribed sphere will be equal to the edge.

3. If an octahedron, the diameter of the circumscribed sphere will be to the edge as the diagonal of a square to its side.

4. If a dodecahedron, describe the right-angled triangle A C B (see 51. *fig.* 4.), having its sides A C, C B to one another in a ratio which is the duplicate of the medial ratio; and the diameter of the circumscribed sphere will be to the edge as the hypotenuse A B to the lesser side A C.

5. If an icosahedron, describe the right-angled triangle A C B (see 51. *fig.* 3.), having its sides A C, C B in the medial ratio; and the diameter of the circumscribed sphere will be to the edge as the hypotenuse A B to the lesser side A C.

We need not enter into the details of these constructions: it will be sufficient, as in the preceding problem, to point out the considerations from which they are derived respectively.

And first, a tetrahedron may be inscribed in a cube, which shall have for its four angles four of the angles of the cube, and for its four edges the diagonals of four faces of the cube (see note, p. 162); and the sphere which is circumscribed about such tetrahedron will be also circumscribed about the cube; but in a cube, the line which joins two opposite angles is the diameter of the circumscribed sphere, and the square of this line is equal to the square of one of the edges of the cube, together with the square of the diagonal of one of the faces: hence, therefore, the constructions for the tetrahedron and cube.

In the octahedron, the line which joins two opposite angles is at once the diameter of the circumscribed sphere, and also the diagonal of a square which has for its four sides four of the edges of the octahedron; hence the construction in this case.

In the dodecahedron (see the figure of p. 160) the triangle L H N is right-angled at H, and the sides L H, H N have to one another a ratio which is the duplicate of the medial ratio, as was shown in the last problem: also L N is the diameter of the circumscribed sphere; therefore, the rule in this case is manifest.

And, lastly, in the icosahedron (see the figure of p. 158) the triangle A F G is right-angled at F, and the sides GF, FA are to one another in the medial ratio, as was shown in the last problem; also A G is the diameter of the circumscribed sphere: whence the construction in this case.

Therefore, &c.

Cor. Every regular solid may be divided into pyramids, having for their bases the several faces of the solid, and for their common vertex the centre of the solid; and the altitude of each of these pyramids will be the same, viz. the radius of the inscribed sphere. By help of this proposition, therefore, we may find the solid content of any given regular solid; for it will be one-third of the product of the above radius and the convex surface of the solid (32. *Cor.* 1.).

Scholium.

The regular solids have ceased to occupy that prominent place in science which was assigned to them for so long a period, from the time of Euclid to that of Kepler*. A volume, replete with the most striking results, might indeed be written upon the subject; but as these figures, with the exception of the cube, have little or no concern with anything besides themselves, such a work would be of value to the curious only. It is not surprising, perhaps, if we regard Euclid as the discoverer of the many elegant relations which characterize them in the 13th, 14th, and 15th Books of the Elements †, that he should have composed his immortal treatise, as is said to have been the case, with the sole object of demonstrating these relations; it is not, perhaps, too much to say that at that epoch, when the properties of numbers and geometrical figures were investigated for their own sake, abstractedly and without reference, as in the present day, to the body of mixed sciences dependant upon them, the five regular solids were even worthy of such a distinction. The fate of this portion of his work (so rarely now perused) is, however, a striking illustration of the lasting and transcendant nature of what is really (though humbly) useful above the merely curious and surprising. So obscure is the rank now assigned to these once interesting and all-important figures, that it may be considered even trifling, in the present treatise, to have established their construction, &c. at a greater length than usual. We must refer the reader, by way of apology, to the properties above alluded to, a few of which, capable of being verified without difficulty with the assistance which we have afforded, are here subjoined.

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* See the Life of Galileo, page 27.

† The two last books are, however, with some probability, ascribed to Apollonius.

In the tetrahedron, the radius of the circumscribed sphere is equal to three times the radius of the inscribed sphere: in the cube (as we have seen) the two radii are to one another in the subduplicate of this ratio.

In the cube, the radii of the inscribed and circumscribed spheres have to one another the same ratio as in the octahedron: and the same is true of the dodecahedron and icosahedron.

In the icosahedron, the distance of the regular pentagon, which passes through five of the solid angles, from the centre of the solid, is equal to half the radius of the circle circumscribed about the pentagon.

These few may serve as a sample of the rest, which are occupied with the mutual inscription and circumscription of these figures, and the proportions of their surfaces and contents when inscribed in one and the same sphere.

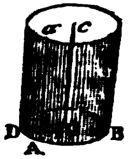
BOOK V.

§ 1. *Surfaces and contents of the Right Cylinder and Right Cone.*—§ 2. *Surface and content of the Sphere.*—§ 3. *Surfaces and contents of certain portions of the Sphere.*

SECTION 1.—*Of the Right Cylinder and Right Cone.*

Def. 1. A *cylinder* is a solid figure, the surface of which is partly plane and partly curved; the plane portions being two equal and parallel circles, and the curved portion such, that any point being taken in the circumference of either circle, the straight line which is drawn through it parallel to the line joining their centres lies wholly in the surface.

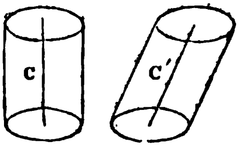
Such a surface may be conceived to be generated by a straight line A a, which is carried round the circumference of a given circle A B D, so as to be always parallel and equal to a given straight line O c at the centre.



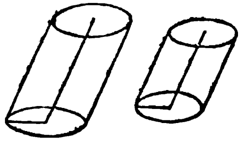
For it is easy to perceive that the upper extremity of such a line will always lie in the circumference of a second circle $a\,b\,d$, which is of the same dimensions with the given circle, and in a plane parallel to it (IV. 13. *Cor.* 2.).

The curved surface of a cylinder is called also the *convex surface;* the circles are called *bases;* and the straight line which joins their centres is called the *axis* of the cylinder.

2. A cylinder is said to be *right* or *oblique*, according as the axis is perpendicular to the bases, or inclined to them. C is a right, and C′ an oblique cylinder.



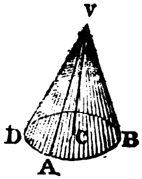
3. *Similar Cylinders* are those whose axes are perpendicular, or equally inclined, to their respective bases, and in the same ratio to the radii of those bases.



4. A *cone* is a solid figure, the surface of which is partly plane, and partly curved; the plane portion being a circle, and the curved portion such, that if any point be taken in the circumference of the circle, the straight line which joins it with a certain point without the plane of the circle, lies wholly in the surface.

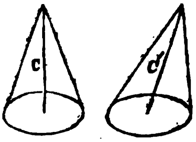
A curved surface of this description

may be conceived to be generated by a straight line V A; which is carried round the circumference of a given circle A B D, so as to pass always through a given point V without the plane of the circle.



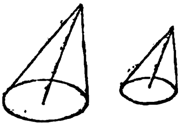
The curved surface of a cone is called also the *convex surface:* the point through which the straight line always passes is called the *vertex* or summit; the circle is called the *base;* and the line which is drawn from the vertex to the centre of the base, the *axis* of the cone.

5. A cone is said to be *right* or *oblique*, according as the axis is perpendicular to the base, or inclined to it. C is a right, and C′ an oblique cone.

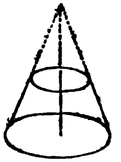


The *slant side* of a right cone is a straight line which is drawn from the vertex to any point in the circumference of the base.

6. *Similar Cones* are those whose axes are perpendicular, or equally inclined to their respective bases, and in the same ratio to the radii of those bases.

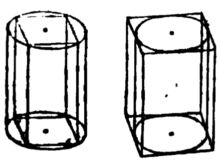


7. If a cone be divided into two parts by a plane parallel to its base, the part next the base is called a *frustum of a cone*, or sometimes a *truncated cone.*

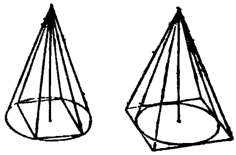


The *axis* of a frustum is that part of the axis of the whole cone, which is intercepted between the cutting plane and the base of the cone.

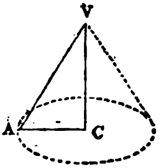
8. A *prism* is said to be *inscribed in a cylinder*, when its bases are inscribed in the bases of the cylinder; and in like manner a prism is said to be *circumscribed about a cylinder*, when its bases are circumscribed about the bases of the cylinder.

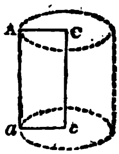


9. A *pyramid* is said to be *inscribed in a cone*, when its vertex is the same with that of the cone, and its base is inscribed in the base of the cone: and in like manner a pyramid is said to be *circumscribed about a cone*, when its vertex is the same with that of the cone, and its base circumscribed about the base of the cone.



The right cylinder, right cone, and sphere, are sometimes styled, by way of pre-eminence, *the three round bodies.* They are also termed *solids of revolution*, because each of them may be conceived to be generated by the revolution of a plane figure about a fixed straight line taken in its plane. Thus we have seen (IV. def. 21.) that a sphere is generated by the revolution of a semicircle about its diameter. If a right angle triangle V C A revolve about one of the sides containing the right angle, as V C, that side will be the axis of a right cone, of which the other side A C will describe the base (IV. 3. *Cor.* 2.) and the hypotenuse V A the convex surface. And if a rectangle A*c* revolve about one of its sides C *c*, that side will be the axis of a right cylinder, of which the two adjoining sides will describe the two bases (IV. 3. *Cor.* 2.), and the side opposite, A *a*, the convex surface.



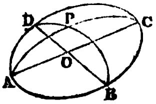


Before we can proceed to consider the convex surfaces of the cone and cylinder, it will be necessary to establish the following lemmas concerning convex surfaces in general.

Lemma I.

A plane surface O A B C D is less than any other surface P A B C D, terminated by the same contour A B C D.

For, in whatever direction we extend the plane BPD, to cut the plane surface in the straight line B D, and the other surface in the curved or bent line B P D, the

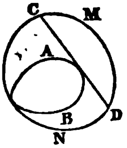


dimension B D of the plane is less than the dimension B P D of the other surface (I. 10. *Scholium*). But if the plane surface were equal to or greater than the other, the dimension of the plane surface could not be, *in every direction, less* than the dimension of the other. Therefore, the plane surface O A B C D is neither equal to nor greater than the surface P A B C D; that is, it is less than the surface P A B C D.

Therefore, &c.

Lemma 2.

If a convex surface A B is enveloped on all sides by another surface M N; whether they have any points, lines, or planes in common, or have no point at all in common, the enveloped surface will always be less than the surface which envelops it.



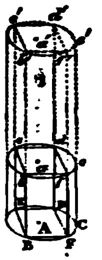
For it is the nature of a convex surface that there is no point of it through which a plane cannot be drawn touching, or at least not cutting, the surface. Therefore, if such a surface A B be enveloped by any other M N, and if A be any point in the former which is not also in the latter surface, a plane CD may be drawn through A, cutting the surface M N, and not cutting the surface A B. And because the plane CD is less than the surface C M D by the preceding lemma, the whole surface CND is less than the whole M N (I. ax. 6.) Therefore, of all the surfaces which envelop the space A B, and are in any part exterior to the surface A B, there is none than which a less cannot be found enveloping the same space A B. But of the surfaces enveloping this space, there must be some (one or more) less than any of the others; for none of them can be less than of some certain magnitude. Therefore, since one or more of the surfaces must be less than any of the others, and since a less may be found than any which is in any part exterior, to the surface AB, such least surface is none other than the surface A B. Therefore, the surface AB is less than the surface MN.

Therefore, &c.

Prop. 1.

A right cylinder is greater than any inscribed prism, and less than any circumscribed prism: also the convex surface of the cylinder is greater than the convex surface of any inscribed prism, and less than the convex surface of any circumscribed prism.

The first part of the proposition is manifest: we have only, therefore, to demonstrate that which relates to the surfaces.

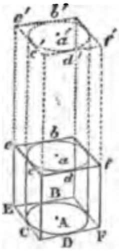


Let A *a* be the axis of a right cylinder, and B C D, *b c d* its two bases; and, first, let E F, *e f* be the bases of any prism inscribed in the cylinder: the convex surface of the cylinder shall be greater than the convex surface of the inscribed prism.

For, if not, it must either be less than the latter surface or equal to it. First, let it be supposed less, viz. by a surface P; let Q be any multiple of P, which is greater than the difference between the two circles B C D, *b c d* taken together, and the two polygons E F, *e f* taken together: take A *a′* the same multiple of A *a* that Q is of P, and complete the cylinder which has the axis A *a′* and the base B C D, and the prism *e′ f′* E F, which is inscribed in it upon the base E F. Then, by making the cylinder which has the axis A *a* coincide successively with the parts of the larger cylinder, it may be shown that the convex surface of the latter is the same multiple of the convex surface of the former that A *a′* is of A *a*, or Q of P; and it is evident that the convex surface of the prism inscribed in it is the same multiple of the convex surface of the prism inscribed in the former cylinder: therefore, because the convex surface of the lesser cylinder together with P is equal to the convex surface of its inscribed prism, the convex surface of the larger cylinder together with Q is likewise equal to the convex surface of its inscribed prism. But the difference between the two circles B C D, *b c d* taken together and the two polygons E F, *e f* taken together (or, which is the same thing, between the two circles B C D, *b′ c′ d′* taken together and the two polygons E F, *e′ f′* taken together) is less than Q. Therefore, the convex surface of the larger cylinder together with this difference is less than the convex surface of its inscribed prism; which is impossible (Lem. 2.), because the convex surface of the cylinder, together with this difference and with the two polygons E F, *e′ f′*, envelops the convex surface of the prism, together with the same two polygons E F, *e′ f′*. Therefore, the convex surface of the cylinder which has the axis A *a* is not less than the convex surface of the inscribed prism.

Neither can the convex surface of the cylinder be equal to the convex surface of the inscribed prism. For, if this be supposed, then, because a polygon may be inscribed in the circle B C D, which shall have a greater perimeter than the polygon E F, a prism may be inscribed in the cylinder which shall have (IV. 29. *Scholium*) a greater convex surface than the prism upon the base EF, that is, than the cylinder has. But this is impossible, as has been already demonstrated.

Therefore, the convex surface of the cylinder must be greater than the convex surface of the inscribed prism.



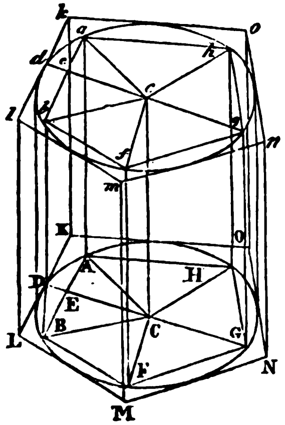
And by a similar course of reasoning applied to the adjoined figure, it may be demonstrated that the convex surface of a cylinder is less than the convex surface of any circumscribed prism.

Therefore, &c.

Prop. 2.

Any right cylinder being given, two prisms may be the one inscribed in it, and the other circumscribed about it, such that the difference of their convex surfaces, or of their solid contents, shall be less than any given difference.

Let A B G, *a b g* be the bases of a given right cylinder; C, *c* their centres. And, first, let P be the given difference of surfaces, and let Q be the convex surface of some circumscribed prism. Then, because (as in III. 31.) a regular polygon may be inscribed in the circle A B G, the apothem C E of which approaches to the radius C D within any given difference, it is evident that a polygon may be inscribed such that C D − C E shall be to C E in a ratio less than any assigned, and, therefore, in a ratio less than that of P to Q. Let ABFGH be such a polygon, and C E its apothem. Join C A, C B, &c., and let the planes A C *c*, B C *c*, &c. cut the upper base in the radii *c a*, *c b*, &c., and join A *a*, B *b*, &c., *a b*, *b f*, &c.; then (def. 8.) it is evident that the polygons A B F G H, *a b f g h* are the bases, and the straight lines A *a*, B *b*, &c., the principal edges of a prism inscribed in the given cylinder. Again, let the polygon K L M N O, similar to A B F G H, be circumscribed about the circle A B D, so that one of its sides, K L, may touch the circle in D (III. 27. *Cor.* 2.): let the plane D C *c* cut the circle *a b d* in the radius *c d*, and let a similar polygon *k l m n o* be circumscribed about this circle, so that the side *k l* may touch it in *d*, and join K *k*, L *l*, &c.; then (def. 8.) it is evident that K L M N O, *k l m n o* are the bases, and the straight lines K *k*, L *l*, &c. the principal edges of a prism circumscribed about the given cylinder.



Now, the convex surfaces of these prisms, inscribed and circumscribed, are the sums of their rectangular faces. And, since these rectangles have all of them the same altitude, the sums of the rectangular faces are as the sums of the bases of the rectangles (II. 35.), which sums are the perimeters of the inscribed and circumscribed polygons, and are, therefore, as the apothems CE, CD (III. 30.). Therefore, the convex surfaces of the prisms are to one another as CE, CD (II. 12.); and their difference is to the convex surface of the inscribed prism as C D − C E to C E (II. 20.), that is in a less ratio than that of P to Q. But even had the ratio of their difference to the convex surface of the inscribed prism been the same with that of P to Q, the difference would have been less than P (II. 18. *Cor.*), because the surface of the inscribed prism is less than Q, which is the surface of some circumscribed prism (IV. 29. *Scholium*): much more, therefore, being less, is the difference less than P.

In the next place, let S be the given difference of solid contents, and let T be the solid content of some circumscribed prism. Then, as before, a polygon may be inscribed in the lower base such that its apothem C E shall approach to the radius C D within any given difference; and, therefore, such also that $CD^2 - CE^2$ may be to CE^2 in a ratio less than any assigned; less, therefore, than that of S to T. Let A B F G H be such a polygon, and let the prisms be inscribed and circumscribed as before. Then, because these prisms have the same altitude, their solid contents are to one another as their bases (IV. 29. *Cor.* 2.), which bases are the inscribed and circumscribed polygons, and, therefore, are to one another as CE^2, CD^2. Therefore, the contents are to one another as CE^2, CD^2 (II. 12.); and their difference is to the content of the inscribed prism as $CD^2 - CE^2$ to CE^2 (II. 20.), that is, in a less ratio than that of S to T. Therefore, as before, because the content of the inscribed prism is less than T, much more is the difference of contents less than S.

Therefore, &c.

Cor. 1. Any right cylinder being given, a regular prism may be inscribed (or circumscribed) which shall differ from the cylinder in convex surface, or in solid content, by less than any given difference. For the difference between the cylinder and either of the prisms, whether in surface or in content, is less than the difference of the two prisms (1.).

Cor. 2. Any two similar right cylinders being given, similar regular prisms may be inscribed (or circumscribed) which shall differ from the cylinders in convex surface or in solid content, by less than any the same given difference.

PROP. 3.

The convex surface of a right cylinder is equal to the product of its altitude by the circumference of its base.

For, if this product be not equal to the convex surface of the cylinder, it must either be greater or less than that surface.

If greater, as by a difference D, it must be greater also than the convex surface of some circumscribed prism; for a prism may be circumscribed, the convex surface of which approaches to that of the cylinder within the difference D (2. *Cor.* 1.): but the convex surface of such a prism is equal to the product of its altitude (which is the same with that of the cylinder) by the perimeter of some circumscribed polygon (IV. 29. *Scholium*); therefore, the circumference of the base of the cylinder is greater than the perimeter of this circumscribed polygon, which is impossible.

And, in the same manner, if less, as by a difference D, it must be less also than the convex surface of some inscribed prism (2. *Cor.* 1.); but the convex surface of such a prism is equal to the product of its altitude (the same with that of the cylinder) by the perimeter of some inscribed polygon (IV. 29. *Scholium*); therefore, the circumference of the base of the cylinder is less than the perimeter of this inscribed polygon, which is impossible.

Therefore the product in question is neither greater nor less than the convex surface of the cylinder; that is, it is equal to it.

Therefore, &c.

Cor. If R is the radius of the base of a right cylinder, and A its altitude, the convex surface of the cylinder is $2\pi RA$ (III. 34. *Scholium.*).

PROP. 4.

The solid content of a right cylinder is equal to the product of its base and altitude.

This proposition is demonstrated in exactly the same manner as the preceding. If the product in question exceed the content of the cylinder, it must likewise exceed the content of some circumscribed prism (2. *Cor.* 1.): but this is impossible, because the prism (IV. 29. *Cor.* 1.) is equal to the product of its altitude, which is the same with that of the cylinder, by its base, which is greater than the base of the cylinder. If, on the other hand, it be less than the content of the cylinder, it must likewise be less than the content of some inscribed prism (2. *Cor.* 1.); but this is impossible, because the prism (IV. 29. *Cor.* 1.) is equal to the product of its altitude, which is the same with that of the cylinder, by its base, which is less than the base of the cylinder. Therefore, the product in question cannot but be equal to the content of the cylinder.

Therefore, &c.

Cor. 1. If R is the radius of the base of a right cylinder, and A its altitude, the solid content of the cylinder is $\pi R^2 A$ (III. 34. *Scholium.*).

Cor. 2. If a right cylinder and a prism have equal bases and altitudes, the cylinder shall be equal to the prism (IV. 29. *Cor.* 1.).

Cor. 3. (Euc. xii. 11. and 14.) Right cylinders which have equal altitudes are to one another as their bases; and right cylinders which have equal bases are to one another as their altitudes: also any two right cylinders are to one another in the ratio which is compounded of the ratios of their bases and altitudes (IV. 29. *Cor.* 2.).

PROP. 5. (Euc. xii. 12.)

The surfaces of similar right cylinders are as the squares of the axes; and their solid contents are as the cubes of the axes.

For, in the first place, there may be inscribed in the cylinders similar prisms; the convex surfaces of which approach more nearly to the convex surfaces of the cylinders than by any the same given difference (2. *Cor.* 2.): and it may easily be shown that the convex surfaces of these prisms are to one another always in the same ratio, viz., as the squares of their principal edges (II. 43. *Cor.* 1.); which are equal respectively to the axes of the cylinders: therefore, the convex surfaces of the cylinders are to one another in the same ratio (II. 28.), viz., as the squares of their axes.

And, in the same manner, because there may be inscribed in the cylinders, similar prisms, the solid contents of which approach more nearly to the solid contents of the cylinders than by any the same given difference (1. *Cor.* 2.); and because the solid contents of these prisms are to one another always in the same ratio, viz., as the cubes of their principal edges (IV. 35.), which are equal respectively to the axes of the cylinders, the solid contents of the cylinders are to one another in the same ratio (II. 28.), viz., as the cubes of their axes.

Otherwise:

Let A, a represent the axes of two similar right cylinders, and R, r the radii of their bases. Then (3. *Cor.*) $2\pi RA$, $2\pi ra$ will represent their convex surfaces respectively, and (4. *Cor.* 1.) πR^2A, πr^2a their solid contents. But, because the cylinders are similar, (def. 3.) $R : A :: r : a$; therefore (p. 47, Rule ii.) $RA : A^2 :: ra : a^2$, *alternando* (II. 19.) $RA : ra :: A^2 : a^2$, and (p. 47, Rule ii.) $2\pi RA : 2\pi ra :: A^2 : a^2$, that is, the convex surfaces of the cylinders are to one another as the squares of their axes.

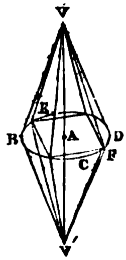
Again, because $R : A :: r : a$, $R^2 : A^2 :: r^2 : a^2$ (II. 37. *Cor.* 4.), and (p. 47, Rule ii.)] $R^2A : A^3 :: r^2a : a^3$; therefore, *alternando* (II. 19.) $R^2A : r^2a :: A^3 : a^3$, and (p. 47, Rule ii.) $\pi R^2A : \pi r^2a :: A^3 : a^3$; that is, the solid contents of the cylinders are to one another as the cubes of their axes.

Therefore, &c.

PROP. 6.

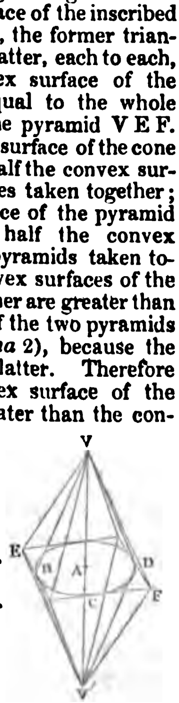
A right cone is greater than any inscribed pyramid, and less than any circumscribed pyramid; also the convex surface of the cone is greater than the convex surface of any inscribed pyramid, and less than the convex surface of any circumscribed pyramid.

The former part of the proposition is manifest: the latter respecting the surfaces may be demonstrated as follows:



Let VA be the axis and B C D the base of a right cone; and, first, let E F be the base of any pyramid inscribed in the cone: the convex surface of the cone shall be greater than the convex surface of the inscribed pyramid.

For, if the axis VA be produced to V′, so that A V′ may be equal to AV, and if a right cone be described which shall have the axis V′A and the base B C D, it may be shown by coincidence that the convex surface of the latter cone V′ B C D is equal to the convex surface of the first cone V B C D; and, because the triangles which form the convex surface of the inscribed pyramid V′ E F have their sides equal respectively to the sides of the corresponding triangles which form the convex surface of the inscribed pyramid V E F (I. 4.), the former triangles are equal to the latter, each to each, and the whole convex surface of the pyramid V′ E F is equal to the whole convex surface of the pyramid V E F. Therefore the convex surface of the cone V B C D is equal to half the convex surfaces of the two cones taken together; and the convex surface of the pyramid V E F is equal to half the convex surfaces of the two pyramids taken together. But the convex surfaces of the two cones taken together are greater than the convex surfaces of the two pyramids taken together (*Lemma* 2), because the former envelop the latter. Therefore (I. ax. 8.), the convex surface of the cone V B C D is greater than the convex surface of the inscribed pyramid VEF.



And by a similar demonstration applied to the adjoined figure, it may be shown that the convex surface of the cone is less than the convex surface of any circumscribed pyramid.

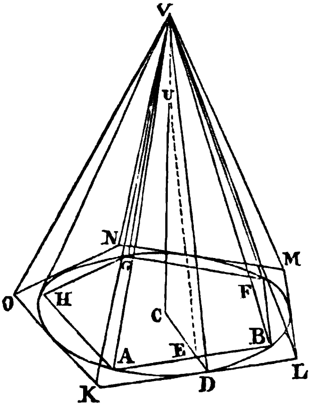
Therefore, &c.

PROP. 7.

Any right cone being given, two pyramids may be the one inscribed in the cone, and the other circumscribed about it, such that the difference of their con-

vex surfaces, or of their solid contents, shall be less than any given difference.

Let V be the vertex of a right cone, A B D its base, and C the centre of the base. And, first, let P be the given difference of the surfaces; and let Q be the convex surface of some circumscribed pyramid. Then, because (as in III. 31.) a regular polygon may be inscribed in the circle A B D, the apothem C E of which approaches to the radius C D within any given difference, a polygon may be inscribed, such that $CD^2 - CE^2$ may be to CE^2 in a ratio less than any assigned; less, therefore, than that of P to Q. Let A B F G H be such a polygon; and let a similar polygon K L M N O be circumscribed about the circle, so that the side K L may touch the circle in D (III. 27. *Cor.* 2.). Join V A, V B, &c., V K, V L, &c., and through E draw E U parallel to D V, to meet C V in U, and join U A, U B, &c.* Then, it may easily be shown, that the convex surfaces of the two pyramids, which have the points U, V, for their vertices, and the inscribed and circumscribed polygons for their bases, are made up of similar triangles, which are to one another in the same ratio, each to each, viz. that of CE^2 to CD^2 (II. 42. *Cor.*). Therefore, the convex surfaces of the pyramids are to one another in the same ratio (II. 23. *Cor.* 1.); and the difference of their convex surfaces is to that of the lesser pyramid as $CD^2 - CE^2$ to CE^2 (II. 20.), that is, in a less ratio than that of P to Q. But the convex surface of the lesser pyramid is less than the surface of the pyramid V A B F G H (I. 26. *Cor.* and I. 12. *Cor.* 2.), and therefore by much more less than Q. Much more, therefore, is the difference of the convex surfaces less than P (II. 18. *Cor.*); and more still is the difference of the surfaces of the inscribed and circumscribed pyramids, which have the common vertex V, less than P.



Next, let S be the given difference of the solid contents; and let T be the solid content of some circumscribed pyramid. As before, let the regular polygon A B F G H be inscribed in the circle A B D, such that $CD^2 - CE^2$ may be to CE^2 in a less ratio than that of S to T; let a similar polygon be circumscribed, and the inscribed and circumscribed pyramids completed. Then, because these pyramids have the same altitude, their solid contents are to one another as their bases, that is, as CE^2, CD^2 (IV. 32.): therefore, the difference of the contents is to that of the inscribed pyramid as $CD^2 - CE^2$ to CE^2 (II. 20.) that is, in a less ratio than that of S to T. Therefore, because the content of the inscribed prism is less than T, much more is the difference of the contents less than S (II. 18. *Cor.*).

Therefore, &c.

Cor. 1. Any right cone being given, a regular pyramid may be inscribed (or circumscribed), which shall differ from the cone in convex surface, or in solid content, by less than any given difference; for the difference between the cone and either of the pyramids, whether in surface or in content, is less than the difference of the two pyramids (6.).

Cor. 2. Any two similar right cones being given, similar regular pyramids may be inscribed (or circumscribed), which shall differ from the cones in convex surface, or in solid content, by less than any the same given difference.

PROP. 8.

The convex surface of a right cone is equal to half the product of its slant side by the circumference of its base.

For, if half this product be not equal to the convex surface of the pyramid, it must either be greater or less than that surface. If greater, it must also be greater than the convex surface of some circumscribed pyramid (7. *Cor.* 1.)— greater, that is, than half the product of the slant side of the cone by the perimeter of the circumscribed polygon (I. 26. *Cor.*), which is the base of the pyramid; for the triangles which form the convex surface of the pyramid have for their bases the sides of the circumscribed polygon, and the lines drawn from their common vertex to the points of contact (which lines are each a slant side of the cone) perpendicular to those bases respectively (III. 2. *Cor.* 1. and IV. 4.). But this is impossible, because the circumference of the base of the cylinder is less than the perimeter of the circumscribed polygon. Again, if half this product be less than the convex surface of the cone, it must be less also than the convex surface of some inscribed pyramid (5. *Cor.* 1.)—less, that is, than half the product of the perpendicular drawn from the vertex to a side of the inscribed regular polygon which is the base of the pyramid, by the perimeter of that polygon; which is impossible, because not only is the slant side greater than the perpendicular (IV. 8.), but the circumference of the base of the cylinder is also greater than the perimeter of the inscribed polygon.

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* The lines U A, U B, &c. are omitted in the re.

Therefore, half the product in question is neither greater nor less than the convex surface of the pyramid; that is, it is equal to it.

Therefore, &c.

Cor. If R is the radius of the base of a right cone, and S its slant side, the convex surface of the cone is π R S (III. 34. *Scholium.*).

Prop. 9.

The solid content of a right cone is equal to one-third of the product of its base and altitude.

This proposition is demonstrated in the same way as the preceding. If a third of the product in question exceed the content of the cone, it must likewise exceed the content of some circumscribed pyramid (7. *Cor.* 1.); but this is impossible, because the latter (IV. 32. *Cor.* 1.) is equal to a third of the product of its altitude, which is the same with that of the cone, by its base, which is greater than the base of the cone. And if, on the other hand, it be less than the content of the cone, it must likewise be less than the content of some inscribed pyramid (7. *Cor.* 1.); but this is impossible, because the latter (IV. 32. *Cor.* 1.) is equal to a third of the product of its altitude, which is the same with that of the cone, by its base, which is less than the base of the cone.

Therefore, a third of the product in question is equal to the content of the cone.

Therefore, &c.

Cor. 1. If R is the radius of the base of a right cone, and A its altitude, the solid content of the cone is $\frac{1}{3}\pi R^2 A$ (III. 34. *Scholium.*).

Cor. 2. (Euc. xii. 10.). If a right cylinder and a right cone have the same base and the same altitude, the cone shall be a third part of the cylinder (4.).

Cor. 3. If a right cone and a pyramid have equal bases and altitudes, the cone shall be equal to the pyramid (IV. 32. *Cor.* 1.).

Cor. 4. (Euc. xii. 11 and 14.) Right cones which have equal altitudes are to one another as their bases; and right cones which have equal bases are to one another as their altitudes: also, any two right cones are to one another in the ratio which is compounded of the ratios of their bases and altitudes (IV. 32. *Cor.* 2.).

Prop. 10. (Euc. xii. 12.).

The surfaces of similar right cones are as the squares of the axes; and their solid contents are as the cubes of the axes.

For, in the first place, there may be inscribed in the cones similar pyramids, the convex surfaces of which approach more nearly to the convex surfaces of the cones than by any the same given difference (7. *Cor.* 2.); and the convex surfaces of these pyramids are, to one another, always in the same ratio, viz. as the squares of the sides of their bases (II. 42. *Cor.* and II. 23. *Cor.* 1.), that is, as the squares of the radii of the circumscribing circles (III. 30.), or (II. 37. *Cor.* 4.) as the squares of the axes of the cones, for the axes of the cones are to one another as the radii of their bases (def. 6.); therefore the convex surfaces of the cones are to one another in the same ratio (II. 28.), viz. as the squares of their axes.

And, in the same manner, because there may be inscribed in the cones similar pyramids, the solid contents of which approach more nearly to the solid contents of the cones than by any the same given difference (7. *Cor.* 2.); and because the solid contents of these pyramids are to one another always in the same ratio, viz. as the cubes of the sides of their bases (IV. 35.), or (IV. 27. *Cor.* 3.) as the cubes of the axes of the cones; the solid contents of the cones are to one another in the same ratio (II. 28.), viz. as the cubes of their axes.

Otherwise:

Let A, a represent the axes of two similar right cones; R, r the radii of their bases; and S, s their slant sides. Then (8. *Cor.*) π R S and $\pi r s$ will represent their convex surfaces respectively, and (9. *Cor.* 1.) $\frac{1}{3}\pi R^2 A$, $\frac{1}{3}\pi r^2 a$ their solid contents. But, because the cones are similar (def. 6.) $R : A :: r : a$; and because the slant sides S, *s* are the hypotenuses of right-angled triangles, which have the sides A, R and *a*, *r* about the right angles proportional (II. 32.) $S : A :: s : a$: therefore (II. 37. *Cor.* 3.), $RS : A^2 :: rs : a^2$, *alternando* (II. 19.) $RS : rs :: A^2 : a^2$, and (p. 47, Rule ii.) $\pi RS : \pi rs :: A^2 : a^2$, that is, the convex surfaces of the cones are as the squares of their axes.

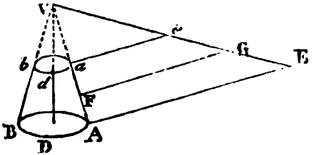
Again, because $R : A : r : a$, $R^2 : A^2 : r^2 : a^2$ (II. 37. *Cor.* 4.), and (p. 47, Rule ii.) $R^2 A : A^3 : r^2 a : a^3$: therefore *alternando* (II. 19.) $R^2 A : r^2 a :: A^3 : a^3$, and (p. 47, Rule ii.) $\frac{1}{3}\pi R^2 A : \frac{1}{3}\pi r^2 a :: A^3 : a^3$, that is, the solid contents of the cones are to one another as the cubes of their axes.

Therefore, &c.

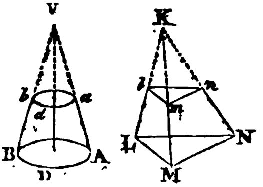
Prop. 11.

The convex surface of a frustum of a right cone is equal to the product of its slant side by half the sum of the circumferences of its two bases; and its solid content is equal to the sum of the solid contents of three cones, which have the same altitude with the frustum, and for their bases its two bases, and a mean proportional between them.

Let A B D, *a b d*, be the bases of the frustum of a cone, which has the vertex V. Draw any slant side V *a* A; from A draw any straight line AE perpendicular to VA (I. 44.); suppose AE to be taken equal to the circumference A B D, and join VE, and through *a* draw *a e* parallel to AE (I. 48.), to meet VE in *e*. Then, because the circumferences A B D, *a b d*, are as their radii (III. 33.), that is, as V A, V *a* (II. 31.), that is, again, as A E, *a e* (II. 31.), and that A E is equal to A B D, *a e* is equal to *a b d* (II. 18.). Now the triangle V A E is equal to the convex surface of the cone V A B D (6.), because (I. 26. *Cor.*) it is equal to half the product of V A the slant side, and A E, which is equal to the circumference A B D; and, for the like reason, the triangle V *a e* is equal to the convex surface of the cone V *a b d*; therefore, the convex surface of the frustum is equal to the trapezoid A E *e a*. And because the trapezoid A E *e a* is equal to the product of its altitude A *a* by half the sum of its parallel sides A E, *a e* (I. 28.), the convex surface of the frustum is equal to the product of its slant side A *a*, by half the sum of the circumferences A B D, *a b d*.



In the next place, with regard to the solid content. Let KLMN be a triangular pyramid, having its base L M N equal to the base ABD, and in the same plane with it, and its vertex K in the same parallel to the base with the vertex of the cone. Then, because the cone and pyramid have equal bases, and the same altitude, they are equal to one another (9. *Cor.* 3.). Let the plane *a b d* be produced to cut the pyramid in the triangle *l m n*: then, because *l m n* is similar to L M N (IV. 12. and IV. 15.), they are to one another as lm^2 and LM^2 (III. 42. *Cor.*), that is, (II. 37. *Cor.* 4.) as Kl^2 and KL^2, or (IV. 14.) as Va^2 and VA^2: but the bases *a b d* and A B D are to one another in the same ratio (III. 33. and II. 31.), and L M N is equal to A B D: therefore, also, *l m n* is equal to *a b d* (II. 12. and II. 18.), and (9. *Cor.* 3.) the cone V *a b d* is equal to the pyramid K *l m n*. Therefore (I. ax. 3.) the frustum of the cone is equal to the frustum of the pyramid. But the latter (IV. 33.) is equal to the sum of three pyramids, having the same altitude with the frustum, and for their bases the bases of the frustum, and a mean proportional between them; and (9. *Cor.* 3.) each of these pyramids is equal to a cone having the same altitude and an equal base. Therefore, also, the frustum of the cone is equal to the sum of three cones, having the same altitude with it, and for their bases the bases of the frustum, and a mean proportional between them.



Therefore, &c.

Cor. If a straight line A *a* be made to revolve about any axis, VC, in the same plane with it, the surface generated by such straight line shall be equal to the product of the straight line and the circumference generated by its middle point F.

For the generated surface is that of a cylinder if the line be parallel to the axis; and, in every other case, that of a frustum of a right cone. In the former case, the reason is sufficiently manifest (3.). In the latter, it may be shown, that if F G be drawn parallel to A E (in the first figure of the proposition) to meet V E in G, F G will be equal to the circumference generated by the point F: also, because V F is equal to half the sum of V A and V *a*, F G is equal to half the sum of A E and *a e* (II. 30. *Cor.* 2.); therefore the circumference generated by the point F is equal to half the sum of the circumferences A B D, *a b d*; and hence by the proposition, the convex surface of the frustum, that is, the convex surface generated by the line A *a* is equal to the product of A *a* and the circumference generated by its middle point F.

Scholium.

Although the propositions of this section have, for greater brevity and simplicity, been stated and demonstrated only with regard to the right cylinder and right cone, it will be found that Props. 2. and 7. apply equally to the oblique cylinder and oblique cone, to which the demonstrations may be without difficulty adapted, and hence it may be demonstrated, almost in the words of Props. 4. 9. 11. 5. and 10. that the solid content of an oblique cylinder is equal to the product of its base and altitude; the solid content of an oblique cone to one-third of the product of its base and altitude; the solid content of a truncated oblique cone to the sum of the solid contents of three cones, having the same altitude with it, and for their bases its two bases, and a mean proportional between them; and, lastly, that the surfaces of similar oblique cylinders and cones are to one another as the squares of their axes, and their solid contents as the cubes of the axes. With regard to the convex surface of the oblique cylinder, it may likewise be shown in a similar manner (see IV. 29. *Scholium*) to be equal to the product of its side (or axis) by the perimeter of a plane section perpendicular to it.

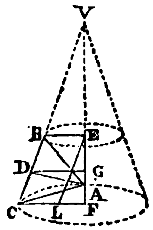
It remains to observe, with regard to Props. 3. and 8., that the remarkable property by which the convex surfaces of the cylinder and cone have been defined, viz. that of containing, the first, straight lines parallel to a given straight line, and the other straight lines diverging from a given point, leads to another property of those surfaces, from which the measures assigned in Props. 2. and 6. may be very readily inferred. This property is, that they are *developable*, that is, they may be conceived to be unfolded and spread out upon a plane. Now, it is easy to perceive, that if the surface of a right cylinder be so developed, it will form a rectangle, which has for its base the circumference of the circle, which is the base of the cylinder, and for its altitude the altitude of the cylinder; whence it follows, that the convex surface of a right cylinder is equal to the product of its altitude by the circumference of its base. In like manner the developed surface of a right cone will form a circular sector, the arc of which is equal to the circumference of the base of the cone, and its radius to the slant side; whence it follows, that the convex surface of a right cone is equal to half the product of its slant side by the circumference of its base.

SECTION 2.—*Surface and content of the sphere.*

PROP. 12.

If an isosceles triangle A B C *be made to revolve about an axis which lies in the same plane with it and passes through the vertex* A, *and if a perpendicular* A D *be drawn from the vertex to the base, and* E F *be that portion of the axis which is intercepted by perpendiculars drawn to it from the extremities of the base; the convex surface generated by the base shall be equal to the product of* E F *by the circumference of a circle having the radius* A D; *and the solid generated by the triangle shall be equal to one-third of the product of this surface by the perpendicular* A D.

First, of the surface generated by the base B C. This is evidently the convex surface of a truncated cone having the axis E F, and E B, F C for the radii of its bases; and is therefore (11. *Cor.*) equal to the product of B C, and the circumference generated by its middle point D, that is, (if D G be drawn parallel to B E to meet E F in G) the circumference which has the radius D G. Now, if E L be drawn



parallel to B C, E L will be equal to B C, (I. 22.), and the triangles L E F, A D G will be similar (I. 18.), because the sides of the one are perpendicular to the sides of the other, each to each; therefore E L or B C is to E F as A D to D G (II. 31.), that is, as the circumference of a circle which has the radius A D to the circumference of a circle which has the radius D G (III. 33. and II. 12.); and therefore (II. 28. *Schol.* Rule I.) the product of B C and the latter circumference is equal to the product of E F and the former. But the convex surface in question is equal to the product of B C and the circumference of which has the radius D G. Therefore (I. ax. 1.) that surface is likewise equal to the product of E F, and the circumference which has the radius A D.

Next, of the solid generated by the triangle A B C. Let C B and F E be produced to meet one another in V. Then the solid in question is the difference of those generated by the triangles A C V and A B V. Now, the solid generated by the triangle A C V is equal to the sum or difference of two cones, having the altitudes A F, V F respectively, and for their common base the circle of which C F is radius—equal, that is (9.), to one-third of the product of A V, which is the sum or difference of the altitudes, by half the radius C F, and the circumference which has the radius C F (III. 32.); or to one-third of the product of AD, half VC, and that circumference (for A D × V C is equal to A V × C F), or lastly, to one-third of the product of A D, and the surface generated by V C (8.). And in the same manner it may be shewn, that the solid generated by the triangle A B V is equal to one-third of the product of A D, and the surface generated by V B. Therefore the difference of these solids, that is, the solid in question, is equal to one-third of the product of A D, and the surface generated by B C.

It has been supposed in the above demonstrations, that E F and B C are not parallel. If B C be parallel to E F, the surface generated by B C will be that of a right cylinder having the axis E F, whence the first part of the proposition is manifest; and the solid generated by the triangle A B C will be equal to two-thirds of the cylinder (9.), whence the second part of the proposition.

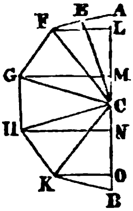
Therefore, &c.

Cor. The proof of the second part of the proposition is equally applicable, whether A B C be isosceles, or otherwise. Therefore, if any triangle A B C be made to revolve about an axis which lies in the same plane with it and passes through its vertex A; and if A D be drawn perpendicular to the base, the solid generated by the triangle shall be equal to one-third of the product of A D and the surface generated by the base B C.

PROP. 13.

If the half A F G H K B *of any regular polygon of an even number of sides revolve about the diagonal* A B; *the whole surface of the solid generated by its revolution shall be equal to the product of* A B *by the circumference of a circle whose radius is the apothem* C E *of the polygon; and its solid content shall be equal to one-third of the product of this surface by the apothem* C E.

From the points F, G, H, K, draw F L, G M, H N, K O perpendicular to A B, (I. 45.) and join CF, CG, CH, C K. Then, because C is the centre of the polygon, the triangles C A F, C F G, &c. are isosceles triangles, having the common vertex C, and the perpendiculars drawn from C to their respective bases equal each of them to the apothem CE. And, because these triangles revolve about the axis AB passing through C, and that A L, L M, &c. are the parts of the axis intercepted by perpendiculars drawn from the extremities of the base of each; the portions of the whole surface in question, generated by A F, F G, &c., are equal, respectively, to the products of AL, LM, &c., by the circumference of the circle which has the radius C E (12.). Therefore the whole surface is equal to the sum of these products, that is, to the product of A B by the circumference of the same circle.

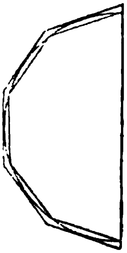


Again, because the portions of the whole solid which are generated by the triangles C A F, C F G, &c. are the third parts, respectively, of the products of the portions of surface generated by A F, F G, &c. by the apothem C E (12.); the whole solid is the third part of the sum of these products, that is, the third part of the product of the whole surface by the apothem C E.

Therefore, &c.

Prop. 14.

If within and about a semicircle there are inscribed and circumscribed any two half-polygons, the one having the diameter of the semicircle for its diagonal, and the other the diameter produced; and if these figures are made to revolve together with the semicircle about the diameter; the sphere generated by the semicircle shall be greater than the inscribed solid of revolution, and less than the circumscribed solid: also the surface of the sphere shall be greater than the surface of the inscribed solid of revolution, and less than that of the circumscribed solid.

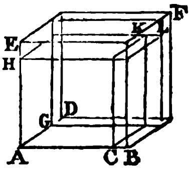


For, with respect to the solid contents, the sphere contains the inscribed solid, and is itself contained in the circumscribed solid: and, with respect to the surfaces, the surface of the sphere envelops the surface of the inscribed solid, and is enveloped by that of the circumscribed solid (*Lemma* 2.).

Therefore, &c.

Lemma 3.

If there be two straight lines, of which one is given, and the other may be made to approach to it within any given difference; the cube of the latter may also be made to approach to the cube of the former within any given difference.



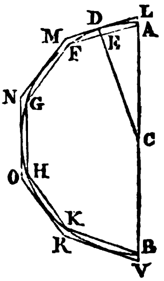
Let AB, AC be the two straight lines, of which AB is given. Upon AB (I. 52.) describe the square BD; from A draw AE perpendicular to the plane BAE (IV. 37.); make AE equal to AB, and complete the cube AF: and in like manner upon AC describe the square CG in the same plane with BD, from AE cut off AH equal to AC, and complete the cube AK: and let the faces HK and GK of the latter cube be produced to meet the faces BF, EF, of the former. Then the difference of the two cubes is equal to the sum of the three parallelopipeds GF, CL, and LE. Of these, the first has its base DF equal to the square of AB (IV. 22.), and its altitude DG equal to the difference of AB, AC; the second has its base CK equal to the square of AC (IV. 22.), and its altitude CB likewise equal to the difference of AB, AC; and with respect to the third, its altitude HE is likewise equal to the difference of AB, AC, and its base HL is a mean proportional between the squares of AB, AC, because its adjacent sides are equal to AB and AC respectively, and (II. 35.) AB^2 is to $AB \times AC$ as $AB \times AC$ to AC^2. Therefore the difference of the cubes is less than a parallelopiped, whose altitude is CB, and its base equal to three times the square of AB. But, because CB may be made less than any given line, this parallelopiped may be made less than any given solid. Much more, therefore, may the difference of the cubes be made less than any given solid, that is, than any given difference.

Therefore, &c.

Cor. It appears from the demonstration, that the difference of the cubes of two straight lines is equal to the product of the difference of the straight lines by the sum of their squares, and a mean proportional between those squares.

Prop. 15.

Any sphere being given, two figures of revolution generated by similar half polygons may be, the one inscribed in the sphere, and the other circumscribed about it, such that the difference of their surfaces, or of their solid contents, shall be less than any given difference.



Let the given sphere be generated by the revolution of the semicircle ADB about the diameter AB. And first, let P be the given difference of surfaces; and let Q be the surface of any solid circumscribing the sphere. Then, since (as in prop. 7.) a regular polygon may be inscribed in the circle ABD, such that, CE being its apothem, $CD^2 - CE^2$ shall be to CE^2 in a ratio less than that of P to Q; let AFGHKB be the half of such a polygon, and let a similar half-polygon LMNORV be circumscribed, so that one of its sides LM may touch the circle in D (III. 27. *Cor.* 2.) Then, if these inscribed and circumscribed half-polygons be made to revolve with the semicircle about the axis AB, they will generate, together with the sphere generated by the semicircle, two figures of revolution, the one inscribed in the sphere, the other circumscribed about it. And because the surfaces of these figures are equal respectively (13.) to the rectangles under A B and the circumference which has the radius C E, and under L V and the circumference which has the radius C D, and that these rectangles are to one another as the squares of C E, C D (II. 37. *Cor.* 1.); the difference of the surfaces is to the surface of the inscribed figure as $CD^2 - CE^2$ to CE^2 (II. 20.); that is, in a less ratio than that of P to Q. But the surface of the inscribed figure is less than Q: much more, therefore, is the difference of the surfaces less than P (II. 18. *Cor.*).

Next, let S be the given difference of contents; and let T be the content of any solid circumscribing the sphere. Then, since a regular polygon may be inscribed in the circle A B D, such that, C E being its apothem, C D—C E shall be less than any given difference, and therefore also such that $CD^2 - CE^2$ shall be to CE^2 in a ratio less than that of S to T (*Lemma* 3.); let A F G H K B be the half of such a polygon, and let the figures of revolution be inscribed in the sphere, and circumscribed about it, as before. Then, because the contents of these figures are (13.) equal respectively to the thirds of two parallelopipeds (IV. 25. *Schol.*), having their bases equal to the surfaces, and their altitudes equal to C E, C D, and that these parallelopipeds are to one another as the cubes of C E, C D, for their bases are, as was shown in the former part of the proposition, as the squares of C E, C D; the difference of the contents is to the content of the inscribed figure as $CD^3 - CE^3$ to CE^3 (II. 20.), that is, in a less ratio than that of S to T. But the content of the inscribed figure is less than T: much more, therefore, is the difference of the contents less than S (II. 18. *Cor.*).

Therefore, &c.

Cor. Any sphere being given, a figure of revolution may be inscribed (or circumscribed), which shall differ from the sphere in surface or in content, by less than any given difference. For the difference between the sphere and either of the figures of revolution, whether in surface or in content, is less than that of the two figures (14.).

Prop. 16.

The surface of a sphere is equal to the product of the circumference and diameter of the generating circle.

For if this product be not equal to the surface of the sphere, it must either be greater or less than it.

If greater, it must be greater also than the surface of some circumscribed solid of revolution (15. *Cor.*), greater, that is (13.), than the product of the diagonal L V by the circumference which has for its radius the apothem C D (see the figure of Prop. 15.); which is impossible, because the diameter is less than LV, and the circumference of the generating circle is the same with the circumference which has the radius C D.

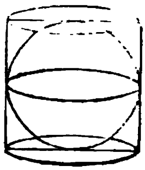
If less, it must also be less than the surface of some inscribed solid of revolution (15. *Cor.*)—less, that is (13.), than the product of the diameter A B, which is the same with the diameter of the generating circle, by the circumference which has for its radius the apothem C E; which is impossible, because the circumference of the generating circle is greater than the circumference which has the radius C E.

Therefore the product in question is neither greater nor less than the surface of the sphere; that is, it is equal to it.

Therefore, &c.

Cor. 1. The surface of a sphere is equal to four times the area of its generating circle. For the area of this circle is equal to half the product of the radius and circumference (III. 32.).

Cor. 2. If a right cylinder be circumscribed about a sphere; the surface of the sphere shall be equal to the convex surface of the cylinder. For the latter is equal to the product of its altitude, and the circumference of its base (3.); and its base is equal to the generating circle of the sphere, and its altitude to the diameter.



Cor. 3. The surface of a sphere is equal to two-thirds of the whole surface of the circumscribing cylinder.

Cor. 4. If D is the diameter of a sphere, its whole surface is equal to πD^2 (III. 34. *Schol.*).

Prop. 17.

The solid content of a sphere is equal to one-third of the product of the radius by the surface.

For the third part of this product cannot be greater than the content of the sphere; since then it would be greater also than the content of some circumscribed solid of revolution (15 *Cor.*)—greater, that is, than one-third of the product of the same radius by the surface of that solid (13.); which is impossible, because the surface of the sphere is less than that of the solid (14.).

Nor, on the other hand, can it be less than the content of the sphere, for then would it be less than some inscribed solid of revolution (15. *Cor.*), less, that is, than one-third of the product of the apothem by the surface of that solid (13.); which is impossible, because not only is the radius greater than the apothem, but the surface of the sphere is likewise greater than the surface of the inscribed solid (14.).

Therefore the product in question is equal to the solid content of the sphere.

Therefore, &c.

Cor. 1. The solid content of a sphere is equal to one-third of the product of the radius by four times the area of the generating circle (16. *Cor.* 1.).

Cor. 2. The solid content of a sphere is two-thirds of the solid content of the circumscribing cylinder. For the latter is equal to twice the product of the radius, and the area of the generating circle (4).

Cor. 3. If any solid contained by planes be circumscribed about a sphere, the content of the sphere will be to the content of the solid as the surface of the sphere to the surface of the solid. For the solid may be divided into pyramids, having the centre of the sphere for their common vertex, and their altitudes equal each to the radius of the sphere; and since each of these pyramids is equal to a third of the product of its base and altitude, their sum is equal to a third of the product of the convex surface of the solid and the radius of the sphere: also, the sphere is equal to a pyramid, having the same altitude, and its base equal to the surface of the sphere (IV. 32.).

Cor. 4. If D is the diameter of a sphere, its whole solid content is equal to $\frac{1}{6}\pi \times D^3$ (16. *Cor.* 4.).

PROP. 18. (Euc. xii. 18.).

The surfaces of spheres are as the squares of the radii, and their solid contents are as the cubes of the radii.

For the surfaces are equal respectively to four times the areas of the generating circles (16. *Cor.* 1.), and these areas are as the squares of the radii (III. 33.).

And the solid contents are to one another in a ratio which is compounded of the ratios of the surfaces and of the radii; that is (because the surfaces are to one another in the duplicate ratio of the radii), in the triplicate ratio of the radii, or (IV. 27. *Cor.* 2.) as the cubes of the radii.

Otherwise:

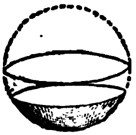
Let D, d be the diameters of two spheres, and R, r their radii. Then (16. *Cor.* 4.) πD^2, πd^2 will represent their surfaces, and (17. *Cor.* 4.) $\frac{1}{6}\pi D^3$, $\frac{1}{6}\pi d^3$ their solid contents; or, since D is equal to 2R and d to $2r$, $4\pi R^2$, $4\pi r^2$ will represent their surfaces, and $\frac{4}{3}\pi R^3$, $\frac{4}{3}\pi r^3$ their solid contents. But (p. 47, Rule ii.) $4\pi R^2$ is to $4\pi r^2$ as R^2 to r^2, and $\frac{4}{3}\pi R^3$ is to $\frac{4}{3}\pi r^3$ as R^3 to r^3. Therefore, the surfaces are as the squares of the radii, and the solid contents as the cubes of the radii.

Therefore, &c.

SECTION 3.—*Surfaces and contents of certain portions of the sphere.*

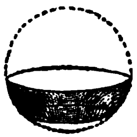
In order to have a clear apprehension of the figures intended in the following definitions, it is necessary to keep in mind, that every section of a sphere which is made by a plane is a circle, the centre of which is the foot of the perpendicular drawn to the plane from the centre of the sphere (IV. 8. *Cor.*).

Def. 10. *A segment of a sphere* is any portion of it which is cut off by a plane, and the circle in which the plane cuts the sphere is called the *base* of the segment.

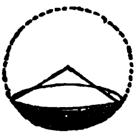


When the plane passes through the centre, the two segments into which the sphere is divided are equal to one another, and are therefore each of them called a *hemisphere.*

The convex surface of a segment is called a *zone.*

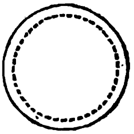


11. A *double-based spherical segment* is a portion of a sphere intercepted between two parallel planes; and the circles in which these planes cut the sphere are called the *bases* of the segment. The convex surface of a double-based segment is likewise called a *zone.*



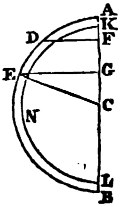
12. A *sector of a sphere* is the solid figure contained by the convex surface of a segment, and that of a right cone, which has the same base with the segment, and for its vertex the centre of the sphere.

The convex surface of the segment is called the *base* of the sector.



13. A *spherical orb* is a portion of a sphere contained between its surface and that of a lesser sphere, which is concentric (or has the same centre) with it.

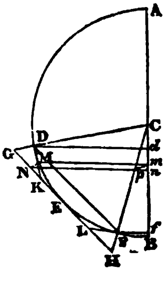
14. A *spherical wedge* or *ungula* is a portion of a sphere intercepted between two planes, each of which passes through the centre of the sphere. The convex surface of an ungula is called a *lune*.



Let A D B be a semicircle, and from the points D, E of the semicircumference, let the straight lines D F, E G be drawn at right angles to the diameter A B; join C E, and let K N L be a second semicircle, having the same centre C; then, if the whole figure revolve about A B, the parts A E G, D E G F, A E C, and A D B L N K will generate a spherical segment, a double-based spherical segment, a spherical sector, and a spherical orb respectively. And if the semicircle A D B, instead of making a complete revolution, revolve only through a certain angle, it will generate a spherical wedge or ungula.

Prop. 19.

If a semicircle be made to revolve about its diameter, the zone which is generated by any arc of the semicircle shall be greater than the surface generated by the chord of that arc, and less than the surface generated by the tangent of the same arc, which is drawn parallel to the chord, and terminated by the radii passing through its extremities.



Let A D B be a semicircle having the diameter A B, and the centre C: let D E F be any arc of the semicircle, D F its chord, and GH a straight line parallel to DF, which touches the arc D E F in E, and is terminated by the radii C D, C F passing through the extremities of the arc: then, if the semicircle be made to revolve about the diameter A B, the zone which is generated by the arc DF, shall be greater than the surface generated by the chord DF, and less than the surface generated by the tangent GH.

From the points D, F draw the straight lines D *d*, F *f*, each of them perpendicular to A B (I. 45.). Then, in the supposed revolution of the figure, these straight lines will generate two circles which have the points *d*, *f* for their centres, and *d* D, *f* F for their radii respectively (IV. 3. *Cor.* 2.). And, because the zone generated by the arc D E F, together with these two circles, forms a convex surface which envelops, and therefore (*Lemma* 2.) is greater than the convex surface consisting of the surface generated by the chord DF and the same two circles, the zone generated by the arc D E F is greater than the surface generated by the chord D F.

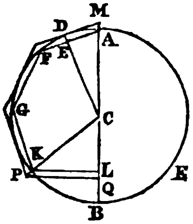
In the next place, from the points D, F draw the tangents D K, F L (III. 56.) to meet GH in the points K, L respectively; bisect D K in M (I. 43.); through M draw M N parallel to C G (I. 48.) to meet G K in N, and from the points M, N draw M *m*, N *n* perpendicular each of them to A B; and, lastly, through *m* draw *m p* parallel to M N to meet N *n* in *p*. Then, because the middle point of D K, in the supposed revolution of the figure about the axis A B, generates the circumference which has the radius M *m*, the surface generated by D K is equal to the product of D K and the circumference which has the radius M *m* (11. *Cor.*). And, in like manner, since N is the middle point of G K (II. 29.), the surface generated by G K is equal to the product of G K and the circumference which has the radius N *n*. But, because (III. 2. *Cor.* 1.) the angle K D G is a right angle, and therefore (I. 8.) the angle K G D less than a right angle, that is, than K D G, D K is less than G K (I. 9.); and, because (I. 22.) M *m* is equal to N *p*, which is less than N *n*, the circumference which has the radius M *m* is less than the circumference which has the radius N *n* (III. 33.). Therefore, upon both accounts, the surface generated by D K is less than the surface generated by G K. And in the same manner it may be shown that the surface generated by L F is less than the surface generated by L H. Therefore, the whole convex surface generated by the three straight lines D K, K L, L F is less than the whole surface generated by the tangent G H. But the zone generated by the arc D E F is less than the surface generated by D K, K L, L F (*Lemma* 2.), for the zone together with the circles which have the radii D*d*, F*f*, is enveloped by the latter surface together with the same two circles. Much more, therefore, is the zone generated by the arc D E F less than the surface generated by the tangent G H.

Therefore, &c.

Prop. 20.

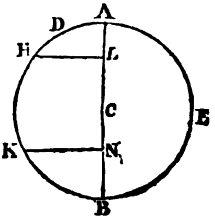
A spherical zone is equal to the product of the circumference of the generating circle and that portion of the axis which is intercepted between its convex surface and base; or, if it be double-based, between its two bases.

Let A D B be a semicircle, and A K any arc, by the revolution of which about the diameter A B a spherical zone is generated, and from K draw K L perpendicular to A B: the zone shall be equal to the product of the whole circumference A D B E by the part A L of the diameter.



The demonstration is in every respect similar to that of prop. 13. For, in the first place, it is evident that the arc A K may be divided into a number of equal arcs, such that, the chords A F, F G, &c. being drawn, their common distance C E from the centre C shall approach to the radius C D within any given difference; and hence it may be shown, as in prop. 15., that there may be inscribed in the zone and circumscribed about it, two surfaces of revolution which differ from each other, and therefore (19.) each of them from the zone, by less than any given difference. Therefore, the product in question cannot be greater than the zone; for then it must be greater also than some circumscribed surface—greater, that is, than the product of M Q and the circumference A D B E; which is impossible, because A L is less than MQ.[*] Neither can it be less than the zone; for then it must be less also than some inscribed surface—less, that is, than the product of A L and the circumference which has the radius C E; which is impossible, because the circumference A D B E is greater than that which has the radius C E. Therefore, it must be equal to the zone; that is, the zone is equal to the product of the circumference A D E B, and the part A L of the diameter.

Next, let H K be any arc, by the revolution of which about the diameter A B a double-based zone is generated; and let H L, K N be drawn perpendicular to A B. Then, because the whole zone generated by the arc A K is equal to the product of A N and the circumference ADBE, and the part generated by the arc A H equal to the product of A L and the same circumference, the remainder, that is, the double-based zone in question, is equal to the product of L N and the same circumference.



Therefore, &c.

Cor. 1. If a cylinder, having the axis A B, be circumscribed about the sphere; any zone having the same axis, shall be equal to that portion of the convex surface of the cylinder which is intercepted between the base of the cylinder and the plane of the base of the zone, or between the planes of its two bases, if it be double-based (3.).

Cor. 2. In the same or in equal spheres, any two zones are to one another as the parts of the axis or axes which are intercepted between their respective bases (II. 35.).

Prop. 21.

Every spherical sector is equal to one-third of the product of its base and the radius of the sphere.

The steps by which this proposition is demonstrated are similar to those indicated in the preceding. In the first place, it may be shown that two solids of revolution may be, one inscribed in the sector, the other circumscribed about it, which approach each of them to the sector more nearly than by any given difference. Hence, the product in question cannot be greater than the sector; for then would it be greater than some circumscribed solid, and therefore (12.) the base of the sector greater than the convex surface of such a solid, which (19.) is impossible: neither can it be less, for then would it be less also than some inscribed solid; which is impossible, because, not only is the base of the sector greater (19.) than the convex surface of such a solid, but the radius C H is likewise greater than the perpendicular C E. Therefore it must be equal to the sector.

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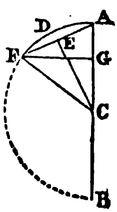
[*] If the point Q lies between A and C, M Q will be the difference, not the sum, of M L and L Q; but, in this case also, A L is less than M Q, because L Q is less than K P, that is, than M A.

Therefore, &c.

Prop. 22.

Every spherical segment, upon a single base, is equal to the half of a cylinder having the same base and the same altitude, together with a sphere, of which that altitude is the diameter.



Let A D F G be any circular half-segment, by the revolution of which about the diameter A B, a spherical segment having the altitude A G is generated: the spherical segment shall be equal to the half of a cylinder having the same base and altitude, together with a sphere, of which A G is the diameter.

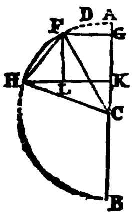
Join A F; and from the centre C draw C E perpendicular to A F. Then, because the solid, generated by the segment A D F, is equal to the difference of the solids generated by the sector C A D F and the triangle C A F, the former of which (21.) is equal to one-third of the product of C A by its base $AG \times 2\pi \times CA$ (20.), and the latter (12.) to one-third of the product of C E by the convex surface generated by A F viz. (12.) $AG \times 2\pi \times CE$; the solid, generated by the segment A D F, is equal to $\frac{2}{3}\pi \times AG\,(CA^2 - CE^2)$, i. e. $\frac{2}{3}\pi \times AG \times AE^2$, or to $\frac{1}{6}\pi \times AG \times AF^2$, because (I. 36. *Cor.* 1.) $CA^2 - CE^2$ is equal to AE^2, and AE^2 (III. 3.) to a fourth of AF^2. To this add the cone generated by the triangle A F G, which (9.) is equal to $\frac{1}{3}\pi \times AG \times FG^2$: therefore, the spherical segment in question is equal to $\frac{1}{6}\pi \times AG\,(AF^2 + 2FG^2)$, i. e. to $\frac{1}{6}\pi \times AG\,(3FG^2 + AG^2)$ because AF^2 is equal to $AG^2 + FG^2$ (III. 36.). And $\frac{1}{6}\pi \times AG\,(3FG^2 + AG^2)$ is equal to $\frac{1}{2}\pi \times AG \times FG^2 + \frac{1}{6}\pi \times AG^3$; the first part of which is (4. *Cor.* 1.) the half of a cylinder having the altitude A G and the same base with the segment, and the other part (17. *Cor.* 4.) a sphere of which A G is the diameter.

Therefore, &c.

Prop. 23.

Every double-based spherical segment is equal to the half of a cylinder having the same altitude with the segment and a base equal to the sum of its two bases, together with a sphere of which that altitude is the diameter.

Let F H K G be any portion of the semicircle A D B, by the revolution of which about the diameter A B, a double-based segment, having the altitude G K, is generated: the segment shall be equal to the half of a cylinder having the same altitude and a base equal to the sum of its two bases, together with a sphere of which G K is the diameter.



For, in the first place, because the solid, generated by the revolution of the circular segment F H, is equal to the excess of the difference of the sectors generated by C A H and C A F above the solid generated by the triangle C F H, it may be shown by the same steps as in the last proposition, that the solid generated by the revolution of F H is equal to $\frac{1}{6}\pi \times GK \times FH^2$, i. e. if F L be drawn parallel to G K, to $\frac{1}{6}\pi \times GK\,(FL^2 + LH^2)$. But (I. 22.) F L is equal to G K, and L H is equal to the difference of F G and H K (I. 22.): therefore (I. 33.), LH^2 is equal to $FG^2 + HK^2 - 2FG \times HK$, and $\frac{1}{6}\pi \times GK\,(FL^2 + LH^2)$, or the solid generated by the segment F H, is equal to $\frac{1}{6}\pi \times GK\,(FG^2 + HK^2 - 2FG \times HK) + \frac{1}{6}\pi \times GK^3$. To this add the truncated cone generated by the trapezoid F G K H; which (11.) is equal to $\frac{1}{3}\pi \times GK\,(FG^2 + HK^2 + FG \times HK)$: therefore, the double-based segment in question is equal to $\frac{1}{6}\pi \times GK\,(3FG^2 + 3HK^2) + \frac{1}{6}\pi + GK^3$; or to $\frac{1}{2}\pi \times GK\,(FG^2 + HK^2) + \frac{1}{6}\pi \times GK^3$; the first part of which is the half of a cylinder (4. *Cor.* 1.) having the altitude G K, and a base $\pi \times FG^2 + \pi \times HK^2$, equal to the sum of the bases of the segment, and the other part a sphere of which G K is the diameter (17. *Cor.* 4.).

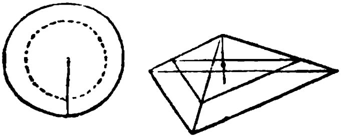
Therefore, &c.

Cor. It appears from the demonstrations of this and the preceding proposition, that the solid generated by the revolution of any circular segment about a diameter of the circle, is equal to $\frac{1}{6}\pi \times GK \times FH^2$; GK being that portion of the diameter, which is intercepted between two perpendiculars drawn to it from the extremities of the segment, and FH the chord which is the base of the segment.

Prop. 24.

Every spherical orb is equal to the sum of three pyramids having their common altitude equal to the thickness of the orb, and for their bases its exterior and interior surfaces, and a mean proportional between them.

For, a spherical orb is the difference between two concentric spheres. Now, if a pyramid be described having its base equal to the exterior surface, or surface of the larger sphere, and its altitude equal to the radius of that surface, this pyramid will be equal to the whole sphere (IV. 32. *Cor.* 1. and 17.). And if, from this, there be cut off, by a plane parallel to the base, a pyramid, having its altitude equal to the radius of the interior surface, the two pyramids will be to one another as the cubes of any two homologous edges (IV. 34. *Cor.*); or, since it may be shown that their altitudes are to one another in the same ratio with the homologous edges, as the cubes of their altitudes, (IV. 27. *Cor.* 3.), that is, as the cubes of the radii of the spheres, or (18.) as the spheres. Therefore, because the larger pyramid is equal to the larger sphere, the smaller pyramid is equal to the smaller sphere (II. 18.); and the difference of the two pyramids is equal to the difference of the two spheres, that is, the frustum is equal to the spherical orb. And, because the larger base of the frustum is equal to the surface of the larger sphere, it may be shown that its smaller base is equal to the surface of the smaller sphere, exactly in the same manner as it has been already shown, that the content of the smaller pyramid is equal to the content of the smaller sphere; also the altitude of the frustum is equal to the thickness of the orb. But the frustum is equal to the sum of three pyramids, having the same altitude with it, and for their bases its two bases and a mean proportional between them (IV. 33.). Therefore, the spherical orb is equal to the sum of three pyramids, having their common altitude equal to the thickness of the orb, and for their bases its exterior and interior surfaces, and a mean proportional between them.

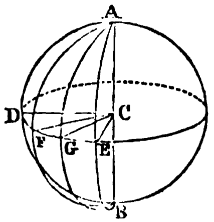


Therefore, &c.

Prop. 25.

Every spherical ungula is to the whole sphere, as the angle between its planes to four right angles; and its lune, or convex surface, is to the surface of the whole sphere in the same ratio.

Let ADBE be an ungula of a sphere having the centre C and the diameter



AB; and from C let CD, CE be drawn in the planes ADB, AEB, perpendicular to AB (I. 44.): the ungula ADBE shall be to the whole sphere, as the angle DCE to four right angles.

For, since the plane DCE is perpendicular to AB (IV. 3.), the angle, which measures the inclination of any two planes passing through AB, may be drawn in that plane at the point C (IV. 17. *Schol.*); and, if any two of these angles at C be equal to one another, the dihedral angles which they measure will be equal (IV. 17.), and therefore the ungulas, which have those dihedral angles, may be made to coincide, and are equal to one another. Now, let the angle DCE be divided into any number of equal angles DCF, FCG, &c.; and therefore the dihedral angle DABE into the same number of dihedral angles by the planes ACF, ACG, &c. (IV. 17.); and the ungula ADBE into as many equal ungulas ADBF, ADBG, &c. by the same planes. Then, if the angle DCF be contained in the four right angles about C any number of times exactly, or with a remainder, the ungula ADBF will be contained in the whole sphere the same number of times exactly, or with a remainder. Therefore the ungula ADBE is to the whole sphere as the angle DCE to four right angles (II. def. 7.).

And a similar proof may be applied to show that the lune A D B E is to the surface of the whole sphere in the same ratio, viz. that of the angle D C E to four right angles.

Therefore, &c.

Cor. 1. Every spherical ungula is equal to one-third of the product of the radius by its lune or convex surface.

Cor. 2. In the same or in equal spheres, any two ungulas are to one another as the angles between their planes. And the same may be said of any two lunes.

Scholium.

We might here add the proportions of similar segments, sectors, orbs, ungulas, and of their convex surfaces. The reader will, however, easily perceive, from the demonstration of prop. 15, that if similar spherical segments and sectors be defined to be such as are generated by similar circular segments and sectors, their surfaces will be as the squares of the radii, and their contents as the cubes of the radii. And the same may be said of similar spherical orbs, defined to be such that the radii of their exterior and interior surfaces are to one another in the same ratio; and of similar ungulas defined to be such as have their dihedral angles equal to one another.

BOOK VI.

§ 1. *Of great and small circles of the Sphere.*—§ 2. *Of Spherical Triangles.*—§ 3. *Of equal Portions of Spherical Surface, and the Measure of solid Angles.*—§ 4. *Problems:*

§ 1.—*Of great and small Circles of the Sphere.*

Def. 1. If a sphere is cut by a plane which passes through the centre, the section is called a *great circle* of the sphere; the radius of such a section being the greatest possible, the same, namely, with the radius of the sphere.

From this definition it is evident that a great circle may be made to pass through any two points in the surface of a sphere; and that, if the two points be not opposite extremities of a diameter, only one great circle can be made to pass through them, for its plane must pass through the centre of the sphere, and only one plane can be made to pass through three points which are not in the same straight line (IV. 1.) But through the two extremities of a diameter, any number of great circles may be made to pass, for they are in the same straight line with the centre of the sphere (IV. 1. *Cor.* 4.).

2. If a sphere is cut by a plane which does not pass through the centre, the section is called a *small circle* of the sphere; the radius of such a section being less than that of the sphere.

A circle, it is plain, may be made to pass through any three points in the sphere's surface; and it will be a great or a small circle, according as its plane passes through the centre of the sphere, or otherwise.

3. The *axis* of any circle of the sphere is that diameter of the sphere which is perpendicular to the plane of the circle; and the extremities of the axis are called the *poles* of the circle.

4. *Parallel circles* of a sphere are such as have their planes parallel.

It is evident that parallel circles have the same axis and poles; for a straight line which is perpendicular to one of two parallel planes is perpendicular to the other likewise (IV. 11.). It may also be observed that two parallel circles cannot both of them pass through the centre of the sphere, that is, they cannot both be great circles of the sphere.

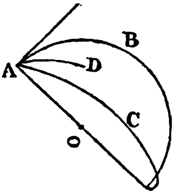
These four definitions may be illustrated by referring to the figure of prop. 1. in which P A P′ is a *great circle*, A B C a *small circle*, P O P′ the *axis*, and P, P′ the *poles* of the circle A B C, and A′ B′ C′, A B C are *parallel circles* of the sphere.

5. Any portion of the circumference of a great circle is called a *spherical arc*.

Two points are said to be *joined* on the surface of the sphere when the spherical arc between them is described; and this arc is called the *spherical distance* of the two points, in order to distinguish it from their direct distance, which is the straight line which joins them. The spherical distance of opposite extremities of a diameter of the sphere is evidently half the circumference of a great circle: but the spherical distance of any other two points is less than a semicircumference, being always the lesser of the two arcs into which they divide the great circle which passes through them.

6. The *polar distances* of any circle of the sphere are the spherical arcs which join any point in the circumference with the two poles of the circle.

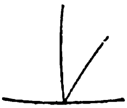
By the *polar distance* (singly) the lesser of these two arcs, or distance from the nearer pole, is generally to be understood.



7. If the arcs A B, AC of two great circles meet one another in a point A, they are said to form at that point a *spherical angle* B A C.

A spherical angle is greater or less, according to the opening between its containing arcs: thus the angle B A C is greater than the angle D A C by the angle B A D.

Every spherical angle is *measured* by the plane angle which measures the inclination of the planes of the containing arcs. For it is easy to perceive, that if this inclination is the same in any two spherical angles, they may be made to coincide, and therefore are equal to one another. If, therefore, the dihedral angle made by the planes of one spherical angle contain any sub-multiple of the dihedral angle made by the planes of another a certain number of times exactly, or with a remainder, the first spherical angle will contain a like submultiple of the other the same number of times exactly or with a remainder; and, therefore, the spherical angles are to one another (II. def. 7.) as the dihedral angles made by their planes, and have the same measures with them.[*]

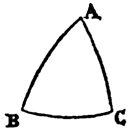


8. When one spherical arc standing upon another makes the adjacent spherical angles equal to one another, each of them is called a *spherical right angle*, and the arc which stands upon the other is said to be *perpendicular*, or *at right angles* to it.

The terms *acute* and *obtuse* are likewise applied to spherical angles, in the same sense as in Book I. def. 11.

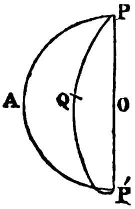
It is evident that a spherical right angle is measured by a rectilineal right angle, a spherical acute angle by a rectilineal acute angle, and a spherical obtuse angle by a rectilineal obtuse angle.

9. A *spherical triangle* is a portion of the sphere's surface included by three arcs of different great circles, as A B C.



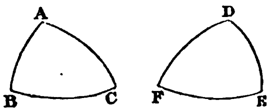
Every spherical triangle, A B C, has three sides, viz. the containing arcs AB, A C, and B C, and three angles A, B, and C.

In the spherical triangles here considered, it is supposed that each of the sides is less than a semicircumference. For, the greatest spherical distance at which two points can be placed is a semicircumference; and if any arc, as P A P′, be taken equal to a semicircumference, its extremities P, P′ will be extremities of a diameter P O P′ of the sphere, and therefore the same great circle will pass through both of them, and any third point Q on the sphere's surface, so that the arcs Q P and Q P′ will be arcs, not of different circles, but of the same circle.

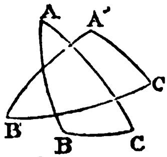


Any three points on the sphere's surface may be assumed for the angles of a spherical triangle (see def. 5.), provided they are not in the same great circle, nor any two of them opposite to one another, that is, opposite extremities of a diameter of the sphere.

10. Two spherical triangles are said to be *symmetrical*, when the sides of the one are equal to the sides of the other, each to each, but in a reverse order, as A B C and D E F.



11. If ABC is any spherical triangle, and the points A′, B′, C′ are those poles of the arcs B C, A C, A B, respectively, which lie upon the same sides of them with the opposite angles A, B, C, and the triangle A′ B′ C′ is completed: this triangle A′ B′ C′ is said to be the *polar triangle* of the triangle A B C.



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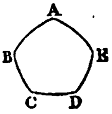
[*] Hence a spherical angle has been defined by some writers to be identical with the dihedral angle of its planes; while others have extended to it the general definition of the angle in which two curves cut one another, considering it the same with the plane rectilineal angle of the tangents at the point A; for the latter angle, being contained by perpendiculars to the common section OA, measures the dihedral angle of the planes.

There are no fewer than eight different triangles which have for their angular points poles of the sides of a given triangle A B C; but there is only one triangle in which these poles A', B', C', lie towards the same parts with the opposite angles A, B, C, and this is the triangle $A'B'C'$, which is known under the name of the polar triangle.

12. A *spherical polygon* is any portion of the sphere's surface included by more than three arcs of different great circles, as A B C D E.

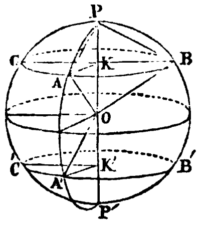


13. *Opposite points* on the surface of the sphere are those which are opposite extremities of a diameter of the sphere. It is evident that the arcs which join two such points with any third point on the sphere's surface, are parts of the same great circle, and are together equal to a semicircumference (see the second figure of def. 9.)

Prop. 1.

Every plane section of a sphere is a circle; the centre of which is either the centre of the sphere, or the foot of the perpendicular which is drawn to the plane from the centre of the sphere.

The substance of this proposition has been already given in the corollary to Book IV. Prop. 8; and the following demonstration is only a statement at greater length of the reasoning from which it was there inferred.



If the plane pass through the centre O of the sphere, as PAP', the distance O A of any point A in the circumference of the section, from the point O, will be the same with a radius of the sphere, and therefore the section will be a circle having the centre O. And if the plane do not pass through the centre, the distances A K, B K, of any two points A, B, in the circumference of the section, from K the foot of the perpendicular O K, will be equal to one another; because the right-angled triangles OKA, O K B have their hypotenuses O A, O B each a radius of the sphere, and the side O K common to both (I.13.). Therefore, in this case the section is a circle having the centre K.

Therefore, &c.

Cor. 1. The radius of a great circle is the same with the radius of the sphere; and the radius-square of a small circle is less than the radius-square of the sphere by the square of the perpendicular, which is drawn to its plane from the centre of the sphere (I. 36. *Cor.* 1.).

Cor. 2. Every diameter of a great circle is likewise a diameter of the sphere.

Prop. 2.

Either pole of a circle of the sphere is equally distant from all points in the circumference of that circle; whether the direct or the spherical distance be understood.

Let A B C (see the figure of prop. 1.) be any circle of a sphere which has the centre O, and let O K be drawn perpendicular to the plane A B C, and produced to meet the surface of the sphere in P; then, if A, B be any two points in the circumference of the circle A B C, and if the straight lines P A, P B, as also the spherical arcs P A, P B be drawn, the line P A shall be equal to the line P B, and the arc P A to the arc PB.

Join K A, K B. Then, because K is the centre of the circle A B C (1.), the right-angled triangles P K A and P K B have the two sides P K, K A of the one equal to the two sides P K, K B of the other, each to each; therefore, (I. 4.) the hypotenuse PA is equal to the hypotenuse P B. And because, in equal circles, the arcs which are subtended by equal chords are equal to one another (III. 12. *Cor.* 1.), the arc P A is likewise equal to the arc PB. And in like manner it may be shown that the other pole P' is also equidistant from A and B.

In this demonstration it is supposed that the point K does not coincide with the point O, or that the circle in question is not a great circle. If, however, ABC is a great circle, the angles P O A, P O B are right angles, and therefore equal to one another (I. 1.), from which the equality of the chords P A, P B and of the arcs P A, P B will follow as before.

Therefore, &c.

Cor. 1. Hence any circle of a sphere may be conceived to be described from either of its poles as a centre with the spherical distance of that pole as a radius. For, if this distance be carried round the pole, its extremity will lie in the circumference of the circle.

Cor. 2. The distances of any circle from its two poles are together equal to a semicircumference.

Cor. 3. A great circle is equally distant from its two poles; but this is not the case with a small circle. For if A B C be supposed to be a great circle, the angles P O A, P′ O A will be right angles, and therefore equal to one another, so that the polar distances P A, P′ A will be likewise equal (III. 12.); but if A B C be a small circle, the angles P O A, P′ O A will be, one of them less, and the other greater than a right angle, and therefore the distances P A, P′ A will be unequal.

Prop. 3.

Equal circles of the sphere have equal polar distances; and conversely.

Let A B C and A′ B′ C′ (see the figure of prop. 1.) be any two equal circles of the sphere; K, K′ their centres, and P, P′ their poles; then, if the radius K A is equal to the radius K′ A′, the polar distance P A shall be equal to the polar distance P′ A′; and conversely.

For, if O be the centre of the sphere, and OK, K P be joined, OK will be perpendicular to the plane A B C (1.), and therefore (def. 3.) O K, K P will lie in the same straight line; and in like manner O K′ will be perpendicular to the plane A′ B′ C′, and O K′, K′ P′ will lie in the same straight line. Join O A, P A and O A′, P′ A′. Then, because the right-angled triangles O K A, O K′ A′ have the hypotenuse O A equal to the hypotenuse O A′, and the side K A equal to the side K′ A′, the angle K O A or P O A is equal to the angle K′ O A′ or P′ O A′ (I. 13.); and therefore, also, the arc PA (III. 12.) is equal to the arc P′A′. And, conversely, if the arc P A be equal to the arc P′ A′, the angle POA will be equal to the angle P′ O A′ (III. 12.); and, therefore, because in the right-angled triangles O K A, O K′ A′, the hypotenuse O A is equal to the hypotenuse O A′, and the angle K O A to the angle K′ O A′, the radius K A is equal to the radius K′ A′ (I. 13.).

In the foregoing demonstration it is supposed that the points K and K′ do not coincide with the point O, that is, that the circles in question are not great circles of the sphere. If, however, the circles are great circles, the angles P O A, P′ O A′ are right angles, and therefore the arcs P A, P′ A′ quadrants: and it is evident that, conversely, circles whose polar distances are quadrants pass through the centre of the sphere, that is, are great circles of the sphere, and are equal to one another.

Therefore, &c.

Cor. Circles whose polar distances are together equal to a semicircumference are equal to one another (2. *Cor.* 2.)

Prop. 4.

Any two great circles of the sphere bisect one another.

For, since the plane of each passes through the centre of the sphere, which is also the centre of each of the great circles, their common section is a diameter of each; and circles are bisected by their diameters.

Therefore, &c.

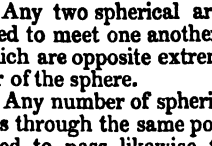
Cor. 1. Any two spherical arcs may be produced to meet one another in two points, which are opposite extremities of a diameter of the sphere.

Cor. 2. Any number of spherical arcs which pass through the same point may be produced to pass likewise through the opposite point.

Prop. 5.

The spherical arc which is drawn from the pole of a great circle to any point in its circumference is a quadrant of a great circle, and is at right angles to the circumference.

Let the point P be the pole of a great circle A B C: let any point A be taken in the circumference A B C, and let P A be joined by the spherical arc P D A: the arc P D A is a quadrant, and at right angles to the circumference A B C.



Take O the centre of the sphere, and join O P, O A. Then, because (def. 3.) O P is at right angles to the plane A B C, the angle P O A is a right angle (IV. def. 1.); and, therefore, the arc P D A is a quadrant. Again, because OP is at right angles to the plane ABC, the plane O P D A is at right angles to the plane A B C (IV. 18.); and, therefore, the arc P D A is at right angles to the circumference A B C (def. 7. and def. 8.).

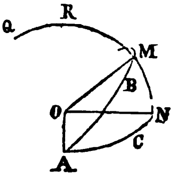
Therefore, &c.

Cor. 1. If two great circles cut one another at right angles, the circumference of each shall pass through the poles of the other.

Cor. 2. If the spherical distances of a point P in the surface of the sphere from two other points A and C in the same surface which are not opposite extremities of a diameter be each of them equal to a quadrant, P shall be the pole of the great circle which passes through the points A and C. For, if O be the centre of the sphere, the angles P O A and P O C will be right angles, because the arcs P A and P C are quadrants; and, therefore, P O is at right angles to the plane O A C (IV. 3.); for which reason P O must be the axis, and P the pole of the great circle which passes through A and C (def. 3.).

Prop. 6.

Every spherical angle is measured by the spherical arc which is decribed from the angular point as a pole, and intercepted between the sides of the angle.



Let B A C be any spherical angle, and from the point A, as a pole, let a great circle be described cutting the sides A B, A C in the points M, N respectively: the spherical angle B A C shall be measured by the arc M N.

Take O the centre of the sphere, and join O A, O M, O N. Then, because A is the pole of the spherical arc M N, the plane M O N is perpendicular to O A (def. 3.), and M O, N O are each of them perpendicular to O A. Therefore the angle M O N measures the dihedral angle M O A N (IV. 17.), or which is the same thing, (def. 7.) the spherical angle MAN or BAC. Therefore, the arc M N which measures the angle M O N, measures also the spherical angle B A C.

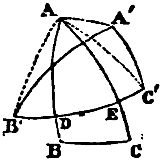
Therefore, &c.

Cor. The angle contained by two spherical arcs is measured by the distance of their poles, which lie towards the same parts of the arcs. For, if the arc N M be produced to R, so that R N may be a quadrant, and to Q, so that Q M may be a quadrant, Q R will be equal to M N (I. ax. 3.). And the points Q, R are the poles of A M, A N respectively, because Q M, Q A, as also R N, R A, are quadrants (5. *Cor.* 2.).

Prop. 7.

If one triangle be the polar triangle of another, the latter shall likewise be the polar triangle of the first; and the sides of either triangle shall be the supplements of the arcs which measure the opposite angles of the other.*

Let A B C be any spherical triangle, and let A′, B′, C′ be those poles of the sides B C, A C, A B, which lie towards the same parts of the arcs B C, A C, A B, with the opposite angles A, B, C, respectively, so that A′ and A lie towards the same parts of B C, B′ and B towards the same parts of A C, and C′ and C towards the same parts of A B: that is, (def. 11.) let A′ B′ C′ be the polar triangle of A B C: the triangle A B C shall, likewise, be the polar triangle of A′ B′ C′, and the sides of either triangle shall be the supplements of the arcs which measure the opposite angles of the other.



For, in the first place, B′ being the pole of A C, A B′ is a quadrant (5.); and C′ being the pole of A B, A C′ is likewise a quadrant: therefore (5. *Cor.* 2.) A is the pole of B′ C′. Also, it is upon the same side of B′ C′ that A′ is: for, because A′ and A are upon the same side of B C, and that A′ is the pole of B C, A′ A is less than a quadrant; and because A is the pole of B′ C′, and that A A′ is less than a quadrant, A and A′ are upon the same side of B′ C′.

And, in the same manner, it may be shown that B is the pole of A′ C′, and B, B′ upon the same side of A′ C′; and that C is the pole of A′ B′, and C, C′ upon the same side of A′ B′. Therefore, the triangle A B C is the polar triangle of A′ B′ C′ (def. 11.).

Next, let the arc B′ C′ be produced both ways, if necessary, to meet the arcs AB, AC (produced likewise if necessary) in the points D, E, (4 *Cor.* 1.). Then, because A is the pole of the arc B′ C′, the spherical angle BAC is measured by D E (6.). Again, because B′ is the pole of A C, B′ E is a quadrant; and for the like reason $C'D$ is also a quadrant: therefore, the sum of $B'E$ and $C'D$, that is, of DE and $B'C'$, is equal to a semicircumference, and the side $B'C'$ is the supplement of DE which measures the spherical angle BAC. And, in the same manner, it may be shown that any other side of either of the triangles ABC, $A'B'C'$ is the supplement of the arc which measures the opposite angle of the other.

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* From this property polar triangles are sometimes called *supplementary triangles*.

Therefore, &c.

Scholium.

If the three sides of a spherical triangle be each of them equal to a quadrant, the polar triangle will coincide with it; for each of the angular points will be the pole of the side opposite to it. (5. *Cor.* 2.) The surface of the sphere may be divided into eight such triangles, by dividing the circumference of any great circle into quadrants, and joining the points of division with the poles of the great circle.

SECTION 2.—*Of Spherical Triangles.*

PROP. 8.

The angles which one spherical arc makes with another upon one side of it are either two right angles, or are together equal to two right angles.

See the Demonstration of Book I. Prop. 2.

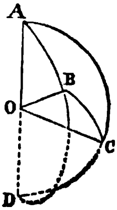
Cor. 1. If two spherical arcs cut one another, the vertical or opposite angles will be equal to one another. See the Demonstration of Book I. Prop. 3.

Cor. 2. If any number of spherical arcs meet in the same point, the sum of all the angles about that point will be equal to four right angles.

PROP. 9.

Any two sides of a spherical triangle are together greater than the third side; and any side of a spherical triangle is greater than the difference of the other two.

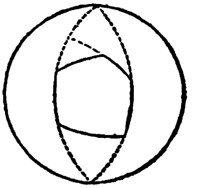
Let ABC be a spherical triangle; the sides BA and AC shall be together greater than BC; and AB alone shall be greater than the difference of AC and BC. Take O the centre of the sphere, and join OA, OB, OC. Then, because the solid angle at O is contained by three plane angles AOB, AOC, and BOC, the two AOB and AOC are together greater than the third BOC (IV. 19.). But these angles are respectively measured by the arcs AB, AC, and BC. Therefore AB and AC are together greater than BC. And hence, taking AC from each, AB alone is greater than the difference of AC and BC.



Therefore, &c.

Cor. 1. The three sides of a spherical triangle are together less than the circumference of a great circle. For, if AB and AC be produced to meet in D, the arcs ABD, ACD will be semicircumferences; but BC is less than BD and DC together; therefore, AB, AC, and BC are together less than ABD and ACD, that is, less than the circumference of a great circle.

Cor. 2. In the same manner it may be shown that all the sides of any spherical polygon are together less than the circumference of a great circle. This is likewise evident from IV. 20.



Scholium.

By help of this proposition, it may be shown that the shortest distance of two points on the surface of a sphere, measured over that surface, is the spherical arc between them. See Book I. prop. 10. *Scholium.*

PROP. 10.

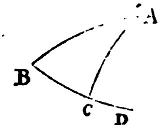
The three angles of a spherical triangle are together greater than two right angles, and less than six right angles.

For the arcs which measure the three angles together with the three sides of the polar triangle are equal to three semicircumferences (7.), or six quadrants: therefore, the former alone are less than six quadrants, and consequently the angles which they measure are less than six right angles. Again, the sides of the polar triangle are less than a whole circumference, or four quadrants (9 *Cor.* 1.): therefore, the arcs before mentioned are greater than two quadrants, and consequently the angles which they measure greater than two right angles.

Therefore, &c.

Cor. 1. A spherical triangle may have two or even three right angles, or two or even three obtuse angles. For, it is evident from the demonstration of the proposition, that the sum of the angles depends upon the magnitude of the sides of the polar triangle, and since the sum of these last may be any whatever less than four quadrants, the sum of the angles of the original triangle may be any whatever greater than two, and less than six right angles.

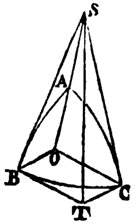
Cor. 2. If one side of a spherical triangle be produced, the exterior angle will be less than the sum of the two interior and opposite angles. For the exterior angle, together with its adjacent interior angle, is only equal to two right angles (8.); but the two interior and opposite angles, together with the same angle, are greater than two right angles.



Prop. 11.

If two sides of a spherical triangle be equal to one another, the opposite angles shall be likewise equal; and conversely.

Let A B C be a spherical triangle, having the side A B equal to the side A C; the angle A C B shall likewise be equal to the angle A B C.



Take O the centre of the sphere, and join O A, O B, O C. From the point C, in the plane A O C, draw C S at right angles to C O (and, therefore (III. 2.), touching the arc C A in C) to meet O A produced in S: at the points B and C draw B T and C T, touching the arc B C, and meeting one another in T, and join B S, S T. Then, because the arc AB is equal to AC, the angle AOB is equal to the angle A O C (III. 12.); and, because the triangles S O B, S O C have two sides of the one equal to two sides of the other, each to each, and the angles S O B, S O C which are included by those sides equal to one another (I. 4.), the base S B is equal to the base S C, and the angle S B O to the angle S C O, that is, to a right angle. Therefore, B S touches the arc A B in B (III. 2.). And, because the spherical angles A B C, A C B are measured by the plane angles of the tangents at B and C (see def. 7. note) they are measured by the angles S B T, S C T respectively. But, because T B and T C are tangents drawn from the same point B to the arc B C, T B is equal to T C (III. 2. *Cor.* 3.). Therefore, the triangles S B T and S C T have the three sides of the one equal to the three sides of the other, each to each, and consequently the angle S B T is equal to the angle S C T (I. 7.). Therefore, also, the spherical angle A B C is equal to the spherical angle A C B.

Next, let the angle A B C be equal to the angle A C B: the side A B shall be equal to the side A C. For, if the polar triangle A′ B′ C′ be described, its sides A′ B′ and A′ C′ which are supplements to the measures of the equal angles (7.) will be equal; and, therefore, by the former part of the proposition, the spherical angle at C′ is equal to the spherical angle at B′. But the sides A B and A C are supplements to the measures of these angles (7.). Therefore, also, A B is equal to A C.

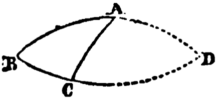
Therefore, &c.

Prop. 12.

If one angle of a spherical triangle be greater than another, the opposite side shall likewise be greater than the side opposite to that other; and conversely.

See the demonstration of Book I. Prop. 11.

Cor. If one side B C of a spherical triangle A B C be produced to D, the exterior angle A C D shall be equal to, or less than, or greater than, the interior and opposite angle A B C, according as the sum of the two sides A B, A C is equal to, or greater than, or less than, the semicircumference of a great circle.

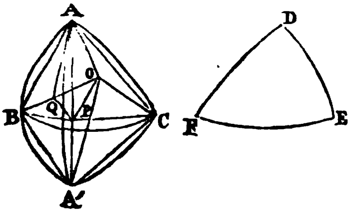


For, if B A and B C be produced to meet one another in D, the angles at B and D will be equal to one another, having for their common measure the measure of the same dihedral angle (def. 7.); and B A D will be a semicircumference. But, by the proposition, the angle A C D is equal to, or less than, or greater than the angle at D, according as A C is equal to, or greater than, or less than A D. Therefore, the angle A C D is equal to, or greater than, or less than the angle at B, according as A B and A C are together equal to, or greater than, or less than a semicircumference.

Prop. 13.

If two spherical triangles have two sides of the one equal to two sides of the other, each to each, and likewise the included angles equal; their other angles shall be equal, each to each, viz. those to which the equal sides are opposite, and the base, or third side, of the one shall be equal to the base, or third side, of the other.

There are here two cases for consideration; first, that in which the equal sides A B, A C and D E, D F lie in the same direction; and, secondly, that in which they lie in opposite directions. The first case may be demonstrated by superposition, after the same manner as Book I. Prop. 4, to which, for brevity's sake, the reader is referred: the second case as follows:—



Let A B C, D E F be two spherical triangles, which have the two sides AB, A C equal to the two sides D E, D F, each to each, viz. A B to D E and A C to DF, but DE lying in a direction from DF, which is the reverse of that in which AB lies from AC, and let them likewise have the angle B A C equal to the angle E D F: their other angles shall be equal, each to each, viz, A B C to D E F; and A C B to D F E, and the base B C shall be equal to the base E F.

Take O the centre of the sphere: from A draw A P perpendicular to the plane O B C, and produce it to meet the surface of the sphere in A′. Join P O, P B, P C, and O B; from P draw P Q perpendicular to OB; join A Q, A B, A C, A′ Q,* A′ B, A′ C; and draw the spherical arcs A′ B, A′ C, A′ A. Then, because in the right-angled triangles APO, A′P O, the hypotenuse A O is equal to the hypotenuse A′ O, and the side P O common to both, the remaining sides A P and A′ P are equal to one another (I. 13.): and because in the right-angled triangles A P Q, A′ P Q, the side A P is equal to the side A′ P, and the side P Q common to both, the hypotenuse A Q is equal to the hypotenuse A′ Q, and the angle A Q P to the angle A′ Q P (I. 4.); and in the same manner it may be shown that A B is equal to A′ B, and A C to A′ C. Now, because A P is perpendicular to the plane O B C, and that P Q is perpendicular to the line O B in that plane, A Q is likewise perpendicular to O B (IV. 4.); and for the like reason A′Q is perpendicular to the same O B. Therefore, the angles AQ P, A′ Q P measure the dihedral angles formed by the planes O A B, O B C, and O A′ B, O B C (IV. 17.), or, which is the same thing, (def. 7.) the spherical angles A B C and A′ B C; and because, as has been already demonstrated, the angle A Q P is equal to the angle A′ Q P, the spherical angle A B C is equal to the spherical angle A′ B C. In the same manner, it may be shown that the spherical angles A C B and A′ C B are equal to one another. And because the straight line A B is equal to the straight line A′ B, the arc A B is equal to the arc A′ B (III. 12. *Cor.* 1.); and, for the like reason, the arc A C is equal to the arc A′ C. Therefore, in the isosceles spherical triangle B A A′, the angle B A′ A is equal to the angle B A A′, and in the isosceles spherical triangle C A A′ the angle C A′ A is equal to the angle C A A′ (11.); and, consequently, the whole (or, if the points B, C, are on the same side of the arc A A′, the remaining) angle B A′ C is equal to the whole or remaining angle B A C. Therefore, the triangles A′ B C, A B C have their several sides and angles equal to one another, but lying in a reverse order.

Now, because A′ B and D E are each of them equal to A B, they are equal to one another; and, for the like reason, A′ C is equal to D F, and the angle B A′ C to the angle E D F. Also, the equal parts lie in the same direction from one another in these two triangles, A′ B C, D E F. Therefore, by the first case, the base B C is equal to the base E F, and the angles A′ B C, A′ C B to the angles D E F and D F E respectively. And, because the angles A B C, A C B are equal to the angles A′ B C, A′ C B, each to each, the former angles are likewise equal to D E F and D F E respectively.

Therefore, &c.

Prop. 14.

If two spherical triangles have two angles of the one equal to two angles of the other, each to each, and likewise the interjacent sides equal; their other sides

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* A′ Q is omitted in the figure.

shall be equal, each to each, viz. those to which the equal angles are opposite, and the third angle of the one shall be equal to the third angle of the other.

The case in which the equal angles lie in the same direction from one another in the two triangles may be demonstrated by superposition, in the same manner as Book I. Prop. 5, to which the reader is referred. The other case, in which they lie in different directions, may be demonstrated as follows:—

Let A B C, D E F (see the figure of Prop. 13.) be two spherical triangles, which have the two angles A B C, A C B of the one equal to the two angles D E F, D F E of the other, each to each, but in a reverse order, and likewise the side B C equal to the side E F: their other sides shall be equal each to each, and the third angle B A C shall be equal to the third angle E D F.

Let there be described, as in the last proposition, upon the other side of the base BC the triangle A′ B C, which has its sides and angles equal to those of the triangle A B C, each to each, but in a reverse order. Then, because the angles A′ B C and D E F are equal, each of them, to the angle A B C, they are equal to one another; and, for the like reason the angle A′ C B is equal to the angle DFE. Also, BC is equal to E F; and these equal parts lie in the same direction from one another in the two triangles A′ B C, D E F. Therefore, by the first case, the angle B A′ C is equal to the angle E D F, and the sides A′ B, A′ C are equal to the sides D E, D F respectively. And, because the angle B A C is equal to the angle B A′ C, and the sides A B, A C to the sides A′ B, A′ C, each to each, the angle B A C is equal to the angle E D F, and the sides AB, AC to the sides DE, DF respectively.

Therefore, &c.

PROP. 15.

If two spherical triangles have two sides of the one equal to two sides of the other, each to each, and have likewise their bases equal, the angle contained by the two sides of the one shall be equal to the angle contained by the two sides equal to them of the other.

The case in which the equal sides lie in the same direction from one another, may be demonstrated in the same manner as Book I. Prop. 7: the other case, in which they lie in different directions, as follows:—

Let A B C, D E F (see the figure of Prop. 13) be two spherical triangles having the two sides of the one equal to the two sides of the other, each to each, and likewise the base B C equal to the base EF. The angle B A C shall be equal to the angle E D F.

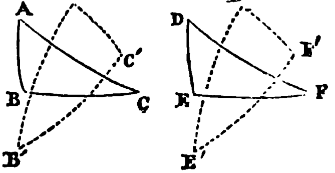
Let there be described, as in Prop. 13, upon the other side of the base B C the triangle A′ B C, which has its sides and angles equal to those of the triangle A B C, each to each, but in a reverse order. Then, because A′ B and D E are each of them equal to A B, they are equal to one another: and, for the like reason, A′ C is equal to D F: also B C is equal to E F; and these equal parts lie in the same direction from one another in the two triangles A′ B C, D E F. Therefore, by the first case, the angle B A′ C is equal to the angle E D F. Therefore, because the angle B A C is equal to the angle B A′ C, the angle B A C is likewise equal to the angle E D F.

Therefore, &c.

PROP. 16.

*If two spherical triangles have the three angles of the one equal to the three angles of the other, each to each, they shall likewise have the three sides of the one equal to the three sides of the other, each to each, viz. those which are opposite to the equal angles.**

Let the spherical triangles A B C, D E F have the three angles A, B, C of the one equal to the three angles D, E, F of the other, each to each. The sides AB, A C and B C shall likewise be equal to the sides D E, D F and E F, each to each.



For, if the polar triangles A′ B′ C′, D′ E′ F′ be described, they will have the three sides of the one equal to the three sides of the other, each to each, because every two corresponding sides, as A′ B′ and D′ E′ are (7.) supplements of the measures of equal angles C and F (I. ax. 3.). Therefore, by the last proposition, these polar triangles have likewise the three angles A', B', C' of the one equal to the three angles D', E', F' of the other, each to each. And hence the triangles A B C, D E F have the three sides of the one equal to the three sides of the other, each to each, because every two corresponding sides, as A B and D E are supplements of the measures of equal angles C' and F' (7.)

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* See the Scholium at the end of this Book.

Therefore, &c.

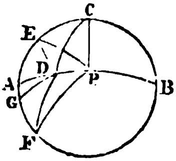
Prop. 17.

If two spherical triangles have two sides of the one equal to two sides of the other, each to each, but the angle which is contained by the two sides of the one greater than the angle which is contained by the two sides which are equal to them of the other; the base of that which has the greater angle shall be greater than the base of the other: and conversely.

See the demonstration of Book I. Prop. 11.

Prop. 18.

If any point be taken within a circle of the sphere which is not its pole; of all the arcs which can be drawn from that point to the circumference, the greatest is that in which the pole is, and the other part of that arc produced is the least; and of any others, that which is nearer to the greatest is always greater than one more remote: and from the same point to the circumference there can be drawn only two arcs that are equal to one another, which two make equal angles, upon either side, with the shortest arc.



Let A B C be any circle (either a great or a small circle) of the sphere, P its pole (if a great circle, either of its two poles; if a small circle, the nearer pole), and D a point on the surface of the sphere, which is upon the same side of the circle A B C D with the pole P. Of all the arcs D A, D P B, D C, &c. which can be drawn from D to the circumference, the arc D P B which passes through P shall be the greatest, and DA the other part of that arc produced the least: and of the others, DC, which is nearer to D P B, shall be greater than DE, which is more remote.

Join P E, P C. Then, because the two sides D P, P C of the triangle D P C are together greater than the third D C (9.), and that PB is equal to PC, the arc D P B is greater than D C. Again, because the triangles PCD, PED have the two sides P C, P D of the one equal to the two sides P E, P D of the other, each to each, but the angle CPD greater than the angle E P D, the base D C is greater than the base DE (17.). Again, because the side E D of the triangle P D E is greater than the difference of P D, P E the other two sides (9.), and that P A is equal to P E, the arc E D is greater than the difference of PD and PA, that is, than AD. Therefore, DPB is the greatest and D A the least of all arcs which can be drawn from D to the circumference, and D C, which is nearer to D P B, is greater than D E, which is more remote.

Also, there can be drawn only two equal arcs D E, D F from the point D to the circumference, and these make equal angles with the shortest arc D A upon either side of it. For, if P F be drawn, making the angle A P F equal to the angle A P E, and D F be joined; then, because the triangles P D F, P D E have two sides of the one equal to two sides of the other, each to each, and the included angles D P F, D P E equal to one another, the bases D F and D E are equal to one another, and the angles P D F, P D E which are adjacent to A D F, A D E, likewise equal (13.). But, besides D F, there cannot be drawn any other arc from D to the circumference equal to D E; for if it were possible, and D G were such an arc, then D G, which is nearer to (or farther from) D P B than D F is, would be equal to D F, which is contrary to what was shown in the former part of the proposition.

Therefore, &c.

Cor. 1. From a point to a spherical arc, the perpendicular is either the least or the greatest distance, the least when it is less than a quadrant, the greatest when it is greater than a quadrant. For the perpendicular arc either passes, or may be produced to pass through the pole of the spherical arc (5. *Cor.* 1.): in the former case it is, as D P B, greater than any other arc which can be drawn from the point E to the circumference of which the spherical arc is a portion, and it is, at the same time, greater than (5.) the quadrant PB; in the latter case it is, as D A, less than any other, and it is, at the same time, less than (5.) the quadrant P A.

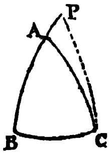
Cor. 2. If there be taken within a circle (that is, anywhere on the surface of the sphere, if it be a great circle, or on the same side with the nearer pole, if a small circle) a point from which there fall more than two equal arcs to the circumference, that point shall be a pole of the circle.

Cor. 3. Hence, also, if there be taken *without* a small circle a point from which there fall more than two equal arcs to the circumference, that point shall be the more distant pole of the small circle. For if the opposite point through which the equal arcs pass (4 *Cor.* 2.) be taken, three distances of the latter from the circumference (I. ax. 3.) will be equal, and therefore, by *Cor.* 2., it will be the nearer pole.

Prop. 19.

In a spherical triangle, which has one of its angles a right angle, either of the other angles is greater or less than a right angle, according as the opposite side is greater or less than a quadrant; and if the sides, which contain the right angle, are both greater or both less than quadrants, the third side, or hypotenuse, shall be less than a quadrant; but if one of these sides be greater than a quadrant, and the other less, the hypotenuse shall be greater than a quadrant.

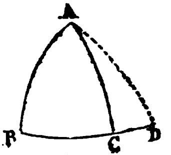
Let A B C be a spherical triangle, having the right angle A B C, and let A C B be one of the other two angles: the angle A C B shall be greater or less than a right angle, according as the opposite side A B is greater or less than a quadrant.

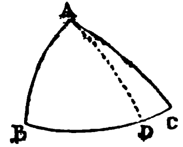


For, if from BA, or from BA produced, B P be cut off equal to a quadrant, and PC joined, the angle P C B will be a right angle, because P is the pole of B C (5.); and it is evident that the angle A C B is greater or less than the angle P C B, according as A B is greater or less than P B.

Also, if the sides A B, B C are both greater or both less than quadrants, the hypotenuse A C shall be less than a quadrant; but if one of them be greater than a quadrant, and the other less, A C shall be greater than a quadrant.

For, if from B C, or from B C produced, B D be cut off equal to a quadrant, and A D be joined, D will be the pole of A B (5.), and, therefore, A D will be equal to a quadrant. And, first, if AB and B C be both less than quadrants, then, A B being (18. *Cor.* 1.) the least arc which can be drawn from A to B C, A C which is nearer to it is less than A D, which is more remote (18.); or, secondly, if AB and BC be both greater than quadrants, then A B being (18. *Cor.* 1.) the greatest arc which can be drawn from A to BC, AC which is further from it is less than AD, which is not so remote (18.); therefore, in both cases, A C is less than a quadrant. But, if either of the sides, as AB, be greater than a quadrant, and the other, B C, less, A B being the greatest arc which can be drawn from A to B C, A C which is nearer to it is greater than A D, which is more remote (18.); therefore, A C is greater than a quadrant.





Therefore, &c.

Cor. 1. It is evident from the demonstration, that if one of the two sides which contain the right angle be *equal* to a quadrant, the opposite angle will be a right angle, and the hypotenuse likewise a quadrant.

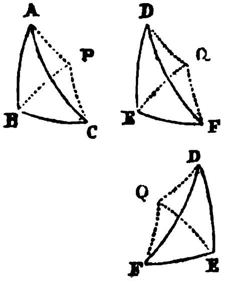
Cor. 2. Hence, in a right-angled spherical triangle, either of the two sides is of the same affection with the opposite angle; and the hypotenuse is, 1°, equal to, or, 2°, less than, or, 3°, greater than, a quadrant, according as, 1°, one of the adjoining angles is a right angle, or these two angles are, 2°, of the same or, 3°, of different affections. See note, page 61.

Section 3. — *Of Equal Portions of Spherical Surface, and the Measure of Solid Angles.*

Prop. 20.

Spherical triangles which have the three sides of the one equal to the three sides of the other, each to each, contain also equal portions of spherical surface, whether the triangles are symmetrical or otherwise.

Let A B C, D E F be two spherical triangles, which have the three sides of the one equal to the three sides of the other, each to each, viz. A B to D E, A C to D F, and B C to E F: the surface of the triangle A B C shall be equal to the surface of the triangle D E F.



Let a plane be drawn through the points A, B, C, to cut the sphere in a circle passing through those points (1.): from the centre of the sphere, draw a perpendicular to the plane A B C, and produce it to meet the surface of the sphere in P: then P is the pole of the circle A B C (def. 3.); and, consequently, (2.) if the spherical arcs P A, P B, P C are drawn, joining the point P with the points A, B, C, respectively, they will be equal to one another. Make the spherical angle D E Q equal to the spherical angle A B P: make also the arc E Q equal to the arc B P, and join Q D, Q F. Then, because the angles D E F (15.) and D E Q are equal to the angles A B C and A B P, each to each, the angle Q E F is equal to the angle P B C (I. ax. 3.). And, because in the triangles Q E F, P B C, two sides of the one are equal to two sides of the other, each to each, and the included angles Q E F, P B C likewise equal to one another (13.) the arc Q F is equal to the arc P C, and the angle E Q F to the angle B P C. In the same manner, it may be shown that the arc Q D is equal to the arc P A, and the angle D Q E to the angle A P B. Therefore, because the angles D Q E and E Q F are respectively equal to the angles A P B and B P C, the angle D Q F is equal to the angle A P C: and because the arcs P A, P B, P C are equal to one another, the arcs Q D, Q E, Q F, which are severally equal to them, are likewise equal to one another (I. ax. 1.).

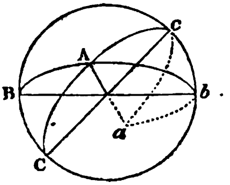
Now, it is evident that isosceles triangles, as P B C and Q E F, which have equal vertical angles, and equal sides containing them, may be made to coincide, and, therefore, are equal to one another. Therefore, the isosceles triangles P A B, P B C, and P A C are respectively equal to the isosceles triangles Q D E, Q E F, and Q D F. But the triangle A B C is equal to the sum of the former three, if the point P falls within the triangle A B C; or to the difference between the sum of two of them and the third, if it fall without the triangle; and the triangle D E F is equal to the sum of the latter three, in the first case, or to the difference between the sum of the corresponding two of them and the third, in the other*. Therefore, the triangle ABC is equal to the triangle D E F (I. ax. 2. 3.).

Therefore, &c.

Prop. 21.

Every spherical triangle is equal to half the difference between the hemispherical surface, and the sum of three lunes which have their angles equal to the three angles of the triangle respectively.

Let A B C be any spherical triangle; its surface shall be equal to half the difference between the hemispherical surface, and the sum of three lunes having their angles respectively equal to the angles A, B, C.



Let the circles be completed, of which the sides A B, A C, and B C are parts, and let A *a*, B *b*, and C *c* be the diameters passing through the points A, B, and C. Then, because C A *c* and A *c a* are semicircumferences (4.), they are equal to one another, and A *c* being taken from each, the remainders A C, *a c* are likewise equal. In the same manner it may be shown that A B is equal to *a b*, and B C to *b c*. And, because the triangles A B C, *a b c* have the three sides of the one equal to the three sides of the other, each to each, their surfaces are equal to one another (20.) Now, the lune A *c a b* A has its angle equal to the angle A of the triangle A B C (8. *Cor.* 1.), and the lunes B A *b* C B, C B *c* A C have for their angles the angles B, C of the same triangle. And these three lunes are together equal to the surface of the hemisphere ~~A B C *b c* to~~gether with the triangles A B C, *a b c*; that is, to the surface of the hemisphere, together with twice the triangle A B C. Therefore, twice the triangle A B C is equal to the difference between the sum of the three lunes and the surface of the hemisphere; and the triangle A B C is equal to half that difference.

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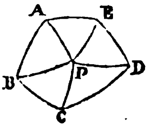
* If the arcs B P, E Q, or these arcs produced, cut the arcs A C, D F in the points R, S, respectively, it may be shewn by prop. 14, that B R is equal to E S: therefore, since B P is equal to E Q, if the point P falls within the triangle A B C, the point Q must likewise fall within the triangle D E F; or if P fall without the triangle A B C, and within the angle A B C, Q must likewise fall without the triangle D E F, and within the corresponding angle D E F.

Therefore, &c.

Cor. 1. Hence, if the angles of a spherical triangle are given, we may find the proportion which its surface bears to the surface of the sphere. For, since the surface of any lune is to the surface of the sphere, as the angle of the lune to four right angles (V. 21.), or as twice its angle to eight right angles; if the surface of the sphere be represented by 8 right angles, or 8 R,* the surfaces of the lunes which have the angles A, B, and C will be represented by 2 A, 2 B, and 2 C; and the hemispherical surface will be represented by 4 R: therefore, the surface of the triangle A B C will be represented by half $2A+2B+2C-4R$, that is, by $A+B+C-2R$. And hence, it is commonly said, that *the surface of a spherical triangle is measured by the excess of the sum of its angles above two right angles;* it being understood that the surface of the sphere is measured by eight right angles.

Cor. 2. The surface of a spherical triangle whose angles are A, B, C is equal to the surface of a lune whose angle is $\frac{A+B+C}{2}-R$. For both surfaces are measured by $A+B+C-2R$.

Cor. 3. The surface of a spherical polygon A B C D E is measured by the excess of the sum of all its angles, together with four right angles, above twice as many right angles as the polygon has sides.



For the polygon may be divided into as many triangles as it has sides, by joining its angles with any point P taken within the polygon; and the surface of each of these triangles is measured by the sum of its angles *minus* two right angles: therefore, the sum of all the triangles, that is, the polygon, is measured by the sum of all their angles *minus* twice as many right angles as there are triangles. But the sum of all the angles of the triangles is equal to the sum of all the angles of the polygon, together with the angles of the point P, that is, (8. *Cor.* 2.) together with four right angles: and there are as many triangles as the polygon has sides. Therefore, the surface of the polygon is measured by the sum of its angles *plus* four right angles *minus* twice as many right angles as the polygon has sides.

Let Σ be the sum of its angles, and n the number of its sides: then the surface of the polygon is measured by $\Sigma+4R-2nR$.

Scholium.

It follows from the last corollary, that *if* S *be the number of the solid angles of a polyhedron,* F *the number of its faces, and* E *the number of its edges,* $S+F-E$ *will be equal to* 2. For, if a sphere be described about any point within the polyhedron as a centre, the lines which are drawn from this point through the solid angles of the polyhedron will cut the surface of the sphere in S points, and the planes which are drawn from the same point through the edges will cut the surface in E spherical arcs; and thus the whole surface of the sphere will be divided into F spherical polygons, corresponding to the F faces of the polyhedron. Now, the surface of each of these spherical polygons is measured by the sum of its angles *plus* four right angles *minus* twice as many right angles as the polygon has sides. Therefore, the surfaces of all the polygons taken together are measured by the sum of all their angles taken together *plus* as many times four right angles as there are polygons *minus* twice as many right angles as all the polygons together have sides. But the sum of all their angles taken together is the sum of all the angles at the S angular points, and, therefore, is equal to S times four right angles (8. *Cor.* 2.): the number of the polygons is F; and the number of the sides of the polygons is 2 E, because the number of different sides is E, and each of these is at once a side of two adjoining polygons. Therefore, the surfaces of all the polygons, taken together, are measured by $(S+F-E)$ times four right angles. Therefore, since the surface of the sphere is made up of the surfaces of all the polygons, and that the surface of the sphere is measured by 8 or 2 × 4 right angles, S + F — E is equal to 2.

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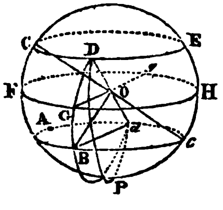
* In which case, R represents the surface of the spherical triangle, which has each of its angular points the pole of the side opposite to it (7. *Scholium.*). We shall see (28. *Scholium.*) that the surface of this triangle measures a *solid right angle* at the centre of the sphere.

[illegible]

And from the equation S + F — E = 2 it may be further shown that *the sum of all the plane angles which contain the solid angles of any polyhedron is* S — 2 *times* 4 *right angles*. For, the plane angles of each face are together equal to twice as many right angles as the face has sides *minus* four right angles. Therefore, the plane angles of all the faces are together equal to twice as many right angles as all the faces taken together have sides, *minus* as many times four right angles as there are faces. But the number of sides of the faces is 2 E, for the reason before mentioned; and F is the number of faces. Therefore, the sum in question is equal to E — F times 4 right angles; that is, because S — 2 = E — F, to S — 2 times 4 right angles.

Prop. 22.

Through any two given points and a third upon the surface of a sphere, which do not lie in the circumference of the same great circle, there may be made to pass two equal and parallel small circles; that is, one of them through the first two given points, and the other through the third given point; and every spherical arc which is terminated by these circles shall be bisected by the circumference of the great circle to which they are parallel.



Let A and B be any two points upon the surface of a sphere, and C any third point, which does not lie in the circumference of the great circle passing through A and B. Through the two points A, B, and the point C, there may be made to pass two equal and parallel small circles of the sphere.

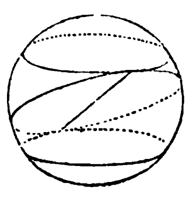
Let O be the centre of the sphere, and let *c* be the opposite extremity of the diameter C O *c*. Then it is evident (IV. 1.) that through the three points A, B, *c* there may be made to pass the small circle A B *c*, and through the point C the circle C D E parallel to A B *c* (IV. 11. *Cor.* 1.). Let P be the common pole of the two circles, and join P C, P *c*. Then, because C *c* is a diameter of the sphere, P C and P *c* are arcs of the same great circle, and are together equal to a semicircumference. And, because the polar distances P C and P *c* of the circles AB*c* and C D E are together equal to a semicircumference, the circles A B *c* and C D E, which have been made to pass through the two points A, B, and the point C, are equal to one another (3. *Cor.*).

Next, let D B be any spherical arc which is terminated by these circles; and let it be cut by the great circle F G H, to which they are parallel, in the point G: D G shall be equal to G B. Join D P; let the great circle D G B be completed, and let its plane cut the plane of the great circle F G H in the diameter G O *g*, and the plane of the great circle D P in the diameter D O *d*. Then, because P D and P *d* are together equal to a semicircumference, and that the point D is in the circle C D E, the point *d* is in the circle A B *c*, which is equal to C D E, and has the same pole with it (3. *Cor.*). Join O G, O B, B *d*: and, because G O, B *d* are sections of parallel planes F G H, A B *c* by the plane D G B *d*, G O is parallel to B *d* (IV. 12.); therefore the angle D O G is equal to the angle D *d* B, and the angle B O G to the angle O B *d* (I. 15.). But the angles O B *d* and O *d* B or D *d* B are equal to one another, because O *d* is equal to O B (I. 6.): therefore, also, the angle D O G is equal to the angle B O G (I. ax. 1.), and the arc D G to the arc G B (III. 12.).

Therefore, &c.

Cor. 1. It is shown in the latter part of the demonstration, that if there be two equal and parallel small circles, and if a great circle meets one of them in any point, it will meet the other in the opposite extremity of the diameter which passes through that point.

Cor. 2. Hence if a great circle cuts one of two equal and parallel small circles, it will cut the other likewise; also

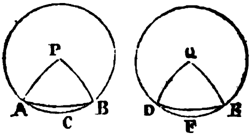


if it touches one of them, it will touch the other likewise.

Prop. 23.

Lunular portions of surface, which are contained by equal spherical arcs with the arcs of equal small circles of the same sphere, are equal to one another; so also are the pyramidal solids, which have these portions for their bases, and their common vertex in the centre of the sphere.

Let A B C, D E F be equal small circles, and let the spherical arcs AB, DE contain with the small arcs A C B, D F E the lunular surfaces ACB, D F E; then, if the arcs AB and DE are equal to one another, the surfaces A C B, D F E shall be likewise equal.



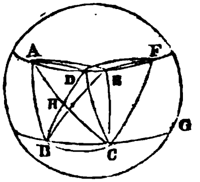
Let P be the nearer pole of the circle ABC, and Q the nearer pole of the circle D E F, and join PA, P B, Q D, Q E. Then, because the circles are equal to one another, the polar distances PA, QD are likewise equal (3.): and, because the triangles P A B, Q D E have the two sides P A, P B equal to the two sides Q D, Q E, each to each, and the base A B equal to the base D E, the angle A P B is equal to the angle D Q E (15.). Therefore, if the pole P be applied to the pole Q, and the arc P A to the arc Q D, the arc P B will coincide with the arc Q E, and the points A and B will coincide with the points D and E respectively, and the arc AB with the arc DE, and the small arc A C B with the small arc D F E; and therefore the lunular surface A C B coincides with the lunular surface D F E, and is equal to it. And because these surfaces may be made to coincide, it is evident that the pyramidal solids, which have these surfaces for their bases, may likewise be made to coincide, and are equal to one another.

Therefore, &c.

Prop. 24.

Spherical triangles, which stand upon the same base and between the same equal and parallel small circles, are equal to one another.

Let the spherical triangles A B C, E B C stand upon the same base B C, and between the same equal and parallel small circles A E F, B C G: the triangle A B C shall be equal to the triangle EBC. Let the circumference of the great circle to which both A E F and B C G are parallel cut the arc A C in H; join BH, and produce it to meet the circumference AEF in D (22. *Cor.* 2.), and join AD. Then, because every spherical arc which is terminated by the circles A E F, B C G is bisected by the circumference of the great circle to which they are parallel (22.), AH is equal to H C, and D H to H B; and because the triangles H A D, H C B have the two sides H A, H D equal to the two sides H C, H B, each to each, and the included angles equal to one another (8. *Cor.* 1.), the base A D is equal to the base C B (13.): and, for the like reason, AB is equal to C D: therefore, the triangles A B C, C D A have the three sides of the one equal to the three sides of the other, each to each, and are equal to one another (20.); so that the quadrilateral A B C D is double of the triangle A B C. In the same manner, the quadrilateral E B C F may be described, which is double of the triangle E B C, and has its sides E F, F C equal to the sides BC, EB, each to each. And, because the spherical arcs AD and E F are each of them equal to B C, they are equal to one another: therefore, also, the chord AD is equal to the chord E F (III. 12. *Cor.* 1.) and the small arc A D to the small arc E F; and, if the small arc E D is added to, or taken from them, the small arc A E is equal to the small arc D F; and hence, again, because the chords of these equal small arcs are equal, the spherical arc A E is equal to the spherical arc D F (III. 12. *Cor.* 1.). And, because the triangles A B E, D C F have the three sides of the one equal to the three sides of the other, each to each, they are equal to one another (20.): but the lunular portions which are contained by the small arcs A E, D F with the equal spherical arcs AE, DF are likewise equal (23.): therefore, the remaining portions of the triangles A B E, D C F are equal to one another. Let each of these equals be taken from the whole surface included by the small arc A E F and the spherical arcs A B, B C, C F, and there remains



the surface included by the small arc E F and the spherical arcs E B, B C, C F, equal to the surface included by the small arc AD and the spherical arcs AB, BC, CD. To each of these equals add the equal lunular portions (23.) which are contained by the small arcs E F, A D, with the equal spherical arcs E F, AD; and the whole quadrilateral EBCF is equal to the whole quadrilateral A B C D. Therefore, because the halves of equals are equal, the triangle E B C is equal to the triangle A B C.

Therefore, &c.

Cor. Hence, if equal triangles A B C, E B C stand upon the same base B C, and the same side of it, the points A, E and B, C lie in the circumferences of two equal and parallel small circles.

For, if the arc B E be produced to meet the circumference A D F in some point as E′ (22. *Cor.* 2.), and if E′ C be joined, the triangle E′ B C will be equal to the triangle A B C, by the proposition, and therefore likewise equal to the triangle E B C, which is impossible, unless the point E′ coincides with the point E. In fact, if the point E were supposed to fall between the two circles, it is evident that a triangle E′ B C might be found having its vertex E′ in the circumference A D F, and including the vertex E of the triangle E B C; or if E were supposed to fall within the circle A D F, a triangle E′ B C might be found having its vertex E′ in the same circumference, and included within the triangle E B C; and in neither case could the triangle E′ B C be equal to the triangle EBC, that is, to the triangle A B C, whereas, by the proposition, such triangle E′ B C must be equal to the triangle A B C.

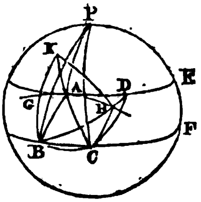
Scholium.

It may at first appear that the preceding proposition does not apply to all triangles which stand upon the base B C, and have their vertices in the circumference A E F; since the arc B A (for instance) may be produced to cut the circumference A E F a second time in a point A′ which may be joined with the point C, and thus, apparently, a second triangle formed, which is greater than A BC: but, if the arc B A produced cuts the circumference A E F in a second point A′, the arc A′ A B will be a semicircumference (22. *Cor.* 1.), and therefore the *triangle* which is apparently contained by A′ B, BC, and A′ C, is not a triangle (def. 9.) but a lune.

PROP. 25.

Of equal spherical triangles upon the same base, the isosceles has the least perimeter.

Let A B C, D B C be two equal spherical triangles upon the same base B C, and upon the same side of it, and therefore also between the same equal and parallel small circles A D E, B C F (24. *Cor.*); and let the triangle A B C be isosceles, having the side A B equal to the side A C: the triangle A B C shall have a less perimeter than the triangle D B C.



Let P be the nearer pole of the circle A D E; and therefore, also, the more distant pole of the circle B C F. Join P A, and through the point A draw the spherical arc GH at right angles to PA: then, because the arc P A, being less than a quadrant, is less than any other arc which can be drawn from P to G H (18. *Cor.* 1.) and that PD is equal to PA, the arc G H will cut the arc BD in some point H which is between D and B. From B draw the arc B G perpendicular to GH, and produce it to meet CA produced in the point K; and join H C, H K, P B, P C. Then, because the spherical triangles P A B, P A C have the three sides of the one equal to the three sides of the other, each to each, the angle P A B is equal to the angle PAC (15.); but the angle PAG is equal to the angle PAH, because they are right angles; therefore, the remaining angle B A G is equal to the remaining angle CAH or (8.*Cor.* 1.) KAG. And, because the triangles B G A, K G A have two angles of the one equal to two angles of the other, each to each, and the interjacent side GA common to both, G K is equal to G B, and A K to A B (14.). Therefore, because the triangles H G B, H G K have two sides of the one equal to two sides of the other, each to each, and the included angles H G B, H G K equal to one another, H K is equal to HB (13.). But in the triangle HKC, the side KC is less than H K and H C together (9.): therefore, A B and A C are together less than H B and H C. And HB, HC together are less than DB, DC together, because HC is less (9.) than DC and D H together. Much more, therefore, are AB and AC together less than D B and D C together. Therefore, if B C is added to each, the perimeter of the triangle A B C is less than the perimeter of the triangle D B C.

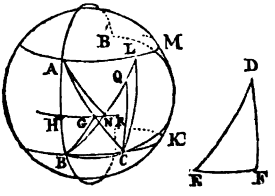
Therefore, &c.

Cor. Hence, of all triangles which are upon the same base, and have equal perimeters, the isosceles has the greatest area. For triangles, which have the same or a greater area than the isosceles, have greater perimeters.

Prop. 26.

If two spherical triangles have two sides of the one equal to two sides of the other, each to each, and the angle which is contained by the two sides of the first equal to the sum of the other two angles of that triangle, but the angle which is contained by the two sides of the other not so; the first triangle shall be greater than the other.

Let the spherical triangles A B C, D E F have the two sides AB, BC equal to the two sides D E, E F, each to each, and let the angle A B C of the first triangle be equal to the sum of the other two angles B C A, B A C, but the angle D E F of the other not equal to the sum of the other two angles E F D, E D F: the triangle A B C shall be greater than the triangle D E F.



Because the angle A B C is equal to the sum of the two angles B C A, B A C, let it be divided into the parts G B C, GBA, equal to them respectively, by the arc B G which meets AC in G. Draw the arc G H, bisecting the angle A G B, and let it meet AB in H. Then, because the angles G B C, G B A are equal to the angles G C B, G A B respectively, G C and G A are equal, each of them, to GB (11.), and therefore, also, equal to one another; and, because in the triangles G H A, G H B the two sides G A, G H are equal to the two sides G B, G H, each to each, and the included angles equal to one another, A H is equal to H B, and the angle A H G to the angle BHG (13.). Now, through the two points B, C and the third point A let there be made to pass two equal and parallel small circles B C K and A L M (22.). And, because in arcs A B and A C, terminated by these circles, the points of bisection H and G lie in the circumference of the great circle to which they are parallel (22.), the arc G H is an arc of that great circle. Make the angle C B L equal to the angle F E D, and let the arc B L cut the arc GH in N, and the circumference A L M in L (22. *Cor.* 2.); and join L C, N A. Then, because the triangles N H A, N H B have two sides of the one equal to two sides of the other, each to each, and the included angles N H A, N H B equal to one another, the arc N A is equal to N B (13.); but NB is equal to N L, because the arc GH produced bisects every arc A L which is terminated by the circles A L M, B C K; therefore N A is equal to N L: but N A and N B are together greater than AB (9.); therefore, L B is likewise greater than A B. Therefore, if B Q be cut off from the arc BL produced, equal to A B or D E, the point Q will be between L and B. Join Q C: and, because the triangles Q B C, D E F have the two sides QB, B C equal to the two sides D E, E F, each to each, and the included angles equal to one another, the third sides QC, DF are equal to one another (13.); and, consequently (20.), the triangle QBC is equal to the triangle D E F. But L B C is greater than QBC; and A B C is equal to L B C, because they are upon the same base BC, and between the same equal and parallel small circles (24.). Therefore, the triangle ABC is greater than the triangle DEF.

Therefore, &c.

Cor. Two given finite spherical arcs A B, B C, together with a third indefinite, inclose the greatest surface possible, when placed so that the included angle ABC may be equal to the sum of the other two angles of the triangle A B C.

This conclusion is analogous to that of III. 39. *Cor.* with regard to the greatest possible area which can be enclosed by two given finite straight lines with a third indefinite; for, although the angle ABC included by the spherical arcs is not a right angle, it is equal to the sum of the other two angles of the triangle A B C, which is the case also with the right angle of a right-angled plane triangle (I. 19. *Cor.* 3.).

Scholium.

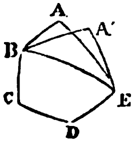
In a triangle A B C which has one of its angles A B C equal to the sum of the other two, the containing sides A B, B C are together less than a semicircumference. For G, the middle point of AC, is the pole of a small circle passing through A, B, and C, because G A, G B and G C are equal to one another: and, because a spherical arc (9. *Schol.*) is the shortest distance between two points along the surface of the sphere, the great arcs AB and B C are together less than the small arcs A B and B C, that is, than the semi-circumference of a small circle: much more, therefore, are A B and B C together less than the semicircumference of a great circle. In fact, when the given sides A B and B C are together equal to a semicircumference, B C coincides with BR, the polar diameter of the circle BCK, and because AC is always bisected by the circumference of the great circle H G N, A G and G C are in this case quadrants, and consequently (19. Cor. 1.) A G C is at right angles to A B; so that A B C is a lune, not a triangle, and equal to a fourth part of the surface of the sphere. And when the given sides A B and B C are together greater than a semicircumference, it is evident, without reference to the figure, that a triangle may be found which shall have A B, B C for two of its sides, and differ by as little as we please from half the surface of the sphere. For, in this case, if A B, B C are placed in the circumference of the same great circle, that is, at an angle equal to two right angles, the arc A C which completes the circle will be less than a semicircumference; and if A B, B C are placed at any angle less than two right angles, the arc AC will be still less (18.); therefore, if A B, B C are placed at any angle less than two right angles, A C being joined will complete a spherical triangle (def. 9.), and this triangle will differ less and less from the hemispherical surface, as the angle A B C approaches to two right angles.

In a triangle A B C of this kind, the angle A B C, which is equal to the sum of the other two, is always greater than a right angle, because the three together are greater than two right angles (10). It is worthy of notice, also, that the triangle B′ A C, which is the difference between the lune B A B′ C, and the triangle ABC has its surface always equal to a fourth part of the surface of the sphere; for it is measured by the difference of 2 B and 2 B — 2 R, that is, by 2 R, and the sphere's surface by 8 R (21. *Cor.* 1.).

It may be observed that a similar course of reasoning to that which is exhibited in Book III. Prop. 40, 41, 42, and 43, founded upon this proposition, will lead to the same conclusions with regard to the surfaces included by spherical polygons and circles of the sphere, which are there stated with regard to the areas of rectilineal figures, and circles on a plane surface: viz.

1. Of all spherical polygons, contained by the same given sides, that one contains the greatest portion of spherical surface, which has all its angles in the circumference of a circle.
2. A circle includes a greater portion of the spherical surface than any spherical polygon of the same perimeter.
3. The lunular surface, which is included by a spherical arc and a small arc, is greater than any other surface which is included by the same perimeter, of which the same spherical arc is a part.

We may also infer from Prop. 25., that, of all spherical polygons having the same number of sides and the same perimeter, the greatest is that which has all its sides equal and all its angles equal. For if a spherical polygon A B C D E have not all its sides equal, and A B, A E be two adjoining sides which are unequal, a greater A′ B C D E may be found with the same perimeter by describing upon the base B E the isosceles triangle A′ B E, which has the same perimeter with A B E. And, as above, if a spherical polygon have not all its angles lying in the circumference of a circle, a greater may be found with the same sides. Therefore, none is greatest, but that which has all its sides equal, and all its angles lying in the circumference of a circle, that is, which has all its sides equal and all its angles equal; and, since there is evidently some greatest, the greatest is that which has all its sides equal and all its angles equal.

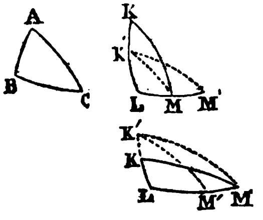


Prop. 27.

Spherical pyramids, which stand upon equal bases, are equal to one another; so, likewise, are their solid angles.

First, let the bases of the pyramids be equal triangles, which have one side of the one equal to one side of the other, and let the equal sides be made to coincide, so that the triangles A B C, E B C (in the figure of Prop. 24.) may represent the bases of the pyramids. Then the triangles A B C, E B C, being equal to one another, lie between the same equal and parallel small circles, and may be completed (as in Prop. 24.) into the quadrilaterals A B C D, E B C F. And it may be shown, as in the same proposition, that the triangles E A B, F D C have the three sides of the one equal to the three sides of the other, each to each, and therefore may be made to coincide (15.): wherefore, also, the pyramids, which have these triangles for their bases, may be made to coincide, and are equal to one another. And hence, as in Prop. 24., it was demonstrated, by the addition and subtraction of the lunular portions of surface, that the quadrilateral A B C D is equal to the quadrilateral EBCF, so, here, it may be demonstrated, by the addition and subtraction of the pyramidal solids (23.), which have these lunular portions for their bases, that the pyramidal solids, which have for their bases the quadrilaterals A B C D and E B C F, are equal to one another. But, because the triangles A B C, C D A have the three sides of the one equal to the three sides of the other, each to each, in the same order, they may be made to coincide (15.), and therefore the pyramids which have these triangles for their bases may be made to coincide, and are equal to one another; and each of them is the half of the pyramidal solid which has the quadrilateral A B C D for its base. And in the same manner it may be shown, that each of the pyramids on the triangular bases, E B C, C F E is the half of the pyramidal solid on the quadrilateral base E B C F. Therefore, because the halves of equals are equal, the pyramid upon the base A B C is equal to the pyramid upon the base E B C.

Next, let the bases of the pyramids be any equal triangles A B C, K L M.



From LK or L K produced cut off L K′ equal to B A, and let the triangle K′ L M′ be taken equal to the triangle A B C or K L M, and join K′ M. Then, by the former case, the pyramids upon the bases A B C, K′ L M′ are equal to one another. And, because the triangle K′ L M′ is equal to the triangle K L M, the triangles K′ M M′ and K K′ M which have the common side K′ M are likewise equal; therefore, by the first case, the pyramids, which have these triangles for their bases, are equal to one another; and, these being added to, or taken from, the pyramid upon the base K′ L M, the whole or remaining pyramid upon the base K′ L M′ is equal to the whole or remaining pyramid upon the base K L M, that is to the pyramid upon the base A B C.

Lastly, let the bases of the pyramids P, P′ be equal polygons. Let a triangle be found which is equal to one, and therefore also to the other of the polygons, and let Q be the pyramid which has this triangle for its base. Then, because this triangle is equal to the base of the pyramid P, it may be divided into triangles which are equal respectively to the triangles into which the base of P is divisible. And it has been already shown that pyramids which have equal triangles for their bases are equal to one another; and the sums of equals are equal; therefore, the pyramids P and Q are equal to one another. In the same manner it may be shown that the pyramids P′ and Q are equal to one another. Therefore P is equal to P′. And it is evident that what has been shown with regard to spherical pyramids, being derived from coincidence, may be shown equally of their solid angles at the centre of the sphere.

Therefore, &c.

PROP. 28.

Any two spherical pyramids are to one another as their bases; and the solid angles of the pyramids are to one another in the same ratio.

Let P, P′ be any two spherical pyramids, and let B, B′ be their bases: the pyramid P shall be to the pyramid P′ as the base B to the base B′.

For if the base B′ be divided into any number of equal parts, then, because pyramids which stand upon equal bases are equal to one another, the pyramid P′ will be divided into the same number of equal parts by planes passing through the arcs of division (27.); and if the base B contain exactly, or with a remainder, a certain number of parts equal to the former, the pyramid P will contain exactly, or with a remainder, the same number of parts equal to the latter (27.): therefore, P is to P′ as B to B′ (II. def. 7.)

And the same may be said of the solid angles.

Therefore, &c.

Cor. Every spherical pyramid is equal to the third part of the product of its base and the radius of the sphere.

For, if the whole sphere be divided into spherical pyramids, these pyramids will be to one another as their bases; and, therefore, any one of them is to the sum of all, as the base of that one to the sum of all the bases (II. 25. *Cor.* 3.); that is, any pyramid is to the whole sphere as its base to the surface of the sphere. Therefore, any pyramid is to the whole sphere as the product of its base and the radius of the sphere to the product of the whole surface and radius (IV. 26.); and because, in this proportion, the second term is equal to the third part of the fourth (V. 17.), the first term is likewise equal to the third part of the third (II. 19. and II. 13.).

Scholium.

Hence it appears that *every solid angle is measured by the spherical surface which is described with a given radius about the angular point, and intercepted between its planes.* For it is shown in the proposition that any two solid angles are to one another as these surfaces.

This measure bears an obvious analogy to the measure of a plane angle, as stated in the Scholium at Book III. Prop. 13. The angular unit of the latter measurement is the right angle; into four of which the whole angular space about any point in a plane is divided by two straight lines drawn at right angles to one another; and each right angle is measured by a quadrant, or fourth part of the circumference of a circle described about that point with a given radius. In like manner, if, through a point in space, three planes be made to pass at right angles to one another, they will divide the whole angular space about that point into eight *solid right angles*, each of which (a solid angular unit) is measured by an octant, or eighth part of the surface of a sphere described about that point with a given radius.

Thus, then, the plane angles, dihedral angles, and magnitude of a solid angle, which has three or a greater number of faces, are represented, upon the surface a sphere described about the angular point, by the sides, angles, and surface of a spherical triangle or polygon: and whatever has been stated with regard to the latter may be understood likewise of the former. It is shown (for instance) in Prop. 13. that if two solid angles, each of which is contained by three plane angles, have two plane angles of the one equal to two plane angles of the other, each to each, and likewise the dihedral angles contained by them equal to one another, the remaining plane and dihedral angles of the one shall be equal to the remaining plane and dihedral angles of the other, each to each. And, generally, all questions which relate to solid angles will be placed in the clearest view before us, when we contemplate only their representation and that of their parts, upon the surface of the sphere.

SECTION 4.—*Problems.*

In the solution of the following problems, it is assumed,

1. That of any two points, which are given upon the surface of a sphere, the direct distance may be obtained; and,

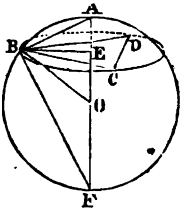
2. That from any given point as a pole, with any given distance less than a semicircumference, a circle may be described upon the surface of a sphere.

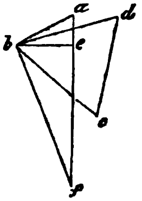
We may observe, however, that the first assumption is only used in Prob. 1 to obtain the diameter of the sphere; and, therefore, if the diameter is supposed to be given, it may be dismissed as unnecessary. Constructions made within the sphere are excluded.

PROP. 29. Prob. 1.

To find the diameter of a given sphere A B C D.

From any point A, as a pole, with any distance AB, describe the circle B C D;





in the circumference B C D take any three points B, C, D; describe the plane triangle *b c d* with its sides *b c*, *c d*, *b d* equal to the direct distances B C, C D, B D respectively; find *e* the centre of the circumscribing circle (III. 59.), and join *e b*; from the point *e* draw *e a* perpendicular to *e b*, and from the centre *b*, with a radius equal to the direct distance B A describe a circle cutting *e a* in *a*; produce *a e* to *f*, and from *b* draw *b f* perpendicular to *b a* (I. 44.) to meet *a f* in *f*: *a f* shall be equal to the diameter of the sphere.

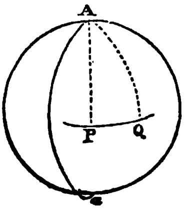
For, if O be the centre of the sphere, and the diameter A O F be drawn cutting the plane B C D in E, and E B be joined, A F will be the axis (def. 3.), and therefore (1.) the point E the centre, of the circle B C D, and the angle A E B will be a right angle. Also, EB is equal to *e b*, because the triangles B C D, *b c d* may be made to coincide, and, then, the points E, *e* being, each of them, the centre of the same circumscribing circle, will likewise coincide. Therefore, because in the right-angled triangles AEB, *a e b*, the hypotenuse A B is equal to the hypotenuse *a b*, and the side E B to the side *e b*, the angle B A E is equal to the angle *b a e* (I. 13.). Join A B, B F, O B. Then, because O B is equal to the half of A F, the angle A B F is a right angle (I. 19. *Cor.* 4.); and, because, in the right-angled triangles A B F, *a b f*, the side A B and angle B A F are equal respectively to the side *a b* and angle *b a f*, the hypotenuses A F and *a f* are equal to one another (I. 5.), and *a f* is equal to the diameter of the sphere.

Therefore, &c.

Cor. Hence, a sphere being given, the quadrant of a great circle may be found; for it is equal to the quadrant of the circle which is described upon the diameter *a f*.

PROP. 30. Prob. 2.

Any point A *being given upon the surface of a sphere, to find the opposite extremity of the diameter which passes through that point.*

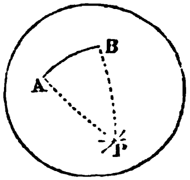


From the pole A, with the distance of a quadrant, (29. *Cor.*) describe a spherical arc P Q (5.); and, from any two points P, Q of this arc, as poles, with the same distance, describe two great circles passing through A, and cutting one another again in the point *a*. Then, because the planes of any two great circles cut one another in a diameter of the sphere, the points A, *a* are opposite extremities of a diameter.

Therefore, &c.

PROP. 31. Prob. 3.

To join two given points A, B *upon the surface of a sphere.*



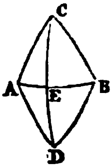
From the pole A, with the distance of a quadrant (29. *Cor.*), describe a great circle, and from the pole B, with the same distance, describe a great circle cutting the former in the point P. From P as a pole, with the same distance, describe the arc A B. Then, because AB is the arc of a great circle (5.) described between the points A, B, it joins those two points on the surface of the sphere (def. 5.).

Therefore, &c.

Cor. In the same manner, any spherical arc being given, the great circle may be completed of which it is a part.

PROP. 32. Prob. 4.

To bisect a given spherical arc A B.



From the pole A, with the distance A B describe a circle, and from the pole B with the same distance AB or BA describe a circle cutting the former circle in the points C, D. Join CD (31.) and let it cut A B in E. AB is bisected in E.

See Book I. Prop. 43.

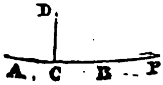
Therefore, &c.

It must here be observed that although the points C, D are determined by the intersection of small circles, the arcs CA, CB, CD, DA, DB, which are the sides of the spherical triangles in the demonstration, are portions of great circles; and the same remark applies to some subsequent problems.

Cor. In the same manner, a spherical arc C D may be drawn which shall bisect any spherical arc A B at right angles.

PROP. 33. Prob. 5.

To draw an arc, which shall be perpendicular to a given spherical arc AB, *from a given point* C *in the same.*



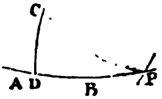
From C B, or C B produced, cut off CP equal to a quadrant, and from the pole P, with the distance PC, describe the spherical arc C D. Then, because P C is drawn from the pole of a great circle C D to the point C in that circle, P C is at right angles to CD (5.); or, which is the same thing, C D is at right angles to P C or AB.

Therefore, &c.

Prop. 34. Prob. 6.

To draw an arc, which shall be perpendicular to a given spherical arc AB, *from a given point* C *without it.*

From the pole C, with the distance of a quadrant, describe a great circle, cutting AB or AB produced in P, and from the pole P, with the same distance, describe the spherical arc C D. For the same reason as in the last problem, C D is the perpendicular required.

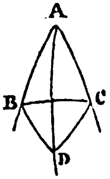


Therefore, &c.

Prop. 35. Prob. 7.

To bisect a given spherical angle B A C.

In A B take any point B; make A C equal to A B, and join B C; from the poles B, C, with the common distance B C, describe two spherical arcs cutting one another in D, and join A D (31.). AD is the bisecting arc required. See Book I. Prop. 46.

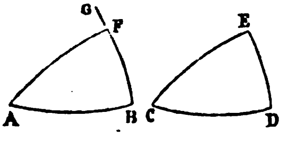


Therefore, &c.

Prop. 36. Prob. 8.

At a given point A, *in a given arc* AB, *to make a spherical angle equal to a given spherical angle* C.

From the pole C, with the distance of a quadrant, describe a circle cutting the sides of the given angle in the points D, E; and from the pole A, with the same distance, describe a circle B F G cutting A B in B; from B F G cut off BF equal to DE, and join AF (31.). The angle B A F is equal to the given angle C; for they are measured by equal arcs B F, D E (6.)

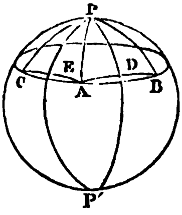


Therefore, &c.

Prop. 37. Prob. 9.

To describe a circle through three given points A, B, C, *upon the surface of a sphere.*

Join (31.) AB and AC, and bisect them at right angles (32. *Cor.*) with the arcs DP, EP, which meet one another in the points P, P′. From either of these points, P, as a pole, with the distance PA, describe a circle. It shall pass through the other two points B and C. For PB is equal to PA, because the triangles P D B, P D A have two sides of the one equal to two sides of the other, each to each, and the included angles P D A, P D B equal to one another (13.); and in the same manner it may be shown that PC, likewise, is equal to P A.



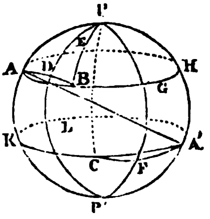
Therefore, &c.

Cor. In the same manner, the poles P, P′ of any given circle A B C may be found.

Prop. 38. Prob. 10.

Through two given points A, B, *and a third point* C *on the surface of a sphere, to describe two equal and parallel small circles; the points* A, B, C *not lying in the circumference of the same great circle.*

Join A B, and draw D E bisecting it at right angles in the point D (32. *Cor.*); find A′ the opposite extremity of the diameter which passes through A (30.); join A′ C (31.), and bisect it at right angles with the arc F P which meets D E in the points P, P′ (32. *Cor.*); join PA, P C, and from the pole P with the distances P A, P C describe two small circles A G H and C K L: the former shall likewise pass through the point B, and they shall be the two equal and parallel small circles required.

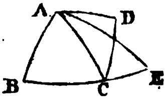


For, if P A and P B be joined, they will be equal to one another, because the triangles PDA, PDB have two sides of the one equal to two sides of the other, each to each, and the included angles PDA, PDB equal to one another (13.). And, for the same reason, if PA′ and PC are joined, PA′ will be equal to PC. But PA and PA′ are together equal to a semicircumference, because AA′ is a diameter of the sphere. Therefore, PA and PC are together equal to a semicircumference; and, consequently (3. *Cor.*), the parallel circles AGH, CKL are equal to one another. Also, because PA is equal to PB, the circle AGH passes through the point B.

Therefore, &c.

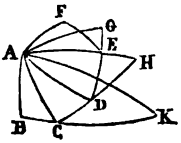
Prop. 39. Prob. 11.

To describe a triangle which shall be equal to a given spherical polygon, and shall have a side and adjacent angle the same with a given side AB *and adjacent angle* B *of the polygon.*



First, let the given polygon be a quadrilateral ABCD. Through the two points A, C, and the third point D, describe two equal and parallel small circles (38.), and let the arc BC, which cuts one of them in C, be produced to cut the other in E (22. *Cor.* 2.), and join AE, AC (31.). Then, because the triangles ACD, ACE lie between the same equal and parallel small circles, they are equal to one another; and, therefore, the triangle ABC being added to each, the triangle ABE is equal to the quadrilateral ABCD.

Next, let ABCDEF be the given polygon. Join AC, AD. Make, as in the former case, the triangle ADG equal to the quadrilateral ADEF, the triangle ACH equal to the quadrilateral ACDG, and the triangle ABK equal to the quadrilateral ABCH. It is evident that the triangle ABK is equal to the polygon ABCDEF.

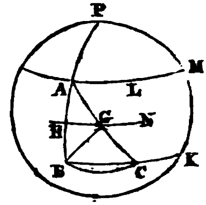


And, in each case, the triangle described has the same side AB and angle B with the given polygon.

Therefore, &c.

Prop. 40. Prob. 12.

Given two spherical arcs AB *and* Q, *which are together less than a semicircumference; to place them so, that, with a third not given, they may contain the greatest surface possible.*



Bisect AB in H (32.), and produce HA to P, so that HP may be equal to a quadrant; from the pole P, with the distance PB, describe the small circle BCK, and from the pole B, with the distance Q, describe a circle cutting the circle BCK in C. Join AC, BC. The triangle ABC shall be the triangle required.

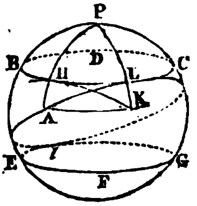
For, if, with the distances PA, PH, there are described from the pole P the small circle ALM and the great circle HGN, the latter cutting the arc AC in G, the circles ALM and BCK will be equal to one another, because the distances PA, PB are together equal to twice the quadrant PH (I. ax. 9.), that is, to a semicircumference (3. *Cor.*); and they are parallel because they have the same poles; and HGN is the great circle to which they are parallel; therefore, AG is equal to GC (22.). But, because P is the pole of the great circle HGN, PH is at right angles to HG (5.); and, because the triangles GHA, GHB have two sides of the one equal to two sides of the other, each to each, and the included angles GHA, GHB equal to one another, GB is equal to GA or GC (13.). Therefore, because in the isosceles triangles GAB, GBC, the angles GBA, GBC are equal to the angles GAB, GCB respectively (11.), the whole angle ABC of the triangle ABC is equal to the sum of the two angles CAB, ACB; wherefore the triangle ABC is the greatest that can be formed with the two sides AB, BC, or the greatest that can be formed with the given sides AB and Q (26. *Cor.*).

Therefore, &c.

Prop. 41. Prob. 13.

Through a given point A *to describe a great circle, which shall touch two given equal and parallel small circles* BCD, EFG.

Find the point P which is the com-



mon pole of the circles B C D, E F G (37. *Cor.*), and join P A (31.): from the pole P, with the distance P A, describe a circle, and from the point H, in which P A cuts the circle B C D, draw H K at right angles to P A (33.), and let it cut the circle, which was described through A, in K: join P K (31.), and let it cut the circle B C D in L: the great circle A L *l* which passes through the points A and L shall be the great circle required.

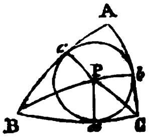
For, because in the triangles P H K, P L A, the two sides P H, P K are equal to the two sides P L, P A, and the included angle L P H common to both triangles, the angle P L A is equal to the angle P H K, (13.) that is, to a right angle; and the arc P L is less than a quadrant; therefore, P L is the least arc which can be drawn from the point P to the circle A L *l* (18. *Cor.* 1.), and if L P be produced to meet the circle in *l*, P *l* is the greatest. But every point of the circle B C D is at the distance P L; and every point in the equal circle E F G at the distance P *l*, because P *l* and P L are together equal to a semicircumference (3. *Cor.*). Therefore, the circle ALM, which has been described through the given point A, touches the given circles B C D and E F G in the points L and *l*.

Therefore, &c.

Prop. 42. Prop. 14.

To inscribe a circle in a given spherical triangle A B C.

Bisect the angle ABC with the arc BP, and the angle A C B with the arc C P which meets the former arc in P (35.); from P draw P *a* perpendicular to B C (34.); make B *c* equal to B *a*, and C *b* equal to C *a*, and join P *c*, P*b*. Then, because the triangles P B *a*, P B *c* have two sides of the one equal to two sides of the other, each to each, and the included angles P B *a*, P B *c* equal to one another, P *c* is equal to P *a* (13.), and the angle P *c* B to the angle P *a* B, that is, to a right angle: therefore, the circle which is described from the pole P, with the distance P *a*, will touch A B in the point *c* (18. *Cor.* 1.). And, in the same manner, it may be shown that the same circle will touch A C in *b*. Therefore, from the pole P, with the distance P *a*, describe the circle *a b c*; and it will be the circle required. Therefore, &c.



Scholium.

The constructions in this section have little or no practical utility, and have, accordingly, been added rather with a view to illustrate the analogies of Plane and Spherical Geometry, than to furnish rules for practice. Some of these we have already had occasion to notice, and others will have offered themselves to the reader; who will readily perceive that these striking points of resemblance (or, as he may be disposed to call them, of identity) are to be ascribed to the circumstance, that spherical triangles, when their sides are but small portions of great circles, and consequently their surfaces small in comparison with the surface of the sphere, become more and more nearly plane, their sides more and more nearly straight lines, and the sum of their angles (the excess of which above two right angles bears the same ratio to eight right angles (21 *Cor.* 1.) as the surface of the triangle to the surface of the sphere) more and more nearly equal to two right angles. Thus every plane triangle may be regarded as a spherical triangle upon the surface of a sphere, the radius of which is indefinitely great; and in this way of viewing the subject, the properties of plane triangles resemble those of spherical triangles, only as a particular case the general one in which it is included.

But it may be asked, has the term *similar*, which introduces us to so wide a field in Plane Geometry, any place in Spherics? Not in propositions which have reference only to the surface of one and the same sphere. Similar figures upon the surface of the same sphere are likewise equal to one another, and may be made to coincide. But, when we consider the surfaces of different spheres, and compare the figures which are formed upon them, here again we shall find room for the application of the term in its full and peculiar sense. Thus, similar spherical triangles are such as are contained by similar arcs upon the surfaces of different spheres. It is easy to perceive that such triangles are equiangular, and have their sides about the equal angles proportionals; and that their surfaces bear the same ratio to one another as the surfaces of their respective spheres, and, therefore, are to one another as the squares of the radii of the spheres, or as the squares of the arcs which are homologous sides of the triangles.

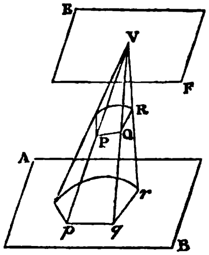
APPENDIX.

Part I.—Of Projection by Lines diverging and by Lines parallel.
Part II.—Of the Plane Sections of the Right Cone, or Conic Sections.
Part III.—Plane Sections of the Oblique Cone, of the Right Cylinder, and of the Oblique Cylinder.

Part I.—*Of Projection by Lines diverging and by Lines parallel.*

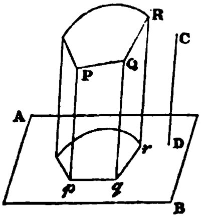
It is not here intended to enter at large upon the subject of perspective, or to anticipate in any manner the rules by which it affords such material assistance to the draughtsman and artist. We propose, on the contrary, no more than the explanation of a few terms, and the statement of a few theorems, occasionally serving to simplify the consideration of lines in different planes, and which will be of immediate service in the account which will be subsequently given of the general properties of the conic sections.

Def. 1. Let AB be a plane given in position, and V a given point without it; then, if through any point P a straight line is drawn from V to meet the plane AB in p, the point p is called the *perspective projection of the point* P upon the plane A B.



The plane A B is called the *plane of projection*, the point V the *vertex*, and the plane E F, which is drawn through the point V parallel to A B, the *vertical plane*.

2. Let A B be a plane given in position, and C D a straight line given in position, not parallel to the plane A B: then, if through any point P a straight line is drawn parallel to C D to meet the plane A B in p, the point p is called the *orthographic projection of the point* P upon the plane AB.



The straight line C D is called the *direction*, and the plane A B, as before, the plane of projection.

3. A line P Q is said to be projected, either perspectively or orthographically, upon a given plane A B, when all its points are so projected; and the line $p\,q$ which contains the projections of the latter is called the *perspective* (fig. of def. 1.) or *orthographic* (fig. of def. 2.) *projection of the line* P Q.

4. A figure P Q R is said to be projected, either perspectively or orthographically, upon a given plane A B, when all its containing lines are so projected; and the figure $p\,q\,r$, which is contained by the projections of the latter is called the *perspective* (fig. of def. 1.) or *orthographic* (fig. of def. 2.) *projection of the figure* P Q R.

5. Any point, line, or figure is called an *original* point, line, or figure with reference to its perspective or orthographic projection.

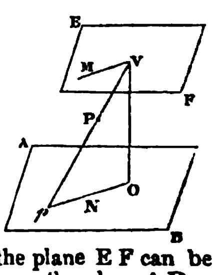
Thus, in the figures of def. 1. and def. 2., the point P is called the original of the point p, the line P Q the original of the line $p\,q$, and the figure P Q R the original of the figure $p\,q\,r$.

It is almost needless to observe that in these definitions the planes E F and A B, (although they necessarily appear circumscribed in the figures, are considered to be of unlimited extent; and the same is to be understood in the following propositions.

Prop. 1.

In perspective projection, no point in the vertical plane E F *can be projected from the vertex* V *upon the plane of projection* A B; *but, of every point which is not in that plane, the projection may be found upon the plane* A B.

For, the straight line which is drawn from the point V through any point in the plane E F, can never meet the plane A B; because it lies entirely in the plane EF, which is parallel to AB. Therefore, no point in the plane E F can be projected from V upon the plane A B.



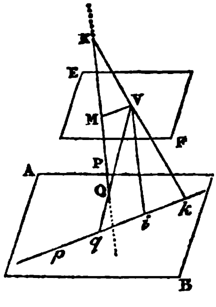
But, if P be a point which is not in the plane E F, draw V O perpendicular to the plane A B (IV. 36.), and let the plane P V O cut the parallel planes E F and A B in the straight lines V M and O N respectively. Then, because the sections of parallel planes by the same plane are parallel straight lines (IV. 12.), V M is parallel to O N; and, because V M is parallel to O N, and that V P cuts V M in V, V P may be produced to cut O N in some point *p* (I. 14. *Cor.* 3.); but if it cuts O N in any point, it must cut the plane A B in the same point, because O N lies in that plane: therefore *p*, the projection of the point P (def. 1.), may be found.

Therefore, &c.

Cor. It is shewn in this proposition, that if a straight line cuts one of two parallel planes, it may be produced to cut the other likewise.

Prop. 2.

The perspective projection of a straight line is a straight line; and if any point of the original straight line be in the vertical plane, the straight line which is its projection shall be of unlimited extent.



Let A B be the plane of projection, V the vertex, P Q any straight line, and *p q* its projection: *p q* shall likewise be a straight line.

Because the points of *p q* lie in straight lines drawn through V and the corresponding points of P Q, and that P Q is a straight line, the points of *p q* lie in the plane V P Q. But they lie also in the plane A B. Therefore they lie in the common section of the planes V P Q and A B, that is (IV. 2.), in a straight line.

Also, if any point M of the original straight line Q P M lie in the vertical plane E F; the straight line *q p*, which is its projection, shall be of unlimited extent towards *p*. For the projection of the point M cannot be found upon the plane A B (1.); and every point in *q p* produced is the perspective projection of some point of Q P M, because the straight line which is drawn from it to the vertex V cuts Q P M in some point between Q and M.

Therefore, &c.

Cor. 1. It is supposed in the proposition that the original straight line P Q does not pass through the vertex V; for, in this case, it is evident that all its points have for their projections the single point in which it cuts the plane of projection.

Cor. 2. The perspective projection of any given straight line is a part of the common section of two planes, viz. the plane which passes through the vertex and given straight line, and the plane of projection.

Cor. 3. The perspective projection of a straight line which is parallel to the plane of projection, is parallel to its original (IV. 10.)

Cor. 4. The perspective projection of a straight line which is not parallel to the plane of projection, shall pass, if produced, through the point in which a parallel to the original drawn through the vertex cuts the plane of projection. For such parallel is in the plane which passes through the vertex and the original straight line, and consequently the point in which it cuts the plane of projection is in the common section of the two planes.

Cor. 5. If the original straight line cuts the vertical plane, in the point M, so that one part, as K M, lies upon one side of that plane, and the other part, as M P Q, upon the other side of it, the projections of the two parts shall together make up the whole of a straight line infinitely produced both ways, except only the finite interval *k q* between the projections of its extreme points K and Q.

Cor. 6. And if such original finite straight line K M P Q be infinitely produced both ways, the projections of the produced parts shall together make up the finite interval *k q* between the projections of its extreme points K and Q.

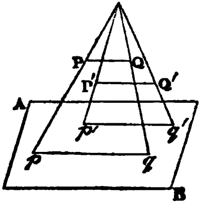
For, if V *i* be drawn parallel to K Q to meet the plane A B in *i*, the projection of every point in the part produced beyond K will be found between *k* and *i*, and the projection of every point in the part produced beyond Q between *q* and *i*.

Prop. 3.

The perspective projections of parallel straight lines, which are likewise parallel to the plane of projection, are parallel straight lines.

Let A B be the plane of projection,

V the vertex, PQ and P′Q′ any two parallel straight lines, which are likewise parallel to the plane AB, and *pq* and *p′q′* their projections: *pq* shall be parallel to *p′q′*.



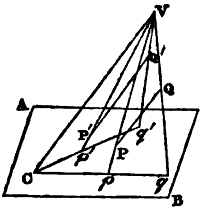
Because PQ is parallel to the plane AB, the projection *pq* is parallel to PQ (2. *Cor.* 3.) and, for the like reason, *p′q′* is parallel to P′Q′. Again, because *pq* and P′Q′ are each of them parallel to PQ, *pq* is parallel to P′Q′ (IV. 6.); and, because *pq* and *p′q′* are each of them parallel to P′Q′, *pq* is parallel to *p′q′*.

And in the same manner it may be shewn that if there are any number of parallel straight lines which are likewise parallel to the plane AB, their perspective projections shall be parallel to one another.

Therefore, &c.

Prop. 4.

The perspective projections of parallel straight lines, which cut the plane of projection, are straight lines, which are not parallels, but which pass, when produced, all of them, through one and the same point, the point, namely, in which a straight line drawn through the vertex parallel to the original straight lines, cuts the plane of projection.



Let AB be the plane of projection, V the vertex, PQ and P′Q′ any two parallel straight lines which are not parallel to the plane AB, and *pq* and *p′q′* their projections; also let VC be drawn through the point V parallel to PQ to meet the plane AB in C: the straight lines *pq*, *p′q′* produced, shall, each of them, pass through the point C.

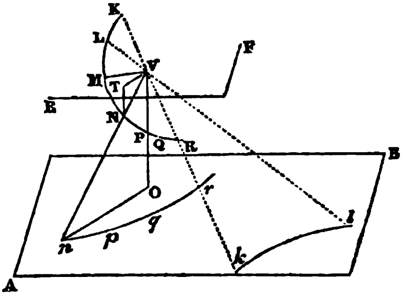
Because straight lines which are parallel to the same straight line are parallel to one another (IV. 6.) VC, which is parallel to PQ, is parallel also to P′Q′. But it has been already shown (2 *Cor.* 4.) that the perspective projection of a straight line which is not parallel to the plane of projection, passes through the point in which a parallel to it drawn through the vertex cuts the plane of projection. Therefore, because VC is parallel both to PQ and P′Q′, the projections *pq* and *p′q′* pass both of them through the point C. And in the same manner, it may be shown that the projection of any other straight line which is parallel to PQ, passes through the same point C; for such straight line is likewise parallel to VC (IV. 6.).

Therefore, &c.

Prop. 5.

The perspective projection of a curved line is a curved line; and, if any point of the original curve be in the vertical plane, the curve which is its projection shall have an arc of unlimited extent corresponding to the arc of the original curve which is terminated in that point.*

Let AB be the plane of projection, V the vertex, PQR any curved line, and *pqr* its projection: *pqr* shall likewise be a curved line.



For, if any part of *pqr*, as *pq*, be a straight line, then, since PQ is the perspective projection of *pq* upon the plane PQR (def. 1.), PQ must likewise be a straight line (2.), which is contrary to the supposition. Therefore, no part of *pqr* is a straight line, that is, *pqr* is a curved line (I. def. 6.).

Also, if any point M of the curve PQR lies in the vertical plane EF, the projection *pqr* shall have an arc of unlimited extent corresponding to the arc MP, which is terminated in M.

Let N be any point in the arc MP, and let VN be joined and produced to meet the plane AB in *n*, which is therefore (def. 1.) the projection of the point N: from V draw VO perpendicular to the plane AB (IV. 36.), and from N draw NT perpendicular to the plane

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* The whole of which (it is also understood) lies in one plane. For, if the parts of a curve lie in different planes, of which one or more pass through the vertex of projection, the projections of the corresponding parts will be straight lines, (see Cor. 1 of this proposition). The demonstration given in the text applies only to a plane curve.

E F, and join O n, V T. Then, because V O is perpendicular to the plane A B, which is parallel to the plane E F, V O is also perpendicular to the plane E F (IV. 11.); but N T is perpendicular to the same plane; therefore N T is parallel to VO (IV. 5.). And, because the plane of the parallels T N, V O (I. def. 12.) cuts the planes E F, AB in the lines TV, O n respectively, TV is parallel to O n (IV. 12.). Therefore, because the triangles V O n, T N V have the sides V O, O n of the first parallel to the sides NT, TV of the other, each to each, and their sides V n, N V in the same straight line, they are equiangular (I. 15.), and, consequently (II. 31.), O n is to OV as TV to TN. Therefore (II. 9.), if TV contains T N any number of times exactly or with a remainder, O n will contain O V the same number of times exactly or with a remainder. But, if the point N be made to approach to the point M, T V will approach in magnitude, as well as position, to M V, and T N, which is the distance of the point N from the plane EF, will be diminished without limit: consequently, there is no limit to the number of times T V may be made to contain T N. Therefore, also, there is no limit to the number of times O n may be made to contain O V, that is, the line O n may be increased without limit, and the point n will describe an arc of unlimited extent corresponding to the arc P N M or M P, which is terminated in M.

Therefore, &c.

Cor. 1. In the demonstration of this proposition it is supposed that the plane of the original curve does not pass through the vertex V; for, in this case, it is evident that its projection upon the plane A B is a straight line (IV. 2.).

Cor. 2. If the original curve cuts the vertical plane in the point M, so that one part, as K L M, lies upon one side of that plane, and the other part, as M P Q, upon the other side of it, the projection shall have two arcs which are extended without limit in opposite directions, corresponding to the two arcs K L M, M P Q, which are terminated in the point M.

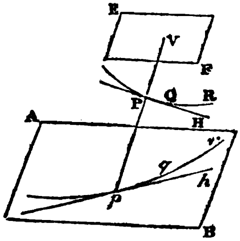
Def. 6. If a curve has an arc of unlimited extent, and if a straight line is drawn which never meets that arc, but which, being produced, may be made to approach nearer to it than by any given distance, such straight line is called an *asymptote* to the arc.

This definition will be best illustrated by the example contained in the following proposition.

Prop. 6.

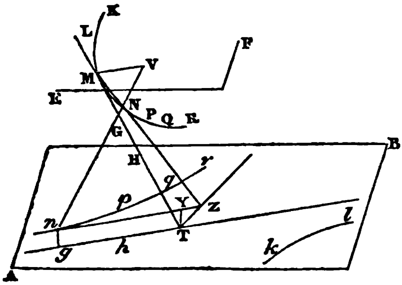
The perspective projection of a straight line which touches any curve is a straight line which touches the projection of that curve, if the point of contact is without the vertical plane; but, if the point of contact is in that plane, the projection of the tangent is an asymptote to the projection of the curve.

Let A B be the plane of projection, V the vertex, and P Q R any curve; and, first, let P be a point of the curve P Q R lying without the vertical plane E F, and P H a straight line which touches the curve P Q R in the point P; then, if the point p be the projection of the point P (def. 1.), because P is a point both in the curve P Q R and the tangent P H, p will be a point in the projection of each; let, therefore, the curve $p\ q\ r$ be the projection of the curve P Q R, and the straight line $p\ h$ the projection of the tangent P H: the straight line $p\ h$ shall likewise touch the curve $p\ q\ r$ in the point p.



Because PH touches the curve PQR, the points of the curve P Q R upon both sides of P lie towards the same part of PH, and therefore also the straight lines drawn from V through those points lie towards the same part of the plane V P H or V $p\ h$. But these straight lines are the same which are drawn from V to the points of the curve $p\ q\ r$ on both sides of p (def. 1.). Therefore, the latter also lie towards the same part of the plane V $p\ h$; and, consequently, the points of the curve $p\ q\ r$ on both sides of p lie towards the same part of the straight line $p\ h$, that is, $p\ h$ touches the curve $p\ q\ r$ in the point p.

But, in the next place, let M be a point of the curve P Q R lying in the vertical plane E F, and let M G H be a straight line touching the curve P Q R in the point M; then, if the curve $p\ q\ r$ be the projection of the curve M P Q R, and the straight line $g\ h$ of the tangent



M G H, the straight line *g h* shall be an asymptote to the curve *p q r*.

Because the point M lies in the plane E F, and therefore (1.) cannot be projected upon the plane A B, and because the tangent M G H does not meet the curve M P Q R in any other point, no point can be found in which the projection *g h* of the tangent meets the projection *p q r* of the curve, to whatever extent both of them may be produced. Again, let D be any given distance: produce the tangent M H to meet the plane A B in T, and therefore also to meet its projection *g h* in the same point T, and let the plane of the curve M P Q R be produced to meet the plane A B in the straight line T Z (IV. 2.); from T, in the plane A B, draw T Y perpendicular to *g* T, and let TY be taken of any length, so that it be less than the given distance D; through Y draw Y Z parallel to *g* T, and let it cut T Z in Z, and join M Z. Then, because M Z falls within the tangent M H T, and that no straight line can be drawn through the point of contact between the curve and its tangent so as not to cut the curve, M Z must cut the curve P Q R in some point N. And, because VM and *g* T are sections of the parallel planes EF and AB by the plane V M T, V M is parallel to *g* T (IV. 12.); but *g* T is parallel to Y Z; therefore (IV. 6.), VM is likewise parallel to Y Z. But the point N is in the plane V M Z, that is, in the plane of V M and Y Z. Therefore, if V N be joined, and produced, it will cut the straight line Y Z in some point *n*; and, because the point *n* is also in the plane A B, it is the projection of N, and therefore a point in the curve *p q r*, which is the projection of M P Q R. Also, because Y Z is parallel to *g h* T, *n* is at the same distance from *g h* T (I. 16.) that Y is, that is, at a less distance than the given distance D. Therefore, *g h* T being produced may be made to approach nearer to the curve *p q r* than by any given distance. And it has been shown that it never meets the curve, to whatever extent it is produced. Therefore, it is an asymptote to the curve *p q r* (def. 6.).

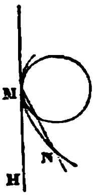
Therefore, &c.

Cor. 1. In that part of the proposition which relates to the tangent at a point M in the vertical plane; it is supposed that the tangent M H does not lie in the vertical plane*; for in this case it is evident that no point of the tangent can be projected upon plane A B, and consequently there is no asymptote.

Cor. 2. The perspective projection of a straight line which cuts a curve is a straight line which cuts the projection of the curve, if the first point of intersection is without the vertical plane.

Scholium.

The assumption made in the demonstration of the foregoing proposition, viz. that "no straight line can be drawn through the point of contact between a curve and its tangent so as not to cut the curve;" or, in other words, that a curve admits of only one tangent at the same point of it, may be regarded as an axiom. That such is the case in the circle has been shown in Book iii. Prop. 2.; and hence, with regard to other curves, generally, it may be illustrated as follows:—Conceive a circle having the same tangent with the curve at the point M, and sufficiently small to fall within the curve, as in the adjoined figure. Then, since no straight line can be drawn through the point M so near to the tangent as not to cut the circumference of this circle, and since the curve lies between the circumference of this circle and the tangent; much less can any straight line be drawn so near to the tangent as not to intercept a part of the curve between itself and the tangent, and, consequently, being produced, to cut the curve.



PROP. 7.

The direction C D *and the plane of projection* A B *being given; the orthographic projection of any point* P *whatever may be found upon the plane* A B.

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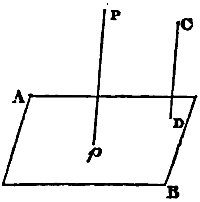
[*] It may, perhaps, appear at first, that if the tangent lies in the vertical plane, the curve must likewise lie in that plane; this, however, is not a necessary consequence; the tangent MH may be the common section of the plane of the curve with the vertical plane, and this is the case which is supposed in the corollary.

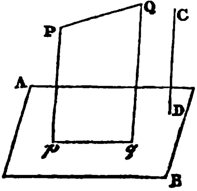


For, the direction C D not being parallel to the plane A B (def. 2.), the straight line which is drawn through P parallel to C D is not parallel to that plane; since, otherwise, the line thus drawn would be parallel to the common section of a plane passing through it with the plane A B (IV. 10.), and therefore, also (IV. 6.), C D would be parallel to the same common section, that is, to a straight line in the plane A B, for which reason C D would be parallel also to the plane A B (IV. 10.), which is contrary to the supposition. Therefore, the straight line which is drawn through P parallel to CD may be produced to meet the plane A B in some point p; and the point p thus found (def. 2.) is the orthographic projection of the point P.

Therefore, &c.

Prop. 8.

The orthographic projection of a straight line is a straight line.



Let C D be the direction, A B the plane of projection, P Q any straight line, and pq its orthographic projection: pq shall likewise be a straight line.

Because P Q is a straight line, and that the parallels to C D, which are drawn through the several points of PQ, are parallel to one another (IV. 6.), these parallels lie in one and the same plane P Q q (IV. 1. *Cor.* 2.): but (def. 2.) the points of pq lie in these parallels respectively; therefore, the points of pq lie in the plane P Q q. But they lie also in the plane AB. Therefore they lie in the common section of the planes P Q q and A B, that is, in a straight line (IV. 2.).

Therefore, &c.

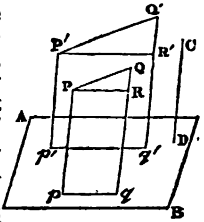
Cor. 1. It is supposed in the proposition that the original straight line P Q is not parallel to C D; for, then, it is evident that all its points have for their projections the single point in which it cuts the plane of projection.

Cor. 2. The orthographic projection of any given straight line is a part of the common section of two planes, viz. a plane which passes through the given straight line parallel to the direction CD and the plane of projection A B.

Cor. 3. The orthographic projection of a straight line, which is parallel to the plane of projection, is parallel to its original (IV. 10.).

Prop. 9.

The orthographic projections of parallel straight lines are parallel straight lines, and have the same ratio to their respective originals.



Let C D be the direction, A B the plane of projection, P Q and P′ Q′ any two parallel straight lines, and pq, $p'q'$ their respective projections: pq shall be parallel to $p'q'$.

Because Q q and Q′ q' are each of them parallel to C D (def. 2.), they are parallel to one another (IV. 6.); also P Q is parallel to P′ Q′; therefore, the plane P Q q is parallel to the plane P′ Q′ q' (IV. 15.). But pq and $p'q'$ are the respective sections of these parallel planes made by the plane of projection A B (8. *Cor.* 2.). Therefore, pq is parallel to $p'q'$ (IV. 12.).

Also, the projections pq and $p'q'$ have the same ratio to the original straight lines P Q and P′ Q′ respectively.

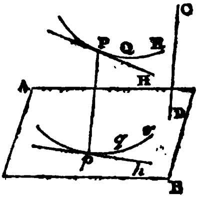
For, if P Q is parallel to pq, then, because P′ Q′ and pq are each of them parallel to P Q, P′ Q′ is parallel to pq (IV. 6.); and because $p'q'$ is likewise parallel to pq by the former part of the proposition, P′ Q′ is parallel to $p'q'$. Also, because Pp and Q q are each of them parallel to C D (def. 2.), Pp is parallel to Q q; and, for the like reason, P′ p' is parallel to Q′ q'. Therefore, the figures P$pq$Q, P′$p'q'$Q′ are, in this case, parallelograms; and, because (I. 22.) the opposite sides of parallelograms are equal to one another, pq and $p'q'$ are equal to P Q and P′ Q′ respectively; that is, the projections have the same ratio to their respective originals, viz. the ratio of equality. But, if P Q is not parallel to pq, draw P R parallel to pq to meet Q q in R, and P′ R′ parallel to $p'q'$ to meet Q′ q' in R′. Then, because P$pq$R and P′ p' q' R′ are parallelograms, P R and P′ R′ are equal to pq and $p'q'$ respectively (I. 22.). But, because P R and P′ R′ are parallel to pq and $p'q'$ respectively, and that pq is parallel to $p'q'$, P R is parallel to P′ R′ (IV. 6.). Therefore, the triangles P Q R, P′ Q′ R′ have the three sides of the one parallel to the three sides of the other, each to each, and (IV. 15.) are equiangular. Consequently, P R is to P Q as P′ R′ to P′ Q′ (II. 31.), that is, since $p\,q$ is equal to P R and $p'\,q'$ to P′ R′, $p\,q$ is to P Q as $p'\,q'$ to P′ Q′.

Therefore, &c.

Cor. If the original straight lines are parallel to the plane of projection, the orthographic projections are equal to their respective originals.

Prop. 10.

The orthographic projection of a curved line is a curved line; and, if a straight line touches the original curve, the projection of that straight line shall likewise touch the projection of the curve.*



Let C D be the direction, A B the plane of projection; P Q R any curved line, and $p\,q\,r$ its projection: $p\,q\,r$ shall likewise be a curved line.

For, if any part of $p\,q\,r$, as $p\,q$, be a straight line, then, because P Q is the orthographic projection of $p\,q$ (def. 2.) upon the plane P Q R, P Q must likewise be a straight line (8.), which is contrary to the supposition. Therefore, no part of $p\,q\,r$ is a straight line, that is, (I. def. 6.) $p\,q\,r$ is a curved line.

Next, let the straight line P H touch the curve P Q R in the point P, and let $p\,h$ be the orthographic projection of P H: $p\,h$ shall touch the curve $p\,q\,r$ in p.

For, CD being parallel to Pp (def. 2.), straight lines which are parallel to C D are parallel to Pp (IV. 6.), and therefore (IV. 10.) parallel to the plane H Pp. Also, the points of the curve P Q R on both sides of P fall, all of them, without and to the same part of the tangent PH. Therefore, the parallels to C D or Pp, which pass through these points likewise fall without and to the same part of the plane H P p. But these parallels pass through the corresponding points of the projection $p\,q\,r$ (def. 2.). Therefore the points of $p\,q\,r$, on both sides of p, lie without and to the same part of the plane H Pp, and consequently also without and to the same part of the straight line $p\,h$ which is in that plane (8. *Cor.* 2). Therefore, $p\,h$ meets the curve $p\,q\,r$ in p, but does not cut it, that is, $p\,h$ touches the curve $p\,q\,r$.

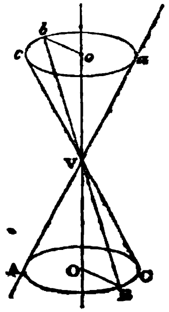
Therefore, &c.

Cor. 1. It is supposed in the proposition that the plane of the original curve is not parallel to the direction C D; for, then, it is evident that the projection of the curve is a straight line, and that the projection of the tangent is confounded with (or, if parallel to the direction C D (8. *Cor.* 1.) is only a point in) the projection of the curve.

Cor. 2. The orthographic projection of a straight line which cuts any curve is a straight line which cuts the projection of that curve.

Part II.—*Of the Plane Sections of the Right Cone, or Conic Sections.*

It is easy to perceive that every section of a right cone which is made by a plane passing through the vertex is rectilineal, and, again, that every section which is made by a plane parallel to the base is a circle. The former follows from the definition (V. def. 4.) of a cone; the latter will be demonstrated at large hereafter (in prop. 11.). But, if a right cone be cut by a plane which neither passes through the vertex nor is parallel to the base, the section will be neither rectilineal nor circular; but will, according to the position of the cutting plane, take one of the three forms mentioned in the following definitions.



Def. 7. If the slant sides of a right cone are produced upwards through the vertex, the produced parts will, it is evident, lie in the surface of another right cone which has the same vertex, and its axis lying in the same straight line with the axis of the first. This cone, with reference to the first, is called the *opposite cone*, and its surface the *opposite surface.*

The two opposite surfaces, infinitely produced, are to be considered as constituting one *complete conical surface;* which may be conceived to be generated by the revolution of a slant side infinitely produced both ways about the axis of the cone.

8. If a complete conical surface is cut by a plane which neither passes through the vertex nor is parallel to the base, the curved line in which such plane cuts the surface is called a *conic section**.

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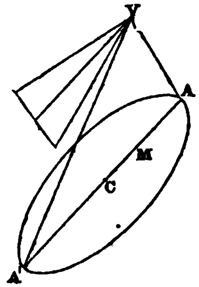
* See note, page 210.

* The plane sections which are here excepted, viz. the straight line and circle, are likewise sometimes called conic sections, inasmuch as they likewise are plane sections of a cone: the term is, however, usually appropriated to the other plane sections, viz. the ellipse (def. 10.), the parabola (def. 11.), and the hyperbola (def. 12.).

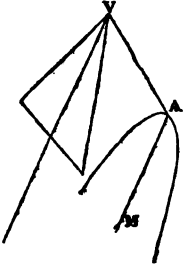
9. The *vertical plane* of a conic section is a plane which passes through the vertex of the cone parallel to the plane of the conic section.

10. If the vertical plane of a conic section falls entirely without the surface, the conic section surrounds the lower cone, and is called an *ellipse*.

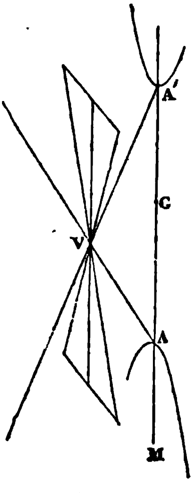


The reason why the section in this case surrounds the lower cone is, that the lower surface only is intercepted between the planes; and because every slant side of this surface cuts the vertical plane in V, it will, if produced, cut also the plane of the conic section which is parallel to the vertical plane (1. *Cor.*).

11. If the vertical plane touches the conical surface in a slant side, the conic section has two infinite arcs, and is called a *parabola*.



The section has in this case two infinite arcs, because one slant side of the cone lies in the vertical plane, and therefore can never be produced to meet the plane of the conic section; it is still, however, only the lower surface which is intercepted between the planes, and because every other slant side of this surface cuts the vertical plane in V, it may be produced to cut the plane of the conic section (1. *Cor.*).



12. If the vertical plane cuts the conical surface in two slant sides, the conic section has four infinite arcs, two lying in one and two in the other of the opposite surfaces, and is called an *hyperbola*.

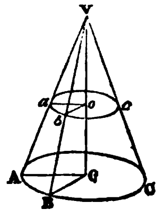
The section in this case cuts both of the opposite surfaces, because a part of each is intercepted between the vertical plane and the plane of the conic section; and because there are two slant sides in each surface which lie in the vertical plane, and therefore cannot be produced to meet the plane of the conic section, the section has two infinite arcs in each surface corresponding to them.

These curves, or the *conic sections* properly so called, different as they are in form, the first a complete figure inclosing an area, the second having two infinite arcs, the third four, are nevertheless very nearly related to one another in their properties, many of which bear a striking analogy to the properties of the circle. Thus, "if, in any conic section two chords are drawn which cut (or are produced to cut) one another, and other two chords parallel to the former respectively which likewise cut one another, the rectangles contained under the segments of the former two shall have the same ratio to one another as the rectangles which are contained under the segments of the latter two;" a property which we have seen (III. 20.) obtains in the circle, the ratio in this case being always that of equality. It is proposed in the present part of the Appendix to demonstrate a few of these properties, among them the one just stated; and it will be found that the demonstrations are considerably aided and abridged by help of the principles laid down in the preceding part.

Prop. 11.

Every section of a right cone which is made by a plane parallel to its base, is a circle having its centre in the axis of the cone.

Let V be the vertex of a right cone, VO its axis, and A B C its base; and let *a b c* be a section which is made by any plane parallel to the base A B C: the section *a b c* shall be a circle having its centre in the axis V O.

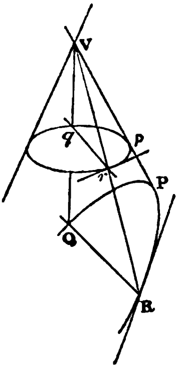


Let the plane *a b c* cut the axis of the cone in the point *o*: in the curve, or circumference, *a b c*, take any two points *a*, *b*; join V *a*, V *b*, and produce them to meet the circumference of the circle A B C in the points A, B respectively, and join *o a*, *o b*, O A, OB. Then, because *o a* and O A are sections of parallel planes, by the plane V O A, *o a* is parallel to O A (IV. 12.), and consequently (II. 30. *Cor.* 2.), *o a* is to O A as V *o* to V O. And, in the same manner, it may be shown that *o b* is to O B as V *o* to V O. Therefore (II. 12.) *o a* is to OA as *o b* to O B ; and, because OA is equal to O B, *oa* is equal to *o b* (II. 18. *Cor.*). In like manner, if *c* be any other point in the circumference *a b c*, and if *o c* be joined, it may be shown that *o c* is equal to *o a* or *o b*. Therefore, every point in the circumference *a b c* is at the same distance from the point *o* ; that is (I. def. 24.), *a b c* is a circle of which *o* is the centre.

Therefore, &c.

PROP. 12.

Every conic section Q P R *is the perspective projection of a circular section* q p r, *upon the plane of the conic section, by straight lines drawn from the vertex* V ; *and the vertical plane of such perspective projection is the vertical plane of the conic section.*



For, every straight line which is drawn from V through a point of the circumference *p q r* to meet the plane of the conic section, meets that plane in some point of the conic section ; and there is no point of Q P R which is not in a straight line with V and some point of *q p r* ; therefore (def. 1.), Q P R is the perspective projection of *q p r* by straight lines drawn from V. And, because the vertical plane of the conic section Q P R is parallel to the plane Q P R (def. 9.), that vertical plane is also the vertical plane of projection (def. 1.).

Therefore, &c.

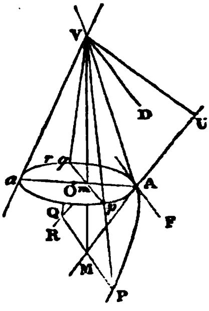
Cor. 1. In like manner, also, every circular section *q p r* may be considered as the perspective projection of the conic section Q P R by straight lines drawn from the vertex V.

Cor. 2. The projection of every point in the conic section may be found in the circular section, whether it be an ellipse, or a parabola, or an hyperbola (1.) : for the plane which passes through V parallel to the circular section, falls entirely without the cone, so that no point of the conic section is found in that plane.

Cor. 3. And so, the projection of every point in the circular section may be found in the conic section ; except, in the case of the parabola, the projection of the point in which the vertical plane touches the circular section, and except, in the case of the hyperbola, the projections of the two points in which the vertical plane cuts the circular section.

PROP. 13.

Every conic section is symmetrically divided by a straight line, which is the common section of the cutting plane, and a plane which passes through the axis of the cone perpendicular to the cutting plane.

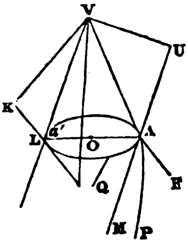


Let V be the vertex, and V O the axis of the cone, and let PQR be the conic section ; from V draw VU, perpendicular to the plane P Q R, and let the plane UVO, which passes through V O, and (IV. 18.) is perpendicular to the plane PQR, cut the latter plane in the straight line A M : the conic section P Q R shall be divided symmetrically by the straight line A M.

Through the point A let there be drawn a plane perpendicular to the axis V O, and let it cut the cone in the circular section *p q r*, having the centre O (11.), and the plane Q P R in the straight line A F (IV. 2.) ; through V draw VD parallel to AF (I. 48.). Take any point P in the conic section, join V P, and let the plane D V P cut the planes of the conic section and circle in the straight lines P Q, and *p q* respectively (IV. 2.) ; also, let these straight lines cut A M, A O in the points M, *m* respectively. Then, because VD is parallel to AF, it is parallel (IV. 10. *Cor.* 1.) to PQ and to *p q*, which are the common sections of planes passing through A F with the plane D V P which passes through D V : therefore, also, P Q and *p q* are parallel to one another (IV. 6.). Now, because the plane *p q r* of the circle is perpendicular to the axis V O, it is perpendicular to the plane U V O, which passes through VO (IV. 18.) ; and the plane P Q R is by the construction perpendicular to the same plane U V O: therefore A F, the common section of the planes *p q r*, P Q R, is perpendicular to the same plane (IV. 18. *Cor.* 2.), and, consequently (IV. def. 1.), perpendicular to each of the straight lines A M, A O which meet it in that plane. Therefore, because P Q is parallel to A F, P Q is perpendicular to A M (I. 14.); and, for the like reason, *p q* is perpendicular to A O. And, because the chord *p q* of the circle *p q r* is perpendicular to the radius A O, it is bisected by A O in the point *m* (III. 3.); but P Q is parallel to *p q;* therefore (II. 30.) P Q is bisected by A M in M. And in the same manner it may be shown, that all other straight lines which are drawn in the conic section parallel to A F, or (which is the same thing) perpendicular to AM, are bisected by A M. Therefore, the straight line A M divides the conic section P Q R symmetrically.

Therefore, &c.

Cor. 1. The straight line A F, to which the bisected chords are parallel, touches both the circular section and the conic section at the point A.



Cor. 2. The straight line A M, which divides a parabola symmetrically, is parallel to the slant side V L, in which the vertical plane of the parabola touches the surface of the cone.

For if the vertical plane cuts the plane of the circle *p q r* in the straight line K L, K L touches the circle, and is therefore (III. 2.) perpendicular to the radius O L; and A F was shewn in the proposition to be perpendicular to A O; also, K L is parallel to A F, because they are sections of parallel planes by the plane of the circle (IV. 12.); therefore (I. 14.) O L and O A are in the same straight line, and V L is in the plane V A O, that is, in the same plane with A M. And, because V L and A M are sections of parallel planes by the same plane, they are parallel to one another.

Def. 13. The straight line A M, which divides a conic section symmetrically, is called the *axis* of the conic section.

14. In the ellipse and hyperbola, the axis A M cuts the curve in two points, A, A′; these points are called the *principal vertices* of the ellipse or hyperbola, and the straight line A A′, which is intercepted between them, is called the *principal diameter* or *transverse axis* of the ellipse or hyperbola.

In the parabola, the axis cuts the curve in one point, A, only; and this point is called the *principal vertex* of the parabola.

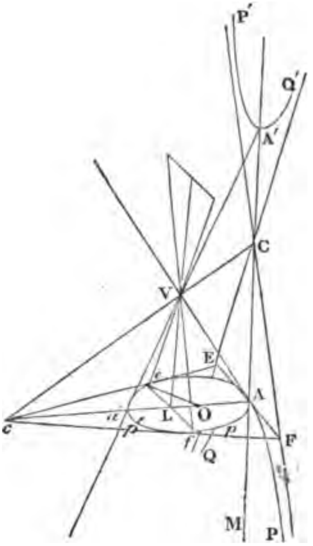
15. The *centre* of an ellipse or hyperbola, is the middle point C of the transverse axis A A′.

A parabola has no centre.

See the figures of defs. 10, 11, 12.

Prop. 14.

The hyperbola has two asymptotes, which pass through the centre, and make equal angles with the axis upon opposite sides of it.



Let V be the vertex of the cone, V O its axis, V A and V A′ the slant sides in which the conical surface is cut by a plane perpendicular to the plane of the hyperbola, A and A′ the principal vertices of the hyperbola, A *e f* the circular section passing through the point A, O its centre, A O *a′* its diameter, and E A F the common section of the planes of the circle and hyperbola which, as has been already seen (13. *Cor.* 1.), touches both the circle and the hyperbola in the point A; also, let the vertical plane of the hyperbola cut the circular section A *e f* in the chord *e f*, and at the points *e, f,* let the straight lines *c* E, *c* F be drawn touching the circle; let cE and cF meet one another in c and the straight line EF in the points E and F respectively, and let the chord ef cut the diameter AOa' in L. Then, because ef and EF are sections of parallel planes by the plane of the circle Aef, ef is parallel to EF (IV. 12.); but EF is perpendicular to the diameter AOa', because it touches the circle in A (III. 2.); therefore ef is perpendicular to the same diameter (I. 14.), and consequently (III. 3.) is bisected by it in the point L; also, because ce, cf, are tangents drawn from c to the circle Aef, they are equal to one another (III. 2. *Cor.* 3.), and the triangle cef is an isosceles triangle: therefore the straight line AOa', which bisects ef at right angles, passes through the point c (I. 6. *Cor.* 3.). Join cV, and produce it to cut AA' in C (I. 14. *Cor.* 3.), and from C, through the points E and F, draw the straight lines CE, CF: the point C shall be the centre of the hyperbola, and the straight lines CE, CF produced, shall be asymptotes, making equal angles with the axis AA' upon either side of it.

Join Oe and VL. Then, because ce touches the circle in e, the triangle Oec is right-angled at e (III. 2.); and it has been shown that eL is perpendicular to the opposite side Oc; therefore, Oe or Oa' is a mean proportional between OL and Oc (II. 34. *Cor.*). But OA is equal to Oa'. Therefore, the straight line Ac is harmonically divided in the points L and a' (II. 46.), and, consequently (II. def. 20. page 68.), the four straight lines VA, VL, Va', and Vc, are harmonicals. But, because VL and AA' are sections of parallel planes by the plane VAA', AA' is parallel to VL (IV. 12.). Therefore (II. 49. *Cor.* 1.), AA' is bisected by Vc produced, and the point C is the centre of the hyperbola (def. 15.).

Again, because the hyperbolic arcs AP, $A'P'$, are the projections of the circular arcs Apf, $a'p'f$, by straight lines drawn from V (12.), and that the point f of the arcs Apf, $a'p'f$, lies in the vertical plane Vef; the projection of the tangent cf, that is, the straight line CF produced, is an asymptote to the hyperbolic arcs AP, $A'P'$ (6.). And, for the like reason, CE produced is an asymptote to the hyperbolic arcs AQ, $A'Q'$.

Lastly, because CE and Ve are sections of parallel planes by the plane CVe, CE is parallel to Ve (IV. 12.), and, for the like reason, CF is parallel to Vf; and it has been already shown that AA' or CA is parallel to VL; therefore the angles ECA and FCA are equal to the angles eVL and fVL respectively (IV. 15.); but, because the slant sides Ve, Vf, are equal to one another (IV. 8.), and that ef is bisected in L, the triangles VeL, VfL, have the three sides of the one equal to the three sides of the other, each to each, and, consequently, the angle eVL equal to the angle fVL (I. 7.); therefore, also, the angle ECA is equal to the angle FCA, that is, the angles which CE and CF make with the axis AA' upon either side of it, are equal to one another.

Therefore, &c.

Cor. 1. The asymptotes of the hyperbola are parallel respectively to the slant sides in which the vertical plane of the hyperbola cuts the surface of the cone.

Cor. 2. And as the existence of asymptotes to the hyperbola has been inferred from Proposition 6, so from *Cor.* 1., of the same proposition, it follows that the infinite arcs of the parabola do not admit of asymptotes; because the tangent of the circle at the point a' (see the figure of 13. *Cor.* 2.) lies in the vertical plane of the parabola.

Prop. 15.

A conic section cannot be cut by a straight line in more than two points; and, if a straight line touches a conic section, it shall meet it in one point only, viz. the point of contact.

For, the circular section being considered (12. *Cor.* 1.) as the projection of the conic section, the projection of any point, P, of the conic section, may be found in the circular section (12. *Cor.* 2.); and the projection of a straight line which cuts any curve in a point that may be projected, is a straight line which cuts the projection of the curve in the projection of that point (6. *Cor.* 2.); therefore, if it were possible that a straight line could cut a conic section in more points than two, the straight line which is its projection (2.) would cut the circular section in more points than two, which is impossible (III. 1.).

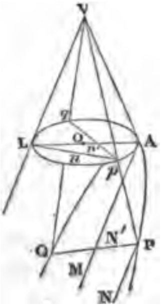
And, in the same manner, we may reason with regard to the tangent. If it were possible that a straight line which touches a conic section should meet it in any other point besides the point of contact; then, because the projections of the point of contact and any such other point may, both of them, be found in the circular section (12. *Cor.* 2.), and because the projection of a straight line which touches a curve in a point that may be projected is a straight line which touches the projection of that curve (6.), the projection of such tangent would be a straight line touching the circular section, and meeting it also in another point besides the point of contact, which is impossible (III. 2.).

Therefore, &c.

Prop. 16.

A straight line which is parallel to the axis of a parabola cannot cut the parabola in more than one point; and a straight line which is parallel to either of the asymptotes of an hyperbola cannot cut the hyperbola in more than one point; but, with these exceptions, if a straight line cuts a conic section in any point, it may be produced to cut it in a second point.

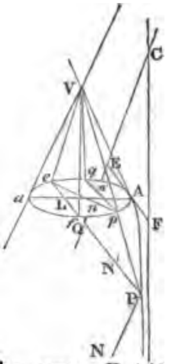
Let PAQ be a parabola, AM its axis, and PN any straight line which is parallel to the axis AM, and cuts the parabola in P: PN shall not cut the parabola in any other point, to whatever distance it may be produced.

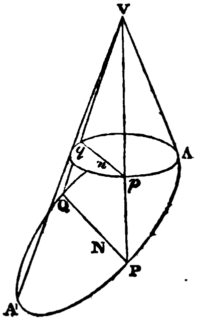


Let V be the vertex of the cone, VL the slant side in which the vertical plane of the parabola touches the surface of the cone, and $A p L q$ the circular section passing through A. Then, because VL (13. *Cor.* 2.) and PN are each of them parallel to AM, VL is parallel to PN (IV. 6.). And, because VL is parallel to PN, the projection $p n$ of the straight line PN on the plane $A p L q$ will, if produced, pass through the point L (2. *Cor.* 4), and the projection of every point in PN produced will, it is evident, be found between p and L, that is, within the surface of the cone. Therefore, every point of PN produced falls within the surface of the cone, and consequently, since the curve of the parabola is always in the surface of the cone, PN cannot be produced to meet the parabola in a second point.

But, if PN′ be not parallel to the axis of the parabola, then neither is it parallel to VL, and consequently the projection $p n'$ does not pass through the point L. But, because PN′ cuts the parabola in P, $p n'$ likewise cuts the circle in p (6. *Cor.* 2.). Therefore, $p n'$, if produced, will cut the circle in a second point q different from the point L. And, because the projection of every point, except L, may be found in the parabola (12. *Cor.* 3.), Q the projection of q may be found, and consequently PN′ produced will meet the parabola in the point Q.

And the same reasoning, in every respect, applies to the case of the hyperbola. For, if a straight line PN be parallel to one of the asymptotes CE, it will also be parallel to the slant side Ve, to which (14. *Cor.* 1.) the asymptote is parallel, and, therefore, its projection $p n$ passing through the point e, every point of PN produced will fall within the surface of the cone. But if a straight line PN′ be not parallel to either of the asymptotes, its projection $p n'$ will not pass through either of the points e, f, but, because PN′ cuts the hyperbola, $p n'$ cuts the circle in p, and will cut it if produced in some other point q, the projection Q of which may be found in the hyperbola, and, consequently, PN produced will meet the hyperbola in a second point Q.

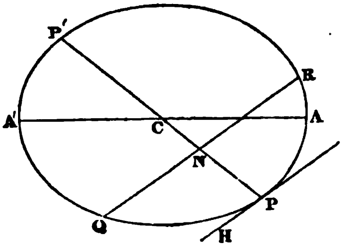


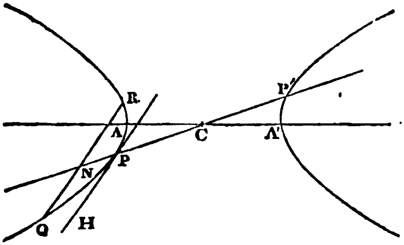


In the case of the ellipse, since the projection $p n$ of any straight line PN which cuts the ellipse must cut the circle (6. *Cor.* 2.), $p n$ may be produced to meet the circumference in a second point q as before; and since the projection of every point in the circle may be found in the ellipse (12. *Cor.* 3.) Q the projection of q may be found, and, consequently, PN produced will meet the ellipse in a second point Q.

Therefore, &c.

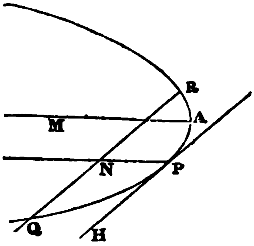
Def. 16. In the ellipse and hyperbola, any straight line PP′ which is drawn through the centre C, and terminated





both ways by the curve, is called a *diameter;* and the points P, P′ in which it meets the curve are called the *vertices* of that diameter.

In the parabola, any straight line PN, which is drawn parallel to the axis AM, is called a *diameter*, and the point P in



which it cuts the parabola is called the *vertex* of that diameter.

17. If at the vertex P of any diameter of a conic section, there be drawn a tangent P H, any straight line Q R which is parallel to the tangent P H and is terminated both ways by the conic section, is called an *ordinate* of that diameter.

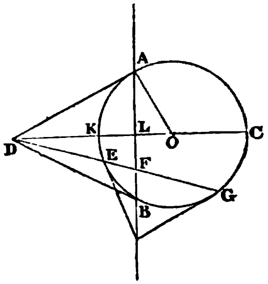
18. In the ellipse and hyperbola, the segments P N, N P′, into which a diameter or a diameter produced is divided by any of its ordinates, are called the *abscissæ* corresponding to that ordinate.

In the parabola, the part of the diameter P N, which is intercepted between the vertex P and the ordinate Q R is called the *abscissa* of that ordinate.

Lemma.

If from any point without a circle two straight lines are drawn to touch the circle, every straight line which is drawn through that point to cut the circle shall be harmonically divided by the circumference and the chord joining the points of contact; and the tangents at the points in which every such straight line cuts the circumference shall meet one another in the chord produced.

Let A B C be a circle, and D any point without it, from which the straight lines D A, D B are drawn to touch the circle in the points A, B respectively: then, if D E F G be any straight line passing through D, and cutting the circumference in the points E, G and the chord A B in the point F, D G shall be divided harmonically in E and F, and the tangents at the points E, G shall meet one another in some point of A B produced.



Take O the centre of the circle, and draw the straight line D O cutting the circumference in the points K, C, and the chord A B in the point L; and join O A. Then, because D A touches the circle, the angle D A O is a right angle (III. 2.); also, because (III. 2. *Cor.* 3.) the tangents D A, D B are equal to one another, D is the centre of a circle passing through the points A, B, and, consequently, D O bisects A B at right angles (III. 3. *Cor.* 3.); therefore, because from the right angle A of the triangle DAO, AL is drawn at right angles to the hypotenuse D O, OA or O K is a mean proportional between OL and OD (II. 34. *Cor.*) But O C is equal to O K. Therefore (II. 46.), D C is harmonically divided in the points K and L. And because D C, the diameter produced of the circle ABC, is harmonically divided, every straight line D G which passes through D is harmonically divided by the circumference and the perpendicular A B which passes through L (III. 52. *Cor.*); and the tangents at the points E, G in which every such straight line cuts the circle meet one another in A B produced (III. 53. *Cor.*).

Therefore, &c.

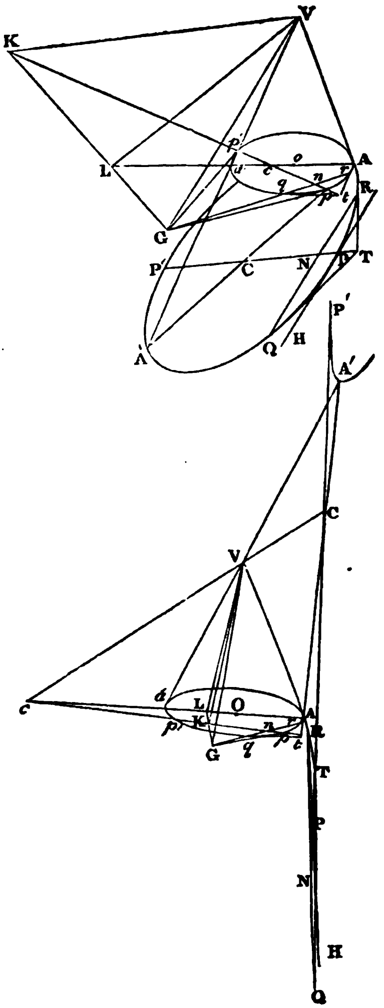
Prop. 17.

Every diameter of a conic section bisects its ordinates; and, if the conic section is an ellipse or an hyperbola, the diameter is itself bisected by the centre.

Let A P Q be any conic section, A M* its axis, P N any diameter, P H a tangent at P, Q R a parallel to P H, and therefore an ordinate to the diameter PN (def. 17.), and let QR cut PN in N: QR shall be bisected in N.

Let the vertex V of the cone, the circle A $p\,q$, its centre O, and diameter A $o\,a'$, as also the line AF*, which is the common section of the planes A P Q, A$p\,q$, remain as in the foregoing propositions; let the lines $q\,r$ and $p\,n$ in the circle be the projections of Q R and P N in the conic section, and the tangent $p\,h$* (6.) the projection of the tangent P H; and let the vertical plane of the conic section cut the plane A $p\,q$ of the circle in the line KG, which cuts Aa' or Aa' produced in L, and join V L. Then, because V L and A M are sections of parallel planes by the same plane, V L is parallel to A M (IV. 12.); and, for the like reason, K G is parallel to AF, and therefore (III. 2.) perpendicular to Aa'.

And first, in the ellipse or hyperbola, let A′ be the other principal vertex and C the centre, join V C*, and let V C or V C produced cut Aa' or Aa' produced in c. Then, because A A′ is parallel to V L, and bisected by V C in C, the four straight lines V L, V A′, V C, V A are harmonicals (II. 49. *Cor.* 2.). Therefore, A L or A L produced is divided by these straight lines harmonically (II. 49.); and, because the mean Aa' is bisected in O, O L, Oa' and Oc are proportionals (II. 46.). Let the planes V P P′, V P H cut the vertical plane in the lines VK, V G, which meet the line K G in the points K, G respectively. Then, because VK and P P′ are sections of parallel planes by the plane V P P′, V K is parallel to P P′ (IV. 12.), and, for the like reason, V G is parallel to P H: therefore, because the straight lines V K, V G are parallel to the original straight lines PP′, P H respectively, and meet the plane of the circle p A q in the points K, G respectively, the projections $p\,p'$, $p\,h$ will, if produced, pass through the points K, G respectively (2. *Cor.* 4.), and, for the like reason, the projection $q\,r$ of the ordinate QR, which is parallel to P H or V G, will also pass through the point G.



Join Gp'*. Then, because $p\,p'$ is a chord of the circle p A q passing through the point c, and that L G is drawn perpendicular to the diameter O c produced from a point L so taken that the diameter produced is divided harmonically, the tangents at the extremities of the chord *p*.*p*′ will meet one another in some point of the line LG (III. 53. *Cor.*); but the straight line G *h p* is the tangent at *p*: therefore G*p*′ is the tangent at *p*′. And, because from the point G, in which the tangents G*p*, G*p*′ meet one another, the line G*q* is drawn to cut the circumference in the points *q*, *r* and the chord *p p*′ in *n*, G *q* is harmonically divided in these points (*Lem.*). Therefore (II. def. 20.) the four straight lines V G, V *r*R, V *n*N, V *q* Q are harmonicals; and, consequently, because Q R is parallel to V G, it is bisected by V N in N (II. 49. *Cor.* 1.).

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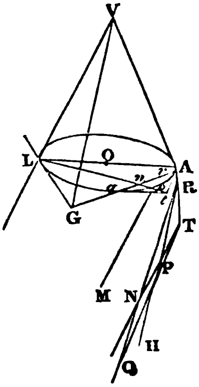
* The line VC is wanting in the upper figure, and VN, VP, VP′, VQ, VR, and VT, are wanting in each of the figures of this proposition. It has been attempted, however, to supply the assistance which might have been derived to the conception from the visible presence of these lines by placing the corresponding letters n, p, p', q, r, and t, in the straight lines VN, VP, VP′, VQ, VR, and VT respectively. The line AF and the letter F; in the two first figures, the letters M and h; in the second the line Gp', and in the third the line Ghp, have likewise been omitted.



In the parabola, the point L coincides with *a*′, and because the diameter P N is parallel to AM, that is (13. *Cor.* 2.) to V L, its projection *pn* passes through L. Therefore, because G*q* is drawn from the intersection G of the tangents G L and G *h p**, G *q* is harmonically divided by the circumference and the chord L*p* (*Lem.*), and Q R is bisected in the point N, as before.

But, further, in the ellipse and hyperbola, the diameter P P′ is bisected by the centre C. For, since K *p* passes through the point *c*, and is cut by the straight line K L, which is drawn perpendicular to the diameter O*c* from a point L so taken that O *c* produced is harmonically divided, K*p* is harmonically divided by K L and the circumference (III. 52. *Cor.*). Therefore, the four straight lines V K, V *p*′P′, V *c*C, V *p* P, are harmonicals (II. def. 20.); and, because P P′ is parallel to V K, it is bisected by V C in C (II. 49. *Cor.* 1.).

Therefore, &c.

Cor. 1. In the ellipse and hyperbola, the tangents at the extremities P, P′ of any diameter are parallel to one another. For they are the originals of G*p* and G*p*′ in the plane of the circular section; and all straight lines, the projections of which pass through G, are parallel to V G and to one another (IV. 10.).

Cor. 2. If any diameter of a conic section bisects a straight line which is not a diameter, the bisected straight line shall be an ordinate of the diameter by which it is bisected.

Prop. 18.

In every conic section the tangents at the extremities of any ordinate, Q R, *meet the diameter,* P N, *in the same point,* T; *and that in such a manner, that, in the ellipse and hyperbola,* C N, C P, *and* C T *are proportionals, and, in the parabola,* N P *is equal to* P T.

Let the construction remain as in the last proposition. Then, in the ellipse and hyperbola, because *q r*, the projection of Q R, passes through G, and that G*p*, G*p*′ are tangents drawn from G to the circle, the tangents at *q* and *r* meet one another in some point of *p p*′ produced (*Lem.*). But these tangents are the projections of the tangents at the points Q and R of the ellipse or hyperbola (6.), and *p p*′ produced is the projection of P P′ produced. Therefore, the tangents at Q and R meet one another in some point, T, of P P′ produced. Again, because *t q* and *t r* are tangents drawn from *t* to the circle, the line *t p* which passes through *t* is harmonically divided by *q r* and the circumference (*Lem.*). Therefore, the four straight lines V *p*′ P′, V *n* N, V *p* P, V *t* T*, are harmonicals (II. def. 20.), and divide P P′ produced harmonically (II. 49.); and, because the mean P P′ is bisected in C, C N, C P, and C T are proportionals (II. 46.).

In the parabola, because the projection *q r* of the ordinate Q R passes through G, and that G L, G*p* are tangents drawn from G to the circle, the tangents at *q* and *r* meet one another in some point *t* of L*p* produced (*Lem.*), and consequently, as before, the tangents at Q and R meet one another in some point, T, of N P produced. Again, because *t q* and *t r* are tangents drawn from *t* to the circle, the line *t p* is harmonically divided by *q r* and the circumference (*Lem.*). Therefore, the four straight lines V L, V *n*N, V *p* P, V *t* T, are harmonicals (II. def. 20.); and because N T is parallel to V L (IV. 6.), it is bisected by V P in P (II. 49. *Cor.* 1)., that is, N P is equal to P T.

When the diameter in question is the axis of the conic section, these demonstrations will be modified, and appear under a more simple form, to which they are easily reduced by substituting A M for P N, A F for P H, &c.

Therefore, &c.

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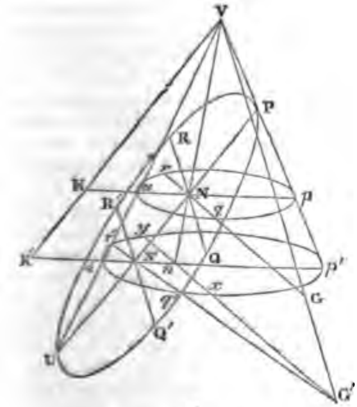
* See note, page 221.

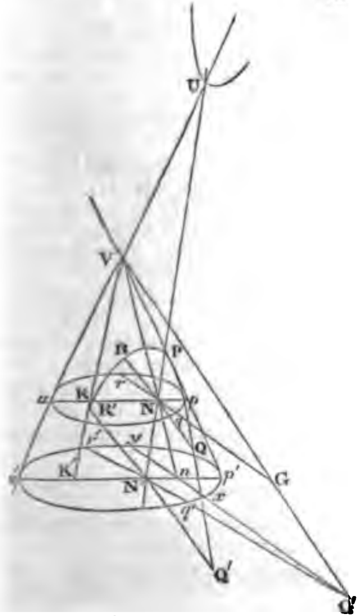
Prop. 19.

In the ellipse and hyperbola, the squares of any two semiordinates of the same diameter are to one another as the rectangles under the corresponding abscissæ: in the parabola, the squares of any two semiordinates of the same diameter, are to one another as the abscissæ.

Let P Q R be an ellipse or hyperbola, P U any diameter, and Q R, Q′ R′ any two ordinates, cutting the diameter P U in the points N, N′ respectively; the square of Q N shall be to the square of Q′ N′ as the rectangle $PN \times NU$ to the rectangle $PN' \times N'U$.

Through N draw a plane parallel to the base of the cone, and, therefore, cutting the cone in a circular section $pqur$ (11.), and let pu, qr be the projections of P U, Q R respectively, upon this plane, by straight lines drawn from the vertex V of the cone; then, because $pqur$ is a circle, the rectangle under qN, Nr is equal to the rectangle under pN, Nu (III. 20.). Through V draw VK parallel to P U to meet the plane of the circle in K, and VG parallel to QR to meet the same plane in G; then, because pu is the projection of P U, it will, if produced, pass through the point K, and, for the like reason, qr produced will pass through the point G (2. *Cor.* 4.).





Then, because V K is parallel to P U, the triangles VKp and VKu are similar to the triangles PNp and UNu respectively (I. 15.); therefore (II. 31.) $PN : pN :: VK : Kp$ and $NU : Nu :: VK : Ku$, and, consequently (II. 37. *Cor.* 3.), $PN \times NU : pN \times Nu :: VK^2 : Kp \times Ku$. And, in the same manner, because the triangles QNq, RNr are similar to the triangles VGq, VGr respectively*, it may be shown that $QN \times NR$ or (17.) $QN^2 : qN \times Nr$ or $pN \times Nu :: VG^2 : Gq \times Gr$. But, from the former proportion, *invertendo* (II. 15.) $pN \times Nu : PN \times NU :: Kp \times Ku : VK^2$. Therefore (II. def. 12.), the ratio of QN^2 to $PN \times NU$ is compounded of the ratios of VG^2 to $Gq \times Gr$ and $Kp \times Ku$ to VK^2.

Now, if through N′ there be likewise drawn a plane parallel to the base of the cone, and which, therefore, likewise cuts the cone in a circle (11.), $p'q'u'r'$, and if VK and VG are produced to meet this new plane in the points K′ and G′ respectively, the projections $p'u'$ and $q'r'$ of the diameter PU and the ordinate Q′R′ upon this plane will pass through the points K′ and G′ respectively (2. *Cor.* 4.), because PU is parallel to VK′ as before, and Q′ R′ to Q R, that is (IV. 6.), to VG′. Therefore, as before, it may be shown, that the ratio of $Q'N'^2$ to $PN' \times N'U'$ is compounded of the ratios of VG'^2 to $G'q' \times G'r'$ and $K'p' \times K'u'$ to VK'^2.

But, if xy is the projection of Q R upon the plane $p'q'u'r'$, xy produced will pass through the point G′, because Q R is parallel to VG′ (2. *Cor.* 4.): and, because $p'q'u'r'$ is a circle, $G'q' \times G'r'$ is equal to $G'x \times G'y$ (III. 20.). Also, because the triangles VKp, VKu are similar to the triangles $VK'p'$, $VK'u'$ respectively, $VK : Kp :: VK' : K'p'$ (II. 31.), and $VK : Ku :: VK' : K'u'$, and, consequently (II. 37. *Cor.* 3.), $VK^2 : Kp \times Ku :: VK'^2 : K'p' \times K'u'$, or, *invertendo* (II. 15.), $Kp \times Ku : VK^2 :: K'p' \times K'u' : VK'^2$; and, in like manner, because the triangles VGq, VGr are similar to the triangles $VG'x$, $VG'y$ respectively, $VG^2 : Gq \times Gr :: VG'^2 : G'x \times G'y$ or $G'q' \times G'r'$. Therefore the ratio of VG^2 to $Gq \times Gr$ is the same with the ratio of VG'^2 to $G'q' \times G'r'$, and the ratio of $Kp \times Ku$ to VK^2 is the same with the ratio of $K'p' \times K'u'$ to VK'^2. Therefore, because ratios which are compounded of the same ratios are the same with one another (II. 27.), the ratio of QN^2 to $PN \times NU$ is the same with the ratio of $Q'N'^2$ to $PN' \times N'U$; and, *alternando* (II. 19.), $QN^2 : Q'N'^2 :: PN \times NU : PN' \times N'U$.

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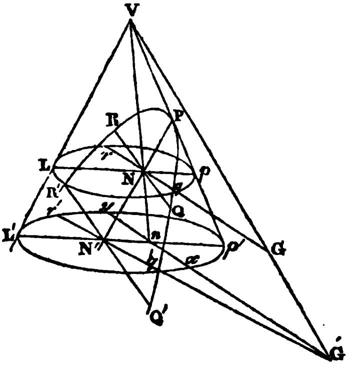
* The straight lines $VqQx$, $VRry$, $Vq'Q'$, and $VR'r'$ are omitted in each of the figures of this proposition.

In the next place, let QPR be a parabola, PN any diameter, and QR, Q′R′ any two ordinates cutting the diameter PN in the points N, N′ respectively: the square of QN shall be to the square of Q′N′ as PN to P′N′.



Through N draw a plane parallel to the base of the cone, and therefore cutting it in a circular section qpr (11.), and let pN, qr be the projections of PN, QR respectively upon this plane, by straight lines drawn from V; also, let VL be the slant side of the cone, which is parallel to the axis of the parabola (13. *Cor.* 2.), and therefore (IV. 6.) likewise parallel to the diameter PN, and let it meet the plane qpr in the point L of the circumference qpr: then, because VL is parallel to PN, pN produced passes through the point L (2. *Cor.* 4.), and, because qpr is a circle, the rectangle under qN, Nr, is equal to the rectangle under pN, NL. (III. 20.) Through V draw VG parallel to QR, to meet the plane of the circle pqr in G; then, because qr is the projection of QR, it will, if produced, pass through the point G (2. *Cor.* 4.).

Now, as in the former part of the proposition, it may be shown that $QN^2 : qN \times Nr$ or $pN \times NL :: VG^2 : Gq \times Gr$; also, because the triangles VLp, PNp are similar, $pN : PN :: Lp : VL$ (II. 31.), and consequently, since $pN \times NL$ is to $PN \times NL$ as pN to PN (II. 35.), $pN \times NL : PN \times NL :: Lp : VL$ (II. 12.); therefore the ratio of QN^2 to $PN \times NL$ is compounded of the ratios of VG^2 to $Gq \times Gr$ and Lp to VL. And in the same manner, if through N′ there be drawn a plane parallel to the base of the cone, and if VL and VG be produced to meet it in the points L′ and G′ respectively, it may be shown that the projections $p'N'$ and $q'r'$ of the diameter PN or PN′ and the ordinate QR upon this plane pass through the points L′ and G′ respectively, and accordingly that the ratio of $Q'N'^2$ to $PN' \times N'L'$ is compounded of the ratios of VG'^2 to $G'q' \times G'r'$, and $L'p'$ to VL′. But if xy is the projection of QR upon the plane of the circle $p'q'u'r'$, it may be shown, as in the former part of the proposition, that xy produced will pass through G′, and that $G'q' \times G'r'$ is equal to $Gx \times Gy$. Also, because the triangle VLp is similar to the triangle $VL'p'$, the ratio of Lp to VL is the same with the ratio of $L'p'$ to VL′ (II. 31.); and, because the triangles VGq, VGr are similar to the triangles $VG'x$, $VG'y$ respectively, the ratio of VG^2 to $Gq \times Gr$ is the same with the ratio of VG'^2 to $G'x \times G'y$ or $G'q' \times G'r'$. Therefore (II. 27.) $QN^2 : PN \times NL :: Q'N'^2 : PN' \times N'L'$, and *alternando* (II. 19.), $QN^2 : Q'N'^2 :: PN \times NL : PN' \times N'L'$. But NL is equal to N′L′ (I. 22.), and consequently (II. 35.), $PN \times NL : PN' \times N'L' :: PN : PN'$. Therefore $QN^2 : Q'N'^2 :: PN : PN'$.

The foregoing demonstrations are not applicable, in the above form, to the case in which the diameter PN is also the axis of the conic section. They become, however, much more simple when they are adapted to this particular case, and the manner in which this is to be done is obvious.

Therefore, &c.

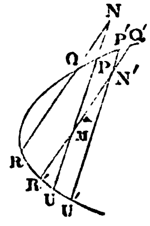
Cor. 1. In the ellipse and hyperbola, the square of the semiordinate varies as the rectangle under the abscissæ; in the parabola, the square of the semiordinate varies as the abscissa (II. 35. *Schol.*).

Cor. 2. If, in the ellipse, a diameter DZ is drawn parallel to the ordinates of the diameter P U, (see the first figure of the Scholium which follows this position,) the square of the semiordinate QN is to the rectangle $PN \times NU$ under the abscissæ, as the square of the semidiameter C P to the square of the semidiameter C D.

Cor. 3. It is not necessary, in the demonstration of the first part of the proposition, that the conic section should be an ellipse or an hyperbola, or P U a diameter having the ordinates Q R, Q′ R′; but only that P U should be a straight line cutting the conic section P Q R in two points, and Q R, Q′ R′ two parallel straight lines likewise cutting the conic section, each in two points, (in which case the part shewing that $QN \times NR$ and $Q'N' \times N'R'$ are equal to QN^2 and $Q'N'^2$, will have to be omitted,) or even one or both touching the conic section in a single point, the only difference being that in this case the points Q and Q′ coincide with the points R and R′ respectively.

Therefore, generally, if a straight line P U cuts a conic section in two points, and is cut by any two parallel straight lines which likewise cut the conic section each in two points, or one or both of them touch the conic section, the rectangle under the segments of one parallel, or its square, if it be a tangent, shall be to the rectangle under the segments of the other, or to its square if it be a tangent, in the same ratio as the rectangles under the corresponding segments of the straight line which is cut by them.

Cor. 4. And hence, in any conic section, if two straight lines, P U, Q R cut one another, and likewise other two P′ U′, Q′ R′, which are parallel to the two first respectively, and if each of them cuts the conic section in two points, or one or more touch it in a single point, the rectangle under the segments of either of the first shall be to the rectangle under the segments of its parallel as the rectangle under the segments of the remaining one of the first to the rectangle under the segments of its parallel; the square of any of the straight lines being understood instead of the rectangle under its segments, when it touches the conic section instead of cutting it.



For, let P U and Q R cut one another in N, and P′ U′ and Q′ R′ in N′; also, because Q′ R′ is parallel to Q R, let it cut P U in M (I. 14. *Cor.* 3.). Then, by the last corollary,

$QN \times NR : Q'M \times MR' :: PN \times NU : PM \times MU$, and

$Q'M \times MR' : Q'N' \times N'R' :: PM \times MU : P'N' \times N'U'$;

therefore, *ex æquali* (II. 24.), $QN \times NR : Q'N' \times N'R' :: PN \times NU : P'N' \times N'U'$.

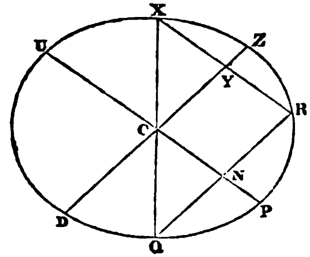
And the same may be directly inferred from the demonstration of the proposition: for the projection of P′ U′ will pass through the point K′ in the same manner as the projection of Q′ R′ passes through G.

Cor. 5. It is indifferent, also, in the second part of the demonstration, whether P Q R be a parabola and P N a diameter, or P Q R be an hyperbola, and PN a straight line parallel to one of the asymptotes, for in this case also PN will be parallel to a slant side of the cone (14. *Cor.* 1. and IV. 6.); and in either case, Q R and Q′ R′ may be any two parallel straight lines cutting the conic section each in two points, or one or both of them touching it in a single point.

Therefore, generally, if a straight line PN, which is a diameter of a parabola or parallel to one of the asymptotes of an hyperbola, be cut by any two parallel straight lines which likewise cut the parabola or hyperbola each in two points, or one (or both of them, as is possible in the case of the hyperbola) touch it in a single point; the rectangle under the segments of one parallel, or its square if it be a tangent, shall be to the rectangle under the segments of the other, or to its square if it be a tangent, in the same ratio as the parts P N, P N′ of the line P N, which are cut off by the parallels respectively.

Scholium.

The diameter D Z, which is supposed in *Cor.* 2. to be drawn in the ellipse parallel to the ordinates of the diameter P U, is said to be *conjugate* to P U. It is easy to perceive that the diameter P U is in this case likewise parallel to the ordinates of D Z: for, let Q R be any ordinate of the diameter PU, and let QC be joined and produced to meet the ellipse in X, and let R X be joined, and let it cut DZ in Y; then RX will be parallel to PU (II. 29.), because Q R and QX are bisected in N and C respectively (17.); and R X is an ordinate to the diameter D Z, (17. *Cor.* 2.) because R Y is to Y X as QC to CX, that is, in a ratio of equality. From this reciprocal relation such diameters P U and D Z are called *conjugate diameters*; and the diameter which is conjugate to the transverse axis (and therefore (13. *Cor.* 1.) perpendicular to that axis) is called the *conjugate axis** of the ellipse, for, being perpendicular to its ordinates, it divides the figure symmetrically, and therefore is a second axis of the figure†.

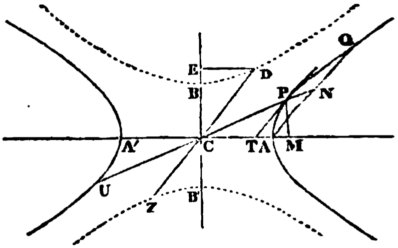


In the hyperbola, which, as we have seen, although so different in form, is very like the ellipse in its properties, there are no diameters, properly speaking, except such as lie in the angle made by the asymptotes. Let us, however, define the *conjugate diameter* of any diameter P U to be a straight line D Z, which is drawn through the centre C parallel to the ordinates of P U, is bisected in the centre, and is such that CD^2 is to CP^2 as the square of the ordinate QN to the rectangle under the abscissæ P N, N U. Such a straight line D Z will, it is evident, as in the ellipse, bisect all straight lines which are drawn parallel to the diameter P U, and terminated by the hyperbola. But there is yet a more striking analogy, to which we are led by observing that, in the ellipse, the curve in which the extremity of a conjugate diameter so defined lies, is a part of the ellipse itself, or, as we may consider it, an ellipse having the same centre and axes with the given ellipse.

It might be expected that the same, or something similar, would obtain in the hyperbola; and such, we are about to demonstrate, is the case; viz. that "the locus of the extremities of all the conjugate diameters of a given hyperbola is an hyperbola which has the same centre and the same axes as the first."

To demonstrate this:

Let C B be drawn parallel to the tangent at the principal vertex A, and therefore perpendicular to the transverse axis A′ C A (13. *Cor.* 1.): let P be any point in the hyperbola, and draw PM likewise parallel to the tangent at A, to meet CA produced in M, so that P M is a semi-ordinate to the transverse axis A′ C A (17. and def. 17.); take C B such that CB^2 shall be to CA^2 as PM^2 to $AM \times MA'$, and make CB′ equal to CB, so that, according to the above definition, B B′ is the conjugate axis of the hyperbola. Let P T be drawn touching the hyperbola in P to meet C A in T; through A draw A Q parallel to P T, and therefore (def. 17.) an ordinate to the diameter P U, by which it is consequently bisected (17.) in the point of intersection N; through C draw C D parallel to P T, and take C D such that CD^2 shall be to CP^2 as QN^2 to $PN \times NU$, and make C Z equal to C D, so that D Z is the diameter which is conjugate to the diameter C P. The points D, Z shall lie in an hyperbola which has B B′ for its transverse axis and A A′ for its conjugate axis.



From D draw D E perpendicular to C B produced. Then, because the sides of the triangles C D E, P T M are parallel, each to each, those triangles are similar (I. 18.): therefore, CE : PM :: CD : PT (II. 31.), and, consequently

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* The conjugate axis of the ellipse being always less than the principal or transverse axis, the former is frequently called the *minor* axis, and the latter the *major* axis of the ellipse.

† There is, however, no other straight line which divides the figure symmetrically, that is, no third axis. For, if C Q be joined in the first figure of prop. 17, and if P C P′ be supposed to represent the transverse axis, then if it were possible that C Q could divide the figure symmetrically, or (which is the same thing) bisect its ordinates at right angles, C Q T would be a right angle, and, consequently, because Q N is perpendicular to C T, C Q would be a mean proportional between C N and C T, (II. 34. Cor.) and therefore equal to C P (18.), so that QN^2 would be to $CP'^2 - CN^2$ or $PN \times NP'$ in a ratio of equality, and consequently (19.) the square of every other semi-ordinate of the axis would be equal to the rectangle under its abscissæ, and the figure would be a circle, not an ellipse.

(II. 37. *Cor.* 4.), $CE^2 : PM^2 :: CD^2 : PT^2$. Now, CD^2 is to PT^2 in a ratio which is compounded of the ratios of CD^2 to QN^2, and QN^2 or NA^2 to PT^2. And because, by supposition, $CD^2 : CP^2 :: QN^2 : PN \times NU$ or (I. 34.) $CN^2 - CP^2$ *alternando* $CD^2 : QN^2 :: CP^2 : CN^2 - CP^2$; also, because NA is parallel to PT, $NA^2 : PT^2 :: CN^2 : CP^2$ (II. 30. *Cor.* 2. and II. 37. *Cor.* 4.); therefore, the ratio which is compounded of the ratios of CD^2 to QN^2 and NA^2 to PT^2 is the same with the ratio which is compounded of the ratios of CP^2 to $CN^2 - CP^2$ and CN^2 to CP^2, the same, that is, with the ratio of CN^2 to $CN^2 - CP^2$, that is, again, since (II. 29.) $CN : CP :: CA : CT$, the same with the ratio of CA^2 to $CA^2 - CT^2$ (II. 37. *Cor.* 4. and II. 20. *Cor.* 1.). Therefore, $CE^2 : PM^2 :: CA^2 : CA^2 - CT^2$ (*a*).

Again, because PM is a semiordinate of the diameter AA' to which BB' is conjugate, $PM^2 : BC^2 :: AM \times MA'$ or $CM^2 - CA^2 : CA^2$. Therefore, combining this with the proportion (*a*), *ex æquo perturbato*, $CE^2 : BC^2 :: CM^2 - CA^2 : CA^2 - CT^2$: and, consequently, because CM, CA, CT are proportionals (18.), $CE^2 : BC^2 :: CM^2 : CA^2$ (II. 22.). And hence, *dividendo*, $CE^2 - CB^2 : CB^2 :: CM^2 - CA^2 : CA^2$ (*b*).

But, because the triangles CDE, PTM are similar, $DE^2 : MT^2 :: CE^2 : PM^2$ (II. 31. and II. 37. *Cor.* 4.); therefore, by the proportion (*a*), DE^2 is to MT^2 as CA^2 to $CA^2 - CT^2$, that is, since CM, CA, CT are proportionals (18.), as CM to $CM - CT$ or MT (II. 37. *Cor.* 2. and II. 20. *Cor.* 1.), or as $CM \times MT$ to MT^2 (II. 35.); therefore, DE^2 is equal to $CM \times MT$, or, since MA, MT, MC, MA' are proportionals (II. 47.), to $AM \times MA'$ (II. 38.), that is (I. 34.) to $CM^2 - CA^2$.

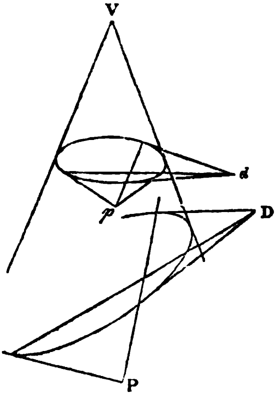
Therefore, by the proportion (*b*), $DE^2 : CA^2 :: CE^2 - CB^2 : CB^2$; and *alternando* (II. 19.) $DE^2 : CE^2 - CB^2$ or (I. 34.) $BE \times EB' :: CA^2 : CB^2$, which shows that the point D is in an hyperbola which has the transverse axis BB' and the conjugate axis AA'. (19.) And, because CZ is equal to CD, and that the diameters of an hyperbola are bisected by the centre, the point Z is in the same hyperbola.

This hyperbola, which has for its transverse axis the conjugate axis of the given hyperbola, is called the *conjugate hyperbola*. Thus the two hyperbolas are mutually *conjugate*, each to the other, and each of them is the locus of the vertices of all the conjugate diameters of the other.

Prop. 20.

If through any point taken within or without a conic section, there are drawn any number of straight lines, each cutting the curve in two points, and if at every such two points tangents are drawn intersecting one another in a third point P, *the locus of the points* P *shall be a straight line; and every straight line which is drawn through the point taken to cut the curve shall be harmonically divided by that straight line and the curve.*

For the point taken, D, may be transferred by a straight line drawn from the vertex of the cone to a corresponding point *d* within or without the circle which is the projection of the conic section (12.): also any straight line passing through the former may, in like manner, be transferred to a corresponding straight line passing through the latter, and the tangents at the points in which the conic section is cut by the former straight line to tangents of the circle (6.) at the points in which it is cut by the latter, and the point P in which the tangents of the conic section intersect one another to a corresponding point *p* in which the tangents of the circle intersect one another. But the points *p* lie in a straight line (III. 53.) because they are the intersections of tangents to a circle at the extremities of chords passing through the same point. Therefore, the projections of the points P are such that their projections lie in a straight line, that is, they also lie in a straight line (2.).



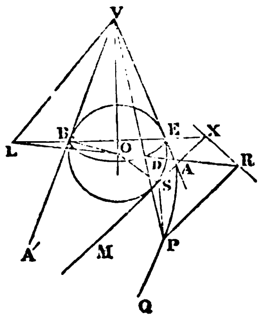
Again, in the circle, any straight line which passes through the projection of the point taken, is divided harmonically by the circumference and the straight line which is the locus of p (III. 53. *Cor.*); and lines harmonically divided are the projections of other lines which are likewise harmonically divided (II. 49.); therefore, also, in the conic section, any straight line which passes through the point taken is divided harmonically by the conic section and the straight line, which is the locus of P.

Therefore, &c.

PROP. 21.

In every conic section A P Q, *if in the right cone of which it is a section there be inscribed a sphere which touches the plane of the conic section in a point* S, *and the conical surface in a circle, the plane of which is produced to cut the plane of the conic section in a straight line* R X; *the distances* S P *and* P R *of any point* P *in the conic section from the point* S, *and the straight line* R X, *shall be to one another in a constant ratio.*[*]

Let V O be the axis of the cone, and A M the axis of the conic section, so



that the plane V A M A′ passes through V O, and is perpendicular to the plane of the conic section A P Q (13. and def. 13.): then, because the axis V O makes equal angles with the slant sides V A, V A′, if the angle V A M be bisected by a straight line cutting V O in O, the point O will be the centre of a circle touching the three straight lines V A, V A′, and A M (III. 59.); and, if OB, OS be drawn perpendicular to V A′, A M respectively, it will touch V A′ in the point B, and A M in the point S. Therefore, if the half of this circle, which is upon the same side of V O with the tangent V B, be made to revolve together with VB about the axis VO of the cone, it will generate a sphere (IV. def. 21.) which touches the conical surface (in which V B always lies) in the circle BDE (IV. 3. *Cor.* 2.), generated by the point B. And the same sphere will touch the plane A P Q in the point S (IV. 8.); for OS, being drawn in the plane V A M perpendicular to A M, which is the common section of the plane V A M with the plane A P Q to which it is perpendicular, is perpendicular to the plane A P Q (IV. 18.). It is supposed, therefore, that the plane of the circle BDE is produced to meet the plane A P Q in the line RX; and it is required to show that, if from any point P of the conic section, P R is drawn perpendicular to R X, and S P is joined, S P shall be to P R in a constant ratio.

Through V draw V L parallel to A M (I. 48.), and, since V L so drawn is in the plane V A M (I. def. 12. and IV. 1.) let it meet the straight line E B, in which the plane of the circle is cut by the plane V A M, or E B produced, in L; join V P, and let it cut the circumference B D E in D, and join L D, D R. Then, because the plane BDE of the circle is perpendicular to the axis V O (11.), and consequently to the plane V A M which passes through V O (IV. 18.), and that the plane A P Q of the conic section is perpendicular to the same plane V A M, the common section R X is perpendicular to the same plane (IV. 18. *Cor.* 2.), and therefore, also, to the line X A M which meets it in that plane (IV. def. 1.). But R X is also perpendicular to R P. Therefore RP is parallel to XAM (I. 14.), that is (IV. 6.), to V L. Therefore the points L, D, R are in the plane of the parallels V L, R P; but they are, also, in the plane of the circle B D E; therefore they are in the common section of these two planes, that is, in a straight line (IV. 2.), and LD, DR are in one and the same straight line. Again, because the plane V P S cuts the sphere in a circle (VI. 1.), and that the straight lines P D, P S meet this circle and do not cut it, they touch it in the points D, S respectively, and consequently, P D is equal to P S (III. 2. *Cor.* 3.). But, because V L is parallel to R P, the triangles V D L, P D R are similar (I. 15.). Therefore, P D is to P R as V D to V L (II. 31.). Therefore, since S P is equal to P D, and V B to V D, S P is to P R as V B to V L, that is, in a constant ratio.

Therefore, &c.

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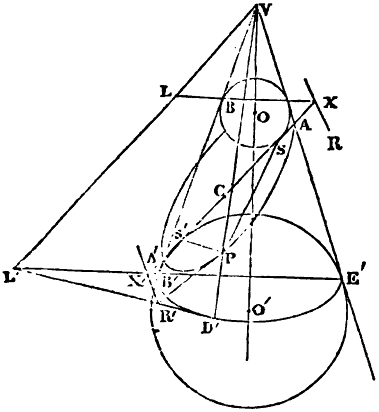
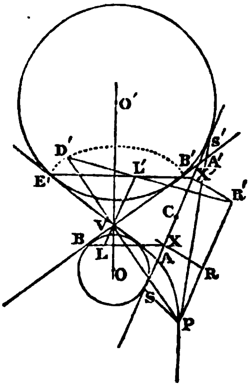
[*] From the "Transactions of the Cambridge Philosophical Society," Vol. III. No. VIII.

Cor. 1. If the conic section be an ellipse or an hyperbola, a second circle may be described in the angle A V B or in the angle vertical to it, touching the straight lines VA, VA′, and A A′, and accordingly, a second sphere inscribed in the cone touching the plane of the ellipse or hyperbola in a second point S′ of the axis, and the surface of the cone in a second circle B′ D′ E′. And if the plane of this circle be produced to cut the plane of the ellipse or hyperbola in a line R′ X′, it may be shown, as in the proposition, that the distances S′ P and P R′ of any point P, from the point S′ and the line R′ X′, are to one another in the constant ratio of V B′ to V L′, that is, in the same constant ratio as before, of V B to V L (II. 29.).





In the case of the parabola no such second circle can be described, because V B is parallel to A M.

Cor. 2. If C is the centre of the ellipse or the hyperbola, CS, CA, CX, as also C S′, C A′, C X′, are proportionals. For, since (by the proposition) S A is to A X as S A′ to A′ X, the straight lines A′ X, A′ A, and A′ S, are in harmonical progression (II. 45. *Cor.*); and consequently, the mean A A′ being bisected in C, C S, C A, and C X are proportionals (II. 46.); and the like demonstration applies to C S′, C A′ and C X′.

Cor. 3. The constant ratio of S P to P R is the same with that of C S to C A, or (which is the same) of C S′ to C A′. For A being a point of the ellipse or hyperbola, S A is to A X in the constant ratio; and because C S, C A, C X are proportionals, C S is to C A as S A to A X (II. 22.) And for the like reason C S′ is to C A′ as S′ A′ to A′ X′, that is, likewise in the constant ratio.

Cor. 4. In the ellipse $SP + PS' = AA'$; and in the hyperbola $SP \sim PS' = AA'$. For, since S P is to P R in the constant ratio of C S to C A, or of C A to C X, or again (II. 23.), of A A′ to X X′, and that S′ P is to P R′ in the same ratio, $SP \pm S'P$ is to $PR \pm PR'$ in the same ratio of A A′ to X X′ (II. 23. and II. 22.). But in the ellipse $PR + PR'$ is equal to R R′, and R R′ is equal to X X′ (I. 22.); therefore (II. 18. *Cor.*), $SP + S'P$ is equal to A A′. And, in the hyperbola, $PR \sim PR'$ is equal to R R′, and R R′ is equal to X X′ (I. 22.); therefore (II. 18. *Cor.*) $SP \sim S'P$ is equal to A A′.

Cor. 5. In the ellipse S P is less than P R; in the hyperbola S P is greater than P R; and in the parabola S P is equal to P R.

Scholium.

The points S and S′ are called the *foci*, and the straight lines R X, R′ X′ the directrix-es, of the conic section: the ellipse and hyperbola having two foci at equal distances from the centre upon either side of it, and two directrix-es; the parabola one focus and one directrix only. From the simple properties which have just been demonstrated with regard to these remarkable points, viz., that

1. In the ellipse $SP + S'P = AA'$.
2. In the hyperbola $SP \sim S'P = AA'$.
3. In the parabola $SP = PR$,

others of very considerable importance are derived. The three conic sections are, indeed, commonly defined by these properties, and from these, by help of the theorems of Plane Geometry, all other properties are derived in order.

PART III.—*Plane Sections of the oblique cone, of the right cylinder, and of the oblique cylinder.*

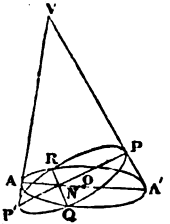
PROP. 22.

Every section of an oblique cone which is made by a plane parallel to its base, is a circle having its centre in the axis of the cone.

See the demonstration of Prop. 11.

PROP. 23.

In an oblique cone, if V A, V A′ *are the slant sides in which the surface of the cone is cut by a plane passing through the axis* V O *perpendicular to the base, and if the cone be cut by a plane* P Q R *which is perpendicular to the plane* V A A′, *and is inclined to either of the slant sides* V A *at the same angle at which the base is inclined to the other slant side* V A′, *the section made by the plane* P Q R *shall be a circle: in other words, every* subcontrary *section of an oblique cone is a circle.*



Let the plane P Q R cut the plane V A A′ in the straight line P P′; take any point Q in the curve P Q R, and through Q draw the plane A Q A′ R parallel to the base of the cone, and let this plane cut the plane V A A′ in the straight line A A′, and the plane P Q R in the straight line Q R (IV. 2.); then A Q A′ is a circle having its centre O in the axis of the cone (22.). And, because the planes A Q A′, P Q R are each of them perpendicular to the plane V A A′, their common section QR is perpendicular to the same plane (IV. 18. *Cor.* 2.), and consequently (IV. def. 1.) to the straight lines AA′, PP′ which meet it in that plane. Now, because, by the supposition, the planes P Q R, A Q A′ are equally inclined to the straight lines V A, V A′ respectively, the angles of inclination V P′ P and V A′ A are equal to one another, and (I. 3.) the vertical angles P N A′ and A N P′ are likewise equal; therefore the triangles P N A′, A N P′ are similar, and (II. 31.) P N is to N A′ as A N to N P′, and consequently (II. 38.), P N × N P′ is equal to A N × N A′. But, because A A′ is the diameter of the circle A Q A′, and is perpendicular to the chord Q R at the point N, Q R is bisected in N (III. 3.), and QN^2 is equal to AN × NA′ (III. 20.). Therefore QN^2 is equal to PN × NP′.

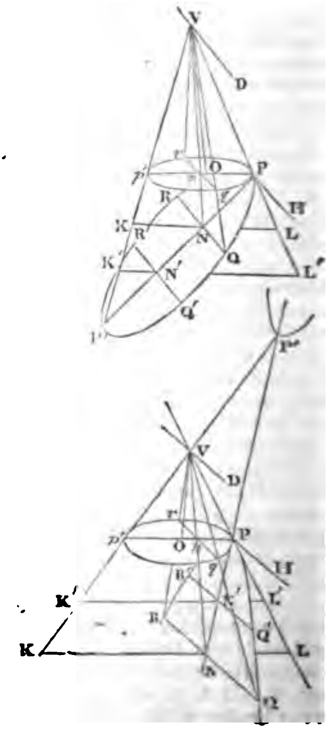
Therefore, because P P′ bisects every straight line Q R, which is drawn perpendicular to it from a point Q of the section P Q R, and that the square QN^2 of the half of such straight line is equal to the rectangle PN × NP′, under the segments of P P′, the section P Q R is a circle, having P P′ for its diameter (III. 3. and III. 20.).

Therefore, &c.

PROP. 24.

Every plane section of an oblique cone which neither passes through the vertex, nor is parallel to the base, nor is subcontrary, is a conic section, that is, either an ellipse or a parabola or an hyperbola.

Let V be the vertex, and VO the axis of an oblique cone, and let it be cut by any plane P Q R: through V draw the straight line VD parallel both to the base of the cone and to the plane P Q R (IV. 40. *Cor.*): through any point in the base of the cone draw a straight line parallel to V D, and (III. 57.) a straight line touching the base parallel to this straight line or (IV. 6.) to V D; and let V P be the slant side of the cone which passes through the point of con-

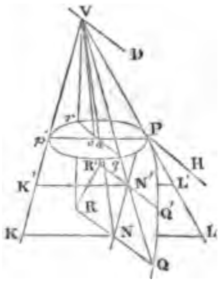


tact, so that the plane DVP touches the conical surface in the slant side VP. Let the plane PVO cut the plane PQR in the straight line PN, and the conical surface in the second slant side Vp'. Then, if PN cuts Vp' in a point P', the section PQR shall be an ellipse; if PN is parallel to Vp', the section shall be a parabola; and if PN cuts Vp' produced beyond the vertex in a point P' the section shall be an hyperbola.

First, let us suppose that PN is not parallel to VP', and therefore, if produced, cuts it in a point P' below or above the vertex V. Take any two points Q, Q' in the curve PQR, and through these points draw the planes DVQ, DVQ' (IV. 1.) cutting the plane PQR in the straight lines QR, $Q'R'$ and the plane VPP' in the straight lines VN, VN' respectively (IV. 2.); also through the point P draw a plane parallel to the base of the cone, and therefore cutting the cone in a circular section Pqr (22.), and let the same plane cut the plane PQR in the straight line PH. Then, because the plane DVP touches the conical surface, the straight line PH touches both the circular section Pqr and the curve PQR, and consequently the diameter POp' of the circle is perpendicular to PH. Let qnr be the projection of QNR on the plane Pqr by straight lines drawn from V, or, which is the same thing, the common section of the plane pqr with the plane DVQ; then, because VD is parallel to the plane of the circle, and likewise to the plane PQR, it is parallel to PH, which is the common section of these two planes (IV. 40. *Cor.*); but qr is parallel to VD, because it is the common section of a plane passing through VD with the plane pqr, to which VD is parallel (IV. 10.): therefore qr is parallel to PH (IV. 6.); and, for the like reasons, QR is likewise parallel to PH, that is (IV. 6.), to qr. Also, because qr is parallel to PH, which touches the circle Pqr in P, it is perpendicular to the diameter POp' (III. 2. and I. 14.), and is therefore (III. 3.) bisected in the point n; wherefore, also, since QR is parallel to qr, it is bisected in the point N (II. 30.). Through N and N' draw KL and $K'L'$, each of them, parallel to POp'. Then, because the triangles VKN, VNL are similar to the triangles $Vp'n$, VnP respectively (I. 15.) $KN : p'n :: VN : Vn$ (II. 31. and II. 19.), and $NL : nP :: VN : Vn$, and consequently (II. 37. *Cor.* 3.), $KN \times NL$ is to $p'n \times nP$ as VN^2 to Vn^2, that is, since the triangles VNQ, Vnq are similar, as QN^2 to qn^2 (II. 37. *Cor.* 4.): but $p'n \times nP$ is equal to qn^2 (III. 20.): therefore, also, $KN \times NL$ is equal to QN^2 (II. 18 *Cor.*). In the same manner, it may be shown that $K'N' \times N'L'$ is equal to $Q'N'^2$. But, because the triangles $P'KN$, PNL are similar to the triangles $P'K'N'$, $PN'L'$ respectively (I. 15.) $KN : NP' :: K'N : N'P'$, and $NL : NP :: N'L' : N'P$ (II. 31.), and consequently, $KN \times NL$ or QN^2 is to $PN \times NP'$ as $K'N' \times N'L'$ or $Q'N'^2$ is to $PN' \times N'P'$ (II. 37. *Cor.* 3.). Therefore, *alternando*, QN^2 is to $Q'N'^2$ as $PN \times NP'$ to $PN' \times N'P'$; and consequently, PQR is an ellipse or an hyperbola having the diameter PP' and tangent PH (19.); an ellipse, if PN cuts VP' below the vertex; an hyperbola, if above.

And, by a similar construction, if PN be parallel to Vp', it may be shown, in the same manner, that $KN \times NL$ is equal to QN^2, and $K'N' \times N'L'$ to $Q'N'^2$. But, because KN' is a parallelogram, KN is equal to $K'N'$ (I. 22.), and, because NL is parallel to $N'L'$, NL is to $N'L'$ as PN to PN' (II. 30. *Cor.* 2.); therefore, $KN \times NL$ is to $K'N' \times N'L'$ as PN to PN' (II. 35.). Therefore, QN^2 is to $Q'N'^2$ as PN to PN'; and, consequently, PQR is a parabola having the diameter PN and tangent PH (19.).

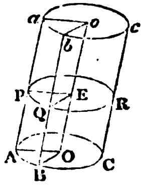
Therefore, &c.



PROP. 25.

Every section of a cylinder which is made by a plane parallel to its base, is a circle having its centre in the axis of the cylinder, whether the cylinder be right or oblique.

Let ABC, abc be the bases of a cylinder, and Oo its axis, and let PQR be a section made by any plane which is parallel to ABC or *a b c*, and cuts the axis O *o* in E. The section P Q R shall be a circle having the centre E.



Let P be any point in the curve P Q R; join P E; through P draw P A parallel to E O, and, consequently (V. def. 1.), lying in the convex surface of the cylinder, to meet the circumference A B C in A, and join O A. Then, because the parallels P A, E O are intercepted between parallel planes, they are equal to one another (IV. 13.); and, because P A and E O are both equal and parallel, E P is equal to O A (I. 21.), that is, to the radius of the circle A B C. And, in the same manner, it may be shown that the straight line drawn from E to any other point Q of P Q R is equal to the same radius. Therefore, the point E is at the same distance from every point of PQR; and, consequently, P Q R is a circle having the centre E.

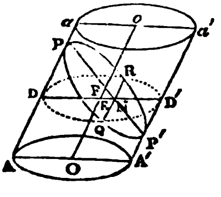
Therefore, &c.

Cor. The radius of every circular section of a cylinder, which is made by a plane parallel to its base, is equal to the radius of the base.

Prop. 26.

In an oblique cylinder, if A a *and* A′ a′ *are the parallel straight lines in which the surface of the cylinder is cut by a plane passing through the axis* O o *perpendicular to the base, and if the cylinder be cut by a plane* P Q R *which is perpendicular to the plane* A a a′ A′, *and is inclined to either of the parallel straight lines,* A a, *at the same angle at which the base is inclined to the other* A′ a′, *the section made by the plane* P Q R *shall be a circle having its centre in the axis of the cylinder; or, in other words, every* subcontrary *section of an oblique cylinder is a circle having its centre in the axis of the cylinder.*

Let the plane P Q R cut the plane A *a a*′ A′ in the straight line P P′ (IV. 2.), take any point Q in the curve P Q R, and through Q draw the plane D Q D′ R parallel to the base of the cylinder (IV. 43.), and let this plane cut the plane A *a a*′ A′ in the straight line D E D′ and the plane P Q R in the straight line Q R; then D Q D′ is a circle having the centre E (25.). And, because the planes D Q D′, P Q R are each of them perpendicular to the plane A *a a*′ A′, their common section Q R is perpendicular to the same plane (IV. 18.), and consequently to the straight lines DD′, PP′, which meet it in that plane (IV. def. 1.). Now, because, by the supposition, the planes P Q R, D Q D′ are equally inclined to the straight lines A *a*, A′ *a*′ respectively, the angles of inclination N P D, N D′ P′ are equal to one another; but N D′ P′ is equal to N D P, because A′ *a*′ is parallel to A *a* (I. 15.); therefore, the angle N P D is likewise equal to N D P, and consequently (I. 6.) the side N P is equal to the side ND. And, for the like reasons, N P′ is equal to N D′. Therefore, the rectangle P N × N P′ is equal to the rectangle D N × N D′. But, because D E D′ is the diameter of the circle D Q D′, and is perpendicular to the chord QR at the point N, QR is bisected in N (III. 3.), and QN^2 is equal to DN × N D′ (III. 20.). Therefore, QN^2 is equal to PN × N P′. Therefore, because PP′ bisects every straight line QR which is drawn perpendicular to it from a point Q of the section P Q R, and that the square QN^2 of the half of such straight line is equal to the rectangle P N × N P′ under the segments of P P′, the section P Q R is a circle having P P′ for its diameter (III. 3. and III. 20.). Also, the middle point of P P′ is the centre of the circle. But, because O *o*, A *a* and A′ *a*′ are parallel, and that A′ O is equal to O A, P′ F is equal to F P (II. 29.), that is, F is the middle point of P P′. Therefore, F is the centre of the circle P Q R.



Therefore, &c.

Cor. The radius of every subcontrary section of an oblique cylinder is equal to the radius of the base of the cylinder.

Prop. 27.

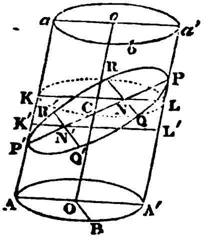
Every plane section of a cylinder which is neither parallel to the axis[*] *nor parallel to the base, nor subcontrary, is an ellipse having its centre in the*

---

[*] A plane which is parallel to the axis of a cylinder, cuts the convex surface in two straight lines which are parallel to the axis.

axis of the cylinder, whether the cylinder be right or oblique.

Let A B A′, *a b a′* be the bases, and O *o* the axis of a cylinder, and let P Q R be any plane section which is neither parallel to the axis O *o* nor to the base



A B A′, nor subcontrary: P Q R shall be an ellipse having for its centre the point C in which its plane cuts the axis of the cylinder.

Through O draw a plane parallel to the plane P Q R (IV. 43.), and let it cut the plane ABA′ in the straight line OB; draw the diameter AA′ perpendicular to O B, and let the plane A O *o* cut the convex surface of the cylinder in the parallel straight lines A *a*, A′ *a′*, and the plane P Q R in the straight line P C P′: in the curve P Q R take any two points Q, Q′, and through these points draw the planes K Q L R, K′ Q′ L′ R′ parallel to the base A B A′ (IV. 43.) and cutting the plane P Q R in the straight lines Q N R, Q′ N′ R′ and the plane A *a a′* A′ in the straight lines K N L, K′ N′ L′ respectively (IV. 2.). Then, because Q R is the common section of two planes which are parallel respectively to the two passing through O B, Q R is parallel to O B (IV. 12. *Cor.*); and, for the like reason, Q′ R′ is parallel to O B or Q R. Also, because K L and A′ A are the common sections of parallel planes by the same plane A *a a′* A′, K L is parallel to A′ A (IV. 12.); and, for the like reason, K′ L′ is parallel to A′ A or K L. But A′ O A is at right angles to O B. Therefore, QR is at right angles to KL. and Q′R′ is at right angles to K′L′ (IV. 15.). And, because the diameter K L of the circle K Q L R is at right angles to the chord Q R, it bisects Q R in the point N (III. 3.); and, for the like reason, Q′ R′ is bisected in N′. Therefore (III. 20.) QN^2 is equal to K N × N L, and $Q'N'^2$ to K′ N′ × N′ L′. But, because the triangles P′ K N, P N L are similar to the triangles P′ K′ N′, P N′ L′ respectively (I. 15.) KN : NP′ :: K′N′ : N′ P′, and N L : P N :: N′ L′ : P N′ (II. 31.), and, consequently, K N × N L or QN^2 : PN × N P′ :: K′ N′ × N′ L′ or $Q'N'^2$: PN′ × N′P′ (II. 37. *Cor.* 3.). Therefore, *alternando*, QN^2 : $Q'N'^2$ :: P N × N P′ : P N′ × N′ P′; and consequently (19.), P Q R is an ellipse having the diameter P P′, and the tangent at P parallel to O B. Also, because A A′ is bisected in O, and that A *a*, O *o* and A′ *a′* are parallel to one another, P P′ is bisected in C (II. 29.). Therefore, C is the centre of the ellipse P Q R.

Therefore, &c.

ERRATA.

* Those marked with an asterisk are of importance.

Page	Col.	Line	
4	2	47	*for* greater *read* greater or less
6	1	29	— side D E *read* sides D E
13	2	27	— same line *read* same side
11	1	12	*for* to the same straight line *read* to the same straight line, and in the same plane
17	1	29	— 22. Cor. *read* 22. Cor. 1
18	1	3 and 4	*for* A D, and *read* A D. And
20	1	13	*for* 32 *read* 31
*23	2	42	— and hence *read* therefore, adding twice the square of A E or A C
24	1	23	— 39. Cor. 2 *read* 29. Cor. 2
25	1	26	— equal to *read* is equal to
26	1	52	— the points D E *read* the points D, E
28	1	33	— triangle A B C *read* triangle *a b c*
28	2	3	from bottom, *for* H C *read* A C
45	2	16	*for* [10]. Cor. *read* [11]. Cor. 2
*46	1	5 and 2	from bottom, *for* A′ : B′ : : A : B *read* A′ : B : : A : B′
*46	1	4 and 3	from bottom, *for* A *read* A′, *and for* A′ *read* A.
*46	1	1	from bottom, *for* B *read* B′, *and for* B′ *read* B
*46	2	1	*for* B′ *read* B, *omitting* because it is supposed to be greater than B, which is greater than Q
47	1	13	of note, *for* and *d* *read* *c* and *d*
50	1	45	*for* def. 7 *read* def. [7]
54	2	12	— A, B, A′, B *read* A, B, A′, B′
56	1	48	— 10. Cor. *read* 11. Cor. 2
62	1	6	— homologous, and *read* homologous; and
64	1	29	— Euc. I. 17 *read* Euc. VI. 17
64	2	33	— I. 38 *read* I. 34
69	2	13	— A C *read* A *c*
70	2	39	— I. 5 *read* I. 6
75	1	19	— D. (I. 48 *read* D. (I. 48)
78	1	2	— 56 *read* 56. Cor.
78	2		in the figure of the Scholium, for C′ *read* G′
82	1	7	*for* 1 *read* 1. Cor. 1
83	1		in the figure of the Scholium, *for the* C *nearest to* A *read c*
92	2	7	*for* A B C D E F *read* A, B, C, D, E, F
95	2	39	— 31 *read* 31. Cor. 1
96	1	17	— is greater *read* is, greater
99	1	1	from bottom, *for* impracticable *read* impracticable in the way of calculation
101	2	2	from bottom, *for* II. 17 *read* II. 13
109	2	41	*for* as may easily be shown *read* (II. 23)
110	1	25	*for* P C D *read* B C D
*110	2	31	— $P A^2 \times A G$ *read* $P A^2 \times E G$
*111	1	16 and 17	*for* (produced if necessary) *read* produced
*111	2	5	*for* B C D *read* B C D and the point A
*111	2		*read* line 19 immediately before line 18
117	1	56	*for* of given *read* of the given
120	1	31	— 12. Cor. 1 *read* 11. Cor.
—	—	41	— 12. Cor. 1 *read* 11. Cor.
*122	2		in the figure the letter C is wanting between N and B
*124	2		in the Scholium case 6, *for* $x^2 + y^2 = b^2$ *read* $x^2 - y^2 = b^2$
126	2	1	of note from bottom, *for* said *read* made
126	2	26	*for* 17 *read* 17. Cor.
128	1	11	— pass *read* passes
134	2	47	— will be *read* will lie
137	2	5	— this *read* that one
139	2	1	from bottom, *for* 20 *read* 21
140	2	3	from bottom, *for* themselves *read* likewise
*143	2	33	*for* altitude G H *read* altitude C H
144	2	38	— 21 *read* 22
145	1	16	A B C D *read* A B C D E
*146	1	27	*omit the reference* (II. 12)
*150	1		in the figure the line A E is dotted by mistake
152	2	30	*for* I. 12 *read* I. 12. Cor. 1.
154	2	16	— G H, being *read* G H being
157	1	4	— A B, A C *read* O B, O C
160	1	1	— A B C D *read* A B C D E
*160	1	4	from bottom, *for* F L *read* E L
160	1		in the lower figure, *for u read* U
167	2	19	*for* right angle *read* right-angled
171	1	21	— 1 Cor. 2 *read* 2 Cor. 2
172	2	22	— IV. 32 *read* IV. 32. Cor. 2
172	2	12	from bottom, *for* pyramid *read* cone
173	1	13	*for* 5. Cor. 1 *read* 7. Cor. 1
173	1	26	— pyramid *read* cone
174	1		in the figure, the letter C is wanting at the centre of A B D
174	1	9	from bottom, *for* 6 *read* 8
175	2	3 and 4	*for* Props. 2 and 6 *read* Props. 3 and 8
181	1	29	*for* 18 *read* 13 and 16
*182	1	12	— C H *read* C D
182	1	5	from bottom, *for* III. 36 *read* I. 36
182	2	13	from bottom, *for* $\frac{1}{3} \pi + G K^3$ *read* $\frac{1}{3} \pi \times G K^3$
184	1	20	*for* 15 *read* 18
190	2	5	from bottom, *for* greater *read* less
190	2	4	from bottom, *for* less *read* greater

ADDENDUM.

Page 93, col. 2, after Book III. prop. 28, add,

Cor. In an isosceles triangle, which has each of the equal angles double of the vertical angle, the sides and base are in extreme and mean proportion; and conversely.

In such a triangle, each of the equal angles is four-fifths of a right angle (I. 19.) As we are enabled, therefore, to describe such a triangle by II. 59, we can thus divide a right angle into five equal parts, as was observed in the scholium of p. 26. And generally, if a regular polygon of *n* sides can be inscribed in a circle, (as, in the present instance, the regular polygon of five sides,) a right angle may be divided into *n* equal parts, by taking for one of those parts (I. 46. Cor.) a fourth of the angle which the side of the polygon subtends at the centre of the circle. (I. 3. Cor.)

INDEX.

The theorems and problems of Plane Geometry will be found under the heads *Straight Line, Angle, Triangle, Square, Rhombus, Rectangle, Parallelogram, Quadrilateral, Rectilineal Figure, Circle*; those of Solid Geometry under *Plane, Dihedral Angle, Solid Angle, Tetrahedron, Cube, Rectangular Parallelopiped, Parallelopiped, Prism, Pyramid, Polyhedron, Regular Polyhedron, Cylinder, Cone, Sphere, Solid of Revolution*; of Spherical Geometry under that head; of Ratios and Proportion under those respective heads; and so of Projection and the Conic Sections.

The parts included in uncis are additions; having been made, either with the view of supplying such connecting links as seemed wanting in the present digest of the whole work, as in "Circle" (E) and (G); or of completing what had been left imperfect, as in the notes on "Proportion" and "Rectangle;" or of extending and generalising where only partial views had been given, as under the heads "similar," "symmetrical," "touch;" or of adding whatever of use or interest had been inadvertently omitted, as in "Annulus," "Lunes," and the note on "Centrolinead."

ABSCISSA (Lat., a part cut off) of a conic section . . . *def.* 220
Acute, (Lat., pointed,) a term applied to angles, whether plane, dihedral, or spherical, which are less than right angles.
Adjacent angles are those which one straight line, or plane, or spherical arc makes with another upon one side of it.
Affection, angles said to be of the same, or of different affections . . *note* 62
Algebra, its signs +, −, ×, &c., borrowed with advantage by geometry . 20
Alternando, a rule in Proportion. *See* "Proportion."
Alternate, certain angles said to be, which are made by two straight lines (or planes), with a third straight line (or plane) 13
Altitude, of any figure, is a perpendicular drawn to the base from the vertex, line, or plane, opposite to it.
Analysis, (Gr., undoing, or taking to pieces,) in geometry, is that mode of demonstrating a theorem, or solving a problem, which searches into the thing proposed, and takes it (as it were) to pieces, in order to discover the more simple truths and constructions upon which it is built: the reverse process is called Synthesis, (Gr., putting together,) and proceeds in a didactic form, by the putting together of truths and constructions already established, to do, or establish the certainty of the thing proposed 107
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  (*c*) All right angles are equal to one another 4
  (*d*) Every angle is measured by the circular arc, which is described about the angular point with a given radius, and is included between the legs, *sch.* 85
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Angle, solid. (*See* "Solid Angle.") *def.* 125
Angle, spherical. (*See* "Spherical Geometry.") . . . *def.* 185
Angle in a segment. (*See* "Circle.") *def.* 79
[*Annulus* (Lat., a ring,) a name given to the space which is the difference of two concentric circles—two annuli are said to be similar, when the radii of the interior and exterior circumferences are to one another in the same ratio. *See* "Circle."]
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Arc, spherical. (*See* "Spherical Geometry.") *def.* 184
Area (Lat., a floor) means, sometimes, the same as surface; but is more properly applied to signify the number of times any surface contains the superficial unit *sch.* 18
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Arithmetical mean, between two magnitudes, is a magnitude which exceeds the lesser of the two by as much as it falls short of the greater.

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(*b*) Is greater than the geometrical mean . . *cor.* 42
(*c*) Arithmetical means being continually taken between two magnitudes, between the new mean and the last, and so on; to arrive at the approximate result immediately, after a certain number have been taken *note* 98

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[*] A more simple instrument, for the same purpose, consisting of three rulers, which are stiffly moveable about a common joint in the same, or in parallel planes, is likewise of frequent use among draughtsmen. Its form is not so convenient, neither from the want of the additional rulers is its adjustment susceptible of the same accuracy as Mr. Nicholson's. It derives its use, however, from the same principle, and the manner of applying it may be explained as follows:—

Let A B and C D be two given lines, converging to the distant point O, and let it be required to draw through any given point P a straight line, which shall pass through the same point O. In A B take any point A, and in C D any point C, and join A C; let Q A, Q P, Q C, represent the three rulers, having the common joint Q, and let the angles P Q A, P Q C be set (or made equal) to the angles A C D and B A C respectively; fix two pins, one at A, the other at C, and move the rulers A Q, C Q, along these pins, until Q P passes through the point P; let the ruler Q P be then steadily fixed, and the line Q P drawn; Q P is the line required. For, if a circle be described through the points A, C, O, the point Q will always lie in its circumference, because A Q C is equal to the sum (fig. 1) or the difference (fig. 2) of P Q A and P Q C, *i. e.* of B A C and A C D, *i. e.* to the supplement of the angle O (fig. 1), or to the angle O (fig. 2); therefore, since the angle P Q C is equal to B A C, Q P produced passes through O.

fig. 1.

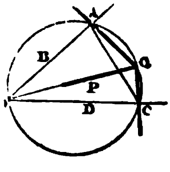
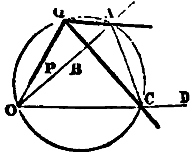


fig. 2.



(*d*) Equal straight lines in a circle are equally distant from the centre; and those which are equally distant from the centre are equal to one another *cor.* 81
(*e*) If a diameter cuts any other chord at right angles, it bisects it; and conversely, if a diameter bisects any other chord, it cuts it at right angles 80
(*f*) Two chords of a circle cannot bisect one another, except they both pass through the centre . *cor.* 80
(*g*) The straight line which bisects any chord at right angles, passes through the centre of the circle . *cor.* 80
(*h*) If two circles have a common chord, it shall be bisected at right angles by the straight line joining their centres *cor.* 80
(*i*) A diameter bisects all chords which are parallel to the tangent at either of its extremities . . *cor.* 80
[(*k*) Every diameter divides the circle symmetrically. *See* "Symmetrically divided."]
(*l*) If a point be taken, from which to the circumference of a circle there fall more than two equal straight lines, that point is the centre of the circle 81
(*m*) From any other point than the centre there cannot be drawn to the circumference of a circle more than two straight lines that are equal to one another, whether the point be within or without the circle . *cor.* 81
(*n*) If a point be taken within a circle which is not the centre, of all the straight lines which can be drawn from that point to the circumference, the greatest is that which passes through the centre, and the other part of that diameter is the least; also, of any others, that which is nearer to the greatest is greater than the more remote, and any two, which are equally distant from the greatest upon either side of it, are equal to one another *sch.* 83
(*o*) If a point be taken without a circle, and straight lines be drawn from it to the circumference, of those which fall upon the concave circumference, the greatest is that which passes through the centre, and of the rest that which is nearer to the greatest is greater than the more remote, and any two which are equally distant from the greatest upon either side of it are equal to one another; also, of those which fall upon the convex circumference, the least is that between the point without the circle and the diameter, and of the rest, that which is nearer to the least is less than the more remote, and any two which are equally distant from the least upon either side of it are equal to one another . . . *sch.* 83

(C) *Of Circles which cut or touch one another*, III. § 1.
(*a*) Circles which are concentric either do not meet one another, or coincide altogether . . . 81
(*b*) Circles which cut or touch one another cannot have the same centre *cor.* 81
(*c*) Circles cannot cut one another in more than two points . . 82
(*d*) If two circles meet one another in a point which is not in the straight line joining their centres, or in that straight line produced, they shall meet one another in a second point upon the other side of that straight line, and shall cut one another . . 82
(*e*) If two circles meet one another in a point which is in the straight line joining their centres, or in that straight line produced, they meet in no other point; the circumference of the circle which has the greater radius falls wholly without the circumference of the other, and the circles touch one another 83
(*f*) Circles which cut one another meet in two points, [and the distance between the two is bisected at right angles by the straight line which joins their centres] . *cor.* 83
(*g*) If two circles cut one another, the straight line which joins their centres is less than the sum, and greater than the difference of their radii *cor.* 83
(*h*) Circles which touch one another meet in one point only; and the straight line which joins their centres, or that straight line produced, passes through the point of contact . *cor.* 82
(*i*) If two circles touch one another, the distance of their centres is equal to the sum or to the difference of their radii; the sum, if they touch externally; the difference, if they touch internally . . . *cor.* 82
(*k*) If the circumferences of two circles do not meet one another in any point, the distance between their centres shall be greater than the sum, or less than the difference of their radii, according as each of the circles is without the other, or one of them within the other; [and the circles approach nearest to one another in the straight line joining their centres, or in that straight line produced] . 83
(*l*) If the distance between the centres of two circles be at once less than the sum, and greater than the difference of their radii, the circles will cut one another; if that distance be equal to the sum, or to the difference of the radii, the circles will touch one another; and, if that distance be greater than the sum, or less than the difference of the radii, the circles will not meet one another . *cor.* 84

(D) *Of Arcs and Angles in a Circle*, III. § 2.

- (*a*) In the same, or in equal circles, equal chords subtend equal angles at the centre, and conversely; and the greater chord subtends the greater angle at the centre, and conversely 84
- (*b*) In the same, or in equal circles, equal angles at the centre stand upon equal arcs, and conversely . . 84
- (*c*) In the same, or in equal circles, equal chords subtend equal arcs; and conversely . . . *cor.* 85
- (*d*) In the same, or in equal circles, any angles at the centre at the arcs upon which they stand; *i. e.*, any angle at the centre is measured by the arc upon which it stands . . 85
- (*e*) The angle at the circumference is equal to half the angle at the centre upon the same base; *i. e.*, any angle at the circumference is measured by half the arc upon which it stands 86
- (*f*) In the same, or in equal circles, equal angles at the circumference stand upon equal arcs, and conversely; also, any angles at the circumference are as the arcs upon which they stand *cor.* 86
- (*g*) Angles in the same segment of a circle are equal to one another 86
- (*h*) The angle in a semicircle is a right angle; the angle in a segment, which is greater than a semicircle, is less than a right angle; and the angle in a segment, which is less than a semicircle, is greater than a right angle *cor.* 86
- (*i*) If a triangle and a circular segment stand upon the same base and upon the same side of it, the vertex of the triangle will fall without, or within, or upon the arc of the segment, according as the vertical angle is less than, or greater than, or equal to, the angle in the segment . *cor.* 87
- (*k*) If any chord be drawn in a circle, the angles contained in the two opposite segments shall be, together, equal to two right angles . . 87
- (*l*) The opposite angles of a quadrilateral in a circle are, together, equal to two right angles . *cor.* 87
- (*m*) If at one extremity of a chord a tangent be drawn, the angles which it makes with the chord shall be equal to the angles in the alternate segments; and conversely, if at one extremity of a chord a straight line be drawn such that the angles it makes with the chord are equal to the angles in the alternate segments, such straight line is a tangent . . 87
- (*n*) Parallel chords intercept equal arcs, and conversely . . 88
- (*o*) If a tangent and a chord be parallel, the arc between them is bisected by the point of contact; and, conversely, if the arc between a chord and tangent be bisected by the point of contact, the chord and tangent shall be parallel . . . *cor.* 88
- (*p*) If any two chords meet one another, the angle contained by them is measured by half the sum, or by half the difference of the intercepted arcs, according as the point in which they meet is within or without the circle 88
- (*q*) If a chord meet a tangent, the angle contained by them is measured by half the difference of the intercepted arcs [and the same measure obtains when two tangents meet one another.] *cor.* 88

(E) *Of Rectangles under the Segments of Chords*, III. § 3.

- [(*a*) If a diameter bisects any chord, the square of half the bisected chord is equal to the rectangle under the segments of the diameter (I. 36, and I. 34.)]
- [(*b*) The tangents at the extremities of a chord meet one another in the same point T of the bisecting diameter, and that in such a manner, that if C is the centre of the circle, N the point of bisection, and CA the radius, CN, CA, and CT, are proportionals (II. 34.)]
- (*c*) If any two chords cut one another, the rectangles under their segments shall be equal, whether they cut one another within or without the circle 88
- (*d*) If two straight lines A B, C D, cut one another in a point E, and if the points A, B, and C, D, are taken, (the two first upon the same side of E, and the two last likewise upon the same side; or the two first upon opposite sides of E, and the two last likewise upon opposite sides,) so that the rectangle under A E, E B, shall be equal to the rectangle under CE, ED, the points A, B, C, D, shall lie in the circumference of the same circle *cor.* 89
- (*e*) If a chord meet a tangent, the square of the tangent shall be equal to the rectangle under the segments of the chord . . . 89
- (*f*) If two straight lines, A B and C E, meet one another in a point E, and if the points A, B, and C, are so taken, that, A and B being upon the same side of E, the square of E C is equal to the rectangle under A E, E B, the straight line EC shall touch the circle which passes through the points A, B, C . . *cor.* 89
- (*g*) If a triangle be inscribed in a circle, and if a perpendicular be drawn from the vertex to the base, the rectangle under the two sides shall be equal to the rectangle under the perpendicular and the diameter of the circle 90
- (*h*) If triangles are inscribed in the same, or in equal circles, the rectangles

under the sides, containing any two angles, shall be to one another as the perpendiculars drawn to the opposite sides . . . *cor.* 90

(*i*) If a quadrilateral be inscribed in a circle, its diagonals are to one another as the sums of the rectangles under the sides terminated in their extremities 90

(*k*) If a quadrilateral be inscribed in a circle, the rectangle under its diagonals shall be equal to the sum of the rectangles under its opposite sides 90

(*l*) If from any point without a circle, two straight lines are drawn to touch the circle, every straight line which is drawn through that point to cut the circle shall be harmonically divided by the circumference, and the chord joining the points of contact; and the tangents at the points in which every such straight line cuts the circumference shall meet one another in the chord produced . *lem.* 220

(*m*) If through any point taken within or without a circle, there are drawn any number of straight lines, each cutting the circle in two points, and if at every such two points tangents are drawn intersecting one another in a point P, the locus of the points P shall be a straight line; and every straight line which is drawn through the point taken to cut the circle shall be harmonically divided by that straight line and the circle *cor.* 112

(F) *Of regular Polygons, inscribed and circumscribed,* III. § 4.

(*a*) If any two adjoining angles of a regular polygon be bisected, the intersection of the bisecting lines will be the common centre of two circles, the one circumscribed about, the other inscribed in, the polygon . 91

(*b*) If the circumference of a circle be divided into any number of equal parts, the chords joining the points of division shall include a regular polygon inscribed in the circle; and the tangents drawn through those points shall include a regular polygon of the same number of sides circumscribed about the circle . . 92

(*c*) If any regular polygon be inscribed in a circle, a similar polygon may be circumscribed about the circle by drawing tangents through the angular points of the former; and, conversely *cor.* 92

(*d*) If any regular polygon be inscribed in a circle, and if a tangent be drawn parallel to one of its sides, and be terminated both ways by radii passing through the extremities of that side, such terminated tangent shall be a side of a similar polygon, circumscribed about the circle . . *cor.* 92

(*e*) The side of a regular hexagon is equal to the radius of the circle in which it is inscribed; the side of a regular decagon is equal to the greater segment of the radius divided medially; and the side-square of a regular pentagon* is greater than the square of the radius by the side-square of a regular decagon inscribed in the same circle 93

(*f*) If K and L represent two regular polygons of the same number of sides, the one inscribed in, and the other circumscribed about the same circle, and if M and N represent the inscribed and circumscribed polygons of twice the number of sides; M shall be a geometrical mean between K and L, and N an harmonical mean between L and M . . . 96

(*g*) If k and l represent the radii of the circles which are inscribed in any regular polygon, and circumscribed about it, and if m and n represent these radii for a regular polygon which has twice as many sides as the former, and an equal perimeter; m shall be an arithmetical mean between k and l, and n a geometrical mean between l and m . . *note* 98

(*h*) If k and l represent the radii of the circles which are circumscribed about any regular polygon and inscribed in it, and m an arithmetical mean between them; and if k' and l' represent these radii for a regular polygon which has twice as many sides as the former and an equal area, k' shall be a mean proportional between k and l, and l' a mean proportional between l and m *note* 98

(*i*) The area of any regular polygon is equal to half the rectangle under its perimeter and the radius of the inscribed circle . . . 93

(*k*) The perimeters of similar regular polygons are as the radii of the inscribed or circumscribed circles, and their areas are as the squares of the radii 94

(*l*) Table of the areas of the regular polygons of 4, 8, 16, &c. sides up to 2^{30}, which are inscribed in and circumscribed about the same circle, as far as the 10th decimal place inclusively; the square of the radius being unity *sch.* 97

(G) *Of the Circumference and Area of the Circle,* III. § 4, 5.

[(*a*) The circumference of a circle is greater than the perimeter of any inscribed polygon, and less than that of any circumscribed polygon; so also

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* The diagonals, also, of a regular pentagon are severally parallel to its sides, and greater than them in extreme and mean ratio. Note, 159.

its area is greater than the area of any inscribed polygon, and less than that of any circumscribed polygon. (I. 10. *sch.*)]

(*b*) Any circle being given, a regular polygon may be inscribed (or circumscribed), which shall differ from the circle in perimeter, or in area, by less than any given difference *cor.* 95

(*c*) Any two circles being given, similar regular polygons may be inscribed, or circumscribed, which shall differ from the circles, in perimeter or in area, by less than any the same given difference . . . *cor.* 95

(*d*) The area of a circle is equal to half the rectangle under the radius and circumference . . . 95

(*e*) The circumferences of circles are as the radii, and their areas are (in the duplicate ratio, or) as the squares of the radii 96

(*f*) The ratio of the circumference to the diameter is 3.1415926535 &c.; to which number we may approximate by any of the theorems (*f*), (*g*), (*h*), in the last division . *sch.* 97

For rules to abridge the calculation, *see note* 97

(*g*) The circumference and diameter have been demonstrated to be incommensurable, so that their approximation has no limit . *sch.* 99

(*h*) If R represents the radius of a circle, and π the number 3.1415 &c. the circumference $= 2\pi R$, and the area $= \pi R^2$ . . . *sch.* 99

(*i*) The circle is a mean proportional between any circumscribed polygon and a similar polygon which has the same perimeter with the circle *note* 102

(*k*) The circle is greater than any (regular polygon, or any rectilineal figure, or any other) plane figure whatever which has the same perimeter; and has a less perimeter than any other plane figure whatever which has the same area . . 102, 105

(H) *Of certain Portions of the Circumference and Area*, III. § 4, 5.

(*a*) Similar arcs of circles are as the radii . . . *cor.* 96

(*b*) In the same, or in equal circles, any two sectors are as the arcs upon which they stand . . 85

[(*c*) A sector of a circle is to the whole circle as the arc upon which it stands to the whole circumference].

(*d*) A sector of a circle is equal to half the product of the radius and the arc upon which it stands . *cor.* 96

(*e*) Similar sectors are as the squares of the radii . . *cor.* 96

(*f*) Similar segments are as the squares of the radii . . *cor.* 96

(*g*) A circular segment is greater than any other plane figure whatever having the same right line for a base and the same perimeter [and has a less perimeter than any other plane figure whatever having the same right line for a base and the same area] *cor.* 106

[(*h*) A circular annulus is equal to the product of its breadth and half the sum of its interior and exterior circumferences.]

[(*i*) Similar annuli are as the squares of their interior or exterior radii.]

(I). *Problems relating to the Circle.*

(*a*) To divide a given circular arc into 2, 4, 8, &c. equal parts . . 112

(*b*) To find the centre of a given circular arc, and to complete the circumference of which it is a part 113

(*c*) To draw a tangent to a given circle from a given point—
  1. When the given point is in the circumference.
  2. When without it . . 113

(*d*) To draw a tangent to a given circle, which shall be parallel to a given straight line . . . 113

(*e*) To draw a common tangent to two given circles—
  1. So as to touch them upon the same side of the line joining their centres.
  2. So as to touch them upon opposite sides of that line 114

(*f*) Upon a given straight line to describe a circular segment which shall contain an angle equal to a given angle 119

(*g*) From a given point, to draw a chord which shall cut off from a given circle a segment containing a given angle—
  1. When the given point is in the circumference . . 119
  [2. When not.][*]

(*h*) To inscribe in, or circumscribe about, a given circle, a triangle similar to a given triangle . 119

(*i*) To inscribe in or circumscribe about a given circle—
  1. An equilateral triangle.
  2. A square.
  3. A regular pentagon.
  4. A regular hexagon.
  5. A regular decagon.
  6. A regular pentedecagon 120

(*k*) To inscribe in or circumscribe about a given circle a regular figure of any required number of sides, very nearly *sch.* 120

(*l*) To describe a circle, in which four given straight lines, of which every one is less than the other three, shall together subtend the whole circumference 91

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[*] Describe a circle about the same centre, touching a chord which is drawn from any point of the circumference to cut off a segment containing the given angle, and through the given point draw a tangent to this circle.

(*m*) To describe a circle, which shall
  1. Pass through three given points, not in the same straight line (to circumscribe a circle about a triangle is the same as this) 114
  2. Pass through two given points, and touch a given straight line 114
  3. Pass through a given point, and touch two given straight lines 114
  4. Touch three given straight lines, of which not more than two are parallel (to inscribe a circle in a triangle is a case of this) 114
  5. Pass through two given points, and touch a given circle *sch.* 116
  6. Pass through a given point, and touch two given circles *sch.* 116
  7. Touch three given circles, of which not more than two are concentric . . *sch.* 117
  8. Pass through a given point, touch a given straight line, and also touch a given circle *sch.* 117
  9. Touch two given straight lines, and also a given circle *sch* 118
  10. Touch a given straight line, and also two given circles *sch.* 118
  In what manner these problems may be modified and varied, see *sch.* 119.
Circle of a sphere (*See* "Spherical Geometry") . . . . . *def.* 184
Circular section of a cone, or cylinder.—*See* "Cone" and "Cylinder."
Circumference of a circle. (*See* "Circle") 3
Circumscribed about a circle, a rectilineal figure said to be . . *def.* 79
——— about a rectilineal figure, a circle said to be . . . *def.* 79
——— about a cone, a pyramid said to be . . . . *def.* 167
——— about a cylinder, a prism, said to be . . . . *def.* 167
Commensurable, magnitudes said to be, 31
  (*a*) Magnitudes, which measure the same magnitude, are commensurable; and if two [or more] magnitudes are commensurable, a magnitude may be found which is measured by them 37
  (*b*) If one magnitude is commensurable with another, which is commensurable with a third, the first is commensurable with the third; and so, of any number of magnitudes . . 37
Common factor of two numbers, is a number which divides each of them.
  (*a*) The greatest may be found by the rule given for the greatest common measure of any two magnitudes *sch.* 36
  (*b*) Every other is contained an exact number of times in the greatest *sch.* 36
  (*c*) The greatest, and the greatest only, dividing the two numbers, gives prime quotients . . *sch.* 36
  (*d*) If $\frac{a}{b} = \frac{c}{d}$, and if a is prime to b, c and d are equimultiples of a and b *note* 47
  (*e*) If $\frac{a}{b} = \frac{c}{d}$, the fractions $\frac{a}{b}$ and $\frac{c}{d}$ have the same lowest terms *note* 47
Common measure of two (or any number of) magnitudes . . . . *def.* 31
  (*a*) Measures their sum and difference 34
  (*b*) If A contains B any number of times, with a remainder R, every common measure of A and B measures R, and every common measure of B and R measures A; also the greatest common measure of B and R is the greatest common measure of A and B 35
  (*c*) Process by which the greatest common measure may be found—
    1. Of two magnitudes.
    2. Of any number of magnitudes 35
  (*d*) Every other common measure measures the greatest . . 36
Common section of any two surfaces is the line in which they cut one another, and which is common to both the surfaces, see p. 128, 131, 151, 214.
Common tangent.—*See* "Circle." (I).
Complement, one angle is said to be the complement of another, when the two together are equal to a right angle.
Complements of parallelograms about the diagonal of a parallelogram . *def.* 3
——— of parallelopipeds about the diagonal plane of a parallelopiped. *See* iv. 22, *Cor.* 2.
Complete conical surface . . *def.* 214
Componendo, a rule in proportion. *See* "Proportion."
Compound ratio, a name given to the ratio which is compounded of two or more ratios . . . *cor.* 45
Compounded, ratios said to be . *def.* 34
——— proportions said to be . *sch.* 48
Concentric, circles (or spheres) are said to be when they have the same centre *def.* 79
Cone, (also its base, axis, slant side, convex surface, frustum) . *def.* 166, 167
  When said to be right, when oblique *def.* 167
  When two cones are said to be similar *def.* 167
  When a pyramid is said to be inscribed in, or circumscribed about a cone *def.* 167
  (*a*) The convex surface of a cone may be supposed to be generated by a straight line which is carried round the circumference of its base, so as to pass always through the vertex *def.* 167
  (*b*) A right cone is generated by the revolution of a right angled triangle about one of the sides . 167
  (*c*) A pyramid may be inscribed in any cone (or circumscribed about it) which shall approach nearer to the cone, in convex surface or in solid content, than by any given difference *cor.* 172 and *sch.* 175

(*d*) A cone is greater than any inscribed pyramid, and less than any circumscribed pyramid; also, the convex surface of the cone is greater than that of any inscribed pyramid, and less than that of any circumscribed pyramid . . 171 and *sch.* 175
(*e*) Any two similar cones being given, similar pyramids may be inscribed (or circumscribed), which shall approach nearer to the cones, in convex surface or in solid content, than by any the same given difference *cor.* 172 and *sch.* 175
(*f*) The convex surface of a right cone is equal to half the product of its slant side and the circumference of its base, or (if R represents the radius of the base, and S the slant side) $=\pi R S$ 172
(*g*) The solid content of every cone is equal to one-third of the product of its base and altitude; or (if R represents the radius of the base, and A the altitude) $=\frac{1}{3}\pi R^2 A$ 173 and *sch.* 175
(*h*) Every cone is equal to the third part of a cylinder which has the same base and the same altitude *cor.* 173 and *sch.* 175
(*i*) Every cone is equal to a pyramid which has an equal base and an equal altitude . *cor.* 173 and *sch.* 175
(*k*) Cones which have equal altitudes are to one another as their bases; and cones which have equal bases, as their altitudes; also any two cones are to one another in that ratio which is compounded of the ratios of their bases and altitudes *cor.* 173 and *sch.* 175
(*l*) The surfaces of similar cones are in the duplicate ratio (or as the squares) of their axes; and their solid contents in the triplicate ratio (or as the cubes) of their axes 173
(*m*) The convex surface of the frustum of a right cone is equal to the product of the slant side of the frustum by half the sum of the circumferences of its two bases; *i. e.* of the slant side and the circumference generated by its middle point . . . 174
(*n*) The solid content of a frustum of a cone, whether it be right or oblique, is equal to the sum of the solid contents of three cones which have the same altitude with the frustum, and, for their bases, its two bases and a mean proportional between them 174
(*o*) If a cone is cut by a plane which is parallel to its base, the section is a circle, having its centre in the axis of the cone . . 215 and 229
(*p*) The subcontrary section of an oblique cone is a circle . . 229
(*q*) If a cone is cut by a plane which is neither parallel to the base nor subcontrary, nor passes through the vertex, the section is a conic section 214 and 230

Conic section . . . *def.* 214
——— vertical plane of . *def.* 215
——— when said to be an ellipse, when a parabola, when an hyperbola *def.* 215

(A) *Of the three Conic Sections.*
(*a*) Every conic section is the perspective projection of a circular section of the cone upon the plane of the conic section by straight lines drawn from the vertex of the cone . 216
(*b*) And in like manner, every circular section may be considered as the perspective projection of the conic section by straight lines drawn from the vertex of the cone . *cor.* 216
(*c*) The projection of every point in the conic section may be found in the circular section, whether it be an ellipse or a parabola, or an hyperbola *cor.* 216
(*d*) The projection of every point in the circular section may be found in the conic section; except in the case of the parabola, the projection of the point in which the vertical plane touches the circular section; and except in the case of the hyperbola, the projections of the two points in which the vertical plane cuts the circular section *cor.* 216
(*e*) A conic section cannot be cut by a straight line in more than two points; and, if a straight line touches a conic section, it shall meet it in one point only, viz., the point of contact 218
[(*f*) If from any point without a conic section, two straight lines are drawn to touch it, every straight line which is drawn through that point to cut the conic section, shall be harmonically divided by the curve and the chord which joins the points of contact; and the tangents at the points in which every such straight line cuts the curve, shall meet one another in the chord produced.—See *lem.* p. 220, *and the demonstration of App.* 20, in p. 227.]
(*g*) If, through any point taken within or without a conic section, there are drawn any number of straight lines, each cutting the curve in two points, and if at every such two points tangents be drawn, cutting one another in a point P, the locus of the points P shall be a straight line, and every straight line which is drawn through the point taken to cut the curve, shall be harmonically divided by that straight line and the curve 227

(B) *Of the Ellipse and Hyperbola.*
(*a*) The curve of the ellipse returns into itself, and incloses an area: the curve of the hyperbola has four infinite arcs. *def.* 215

(*b*) In both, the curve is symmetrically divided by a certain straight line, which cuts it in two points, and is perpendicular to the tangents at those points . . . 216, 217
Transverse (or major) axis, principal vertices, and centre . *def.* 217
(*c*) The hyperbola has two asymptotes, which pass through the centre, and make equal angles with the axis upon opposite sides of it, being parallel respectively to the slant sides in which the vertical plane of the hyperbola cuts the surface of the cone 217
(*d*) If a straight line (not parallel to either asymptote, in the hyperbola) cuts the curve in any point, it may be produced to cut it in a second point; but, in the hyperbola, a parallel to either of the asymptotes cannot meet the curve in more than one point 219
Diameter, vertices, ordinate, abscissæ *def.* 219, 220
(*e*) Every diameter bisects its ordinates, and is itself bisected by the centre 221
(*f*) If a straight line, which cuts the curve in two points, but does not pass through the centre, be bisected by any diameter, it is an ordinate to that diameter . . *cor.* 222
[(*g*) Two straight lines cannot bisect one another, except they both pass through the centre *.]
(*h*) Tangents at the extremities of a diameter are parallel . *cor.* 222
(*i*) Tangents at the extremities of any ordinate meet the diameter in the same point, and that in such a manner (*fig.* p. 221) that C N, C P, C T, are proportionals . . 222
(*k*) The squares of any two semiordinates, of the same diameter, are to one another as the rectangles under the abscissæ . . . 223
(*l*) If two straight lines cut one another, and likewise other two, which are parallel to the two first respectively, and if each of them cuts the curve in two points, or if one or more touch it in a single point, the rectangle under the segments of either of the two first, shall be to the rectangle under the segments of its parallel, as the rectangle under the segments of the remaining one of the first to the rectangle under the segments of its parallel; the square of any of the straight lines being understood, instead of the rectangle under its segments, when it touches the curve instead of cutting it . . . *cor.* 225
(*m*) If, in the hyperbola, a parallel to either of the asymptotes be cut by any two parallel straight lines, the segments of such parallel shall be to one another as the rectangles under the segments of the straight lines; the square being understood instead of the rectangle, if one or both touch the hyperbola . . *cor.* 225
Conjugate diameter of an ellipse, and conjugate axis . *sch.* 226
There is no other axis *note* 226
Conjugate diameter of an hyperbola, and conjugate axis . *sch.* 226
(*n*) The vertices of the conjugate diameters of an hyperbola lie in another hyperbola, which has the same centre and axes as the first . *sch.* 226
Conjugate hyperbolas *sch.* 227
The two are mutually conjugate.
(*o*) If two spheres are described, touching the plane of the curve in two points S, S′, and the conical surface in two circles, the planes of which being produced, cut the plane of the curve in two straight lines X R, X′ R′; then, if P R, P R′ are the perpendiculars drawn from any point P in the curve to the lines X R, X′ R′, S P is to P R, and S′ P to P R′, in the same constant ratio; S P being less than P R in the ellipse, and greater in the hyperbola 229
(*p*) The same being supposed, in the ellipse S P + S′ P = A A′; in the hyperbola, S P ∾ S′ P = A A′. *cor.* 229
The points S, S′ are called the foci, and each of the lines X R, X′ R′, a directrix . . . *sch.* 229

(C) *Of the Parabola.*
(*a*) The curve of the parabola has two infinite arcs . . *def.* 215
(*b*) It is symmetrically divided by a certain straight line, which cuts it in one point only, and is perpendicular to the tangent of that point . 216
Axis and principal vertex, *def.* 217
(*c*) The infinite arcs of the parabola do not admit of asymptotes *cor.* 218
(*d*) The axis of a parabola is parallel to the slant side in which the vertical plane touches the surface of the cone *cor.* 217
(*e*) If a straight line (not parallel to the axis) cut the curve in any point, it may be produced to cut it in a second point; but a parallel to the axis cannot meet the curve in more than one point 219
Diameter, vertex, ordinate, abscissa *def.* 219, 220
(*f*) Every diameter bisects its ordinates 221
(*g*) If a straight line, which cuts the curve in two points, be bisected by any diameter, it is an ordinate to that diameter . . . *cor.* 222
[(*h*) No two straight lines, in a parabola, can bisect one another*.]

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* For each would be parallel to the tangent at the extremity of the diameter (*f*), which is impossible (I. 14. Cor. 2).

(*i*) Tangents at the extremities of any ordinate meet the diameter in the same point, and that in such a manner (see *fig.* p. 222) that N P is equal to P T 221
(*k*) The squares of any two semiordinates of the same diameter are to one another as the abscissæ 223
(*l*) If two straight lines cut one another, and likewise other two, which are parallel to the two first respectively, &c. (see (*l*) of the last division) *cor.* 225
(*m*) If a diameter be cut by any two parallel straight lines, the segments of the diameter are to one another as the rectangles under the segments of the parallels . . *cor.* 225
(*n*) If a sphere is described, touching the plane of the parabola in a point S, and the conical surface in a circle, the plane of which (being produced) cuts the plane of the parabola in a line R X; then, if P R is drawn perpendicular from any point P in the parabola to the line R X, S P is equal to P R. . . 228 and *cor.* 229
The point S is called the focus, and the line R X the directrix *sch.* 229
Conjugate axis of an ellipse or hyperbola *sch.* 226
Conjugate diameters of an ellipse or hyperbola . . . . *sch.* 226
Conjugate hyperbolas . . *sch.* 227
Consequent of a ratio, is the second term.
Consequents of a proportion, are the second and fourth terms.
Construction of a geometrical proposition 3
When said to be a plane construction *sch.* 26
Contact of two circles, when said to be internal, when external . . *note,* 118
——— point of . . . *def.* 79
——— problems of. *See* "Circle." (I) (*m*).
Continued proportion, magnitudes said to be in (*See* "Geometrical Progression.") 34
Content of a solid, is the number of times it contains the cubical unit, or unit of solidity *sch.* 142
The terms "capacity" and "volume" are used in the same sense.
Concave. (*See* "Convex Side.") *def.* 1
Convertendo, a rule in proportion. *See* "Proportion."
Convex, a line or surface is said to be, when its roundness or bulging is everywhere towards the same parts; the test of which, whether it be a line or a surface, is, that it cannot be cut by any the same straight line in more than two points.
Convex side of a line or surface, is that side upon which is the roundness or bulging out; and the other side is called the concave side. . . . . *def.* 1
Convex surface of a prism or pyramid, *def.* 127
——— of a cylinder or cone, *def.* 166, 167
(*a*) A convex surface is greater than a plane surface which has the same contour . . . 167
(*b*) Of two surfaces, one of which is convex and is enveloped by the other, the enveloping surface is greatest 168
Corollary of a proposition . . 3
Cube. (*Also* "Cube of a straight Line.") *def.* 126
(*a*) Cubes are to one another in the triplicate ratio of their edges; *i. e.* the triplicate ratio of two straight lines is the same with the ratio of their cubes. . . . *cor.* 144
(*b*) The difference of two cubes is equal to the sum of three parallelopipeds having the same altitude, viz., the difference of the edges, and for their bases the respective bases of the cubes and a mean proportional between them; *i. e.,* $R^3 - r^3 = (R - r) \times (R^2 + r^2 + R\,r)$ *. . *lem.* 177
(*c*) If the difference of the edges of two cubes may be made less than any given difference, the difference of the cubes may likewise be made less than any given difference . *lem.* 177
(*d*) The cubes of proportional straight lines are proportionals, [and conversely] . . . . *cor.* 144
(*e*) The cube is one of the regular solids, and, as such, is sometimes called the hexahedron . . . 160
Cuboctahedron, a solid derivable from either the cube or the octahedron *sch.* 163
Curve, or *curved line* . . *def.* 1
Upon which side concave, upon which convex . . . *def.* 1
(*a*) A curve is greater than a straight line which has the same extremities, and less (if convex) than the sum of any number of straight lines which are exterior to it, and joins those extremities . . . *sch.* 8
(*b*) Of two curves, one of which is convex and is enveloped by the other, the enveloping curve is greatest *sch.* 8
Cylinder (also its bases, axis, convex surface) *def.* 166
When said to be right, when oblique: when two cylinders are said to be similar: when a prism is said to be inscribed in, or circumscribed about a cylinder . . . 166, 167
(*a*) The convex surface of a cylinder may be supposed to be generated by a straight line which is carried round the circumference of either base so as to be always parallel to the axis 166
(*b*) A right cylinder is generated by the revolution of a rectangle about one of its sides . . . 167
(*c*) A cylinder is greater than any inscribed prism, and less than any circumscribed prism; and the convex

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* Hence the form under which the difference of two magnitudes, which are as the cubes of certain straight lines, appears in IV. 33., V. 11. and V. 24.

surface of the cylinder is greater than that of any inscribed prism, and less than that of any circumscribed prism 168 and *sch.* 175
(*d*) A prism may be inscribed in any cylinder (or circumscribed about it) which shall approach nearer to the cylinder, in convex surface or in solid content, than by any given difference 169 and *sch.* 175
(*e*) Any two similar cylinders being given, similar prisms may be inscribed (or circumscribed), which shall approach nearer to the cylinders, in convex surface or in solid content, than by any the same given difference *cor.* 170 and *sch.* 175
(*f*) The convex surface of a right cylinder is equal to the product of its altitude and the circumference of its base; or (if R represents the radius of the base, and A the axis) $= 2\pi R A$. 170
(*g*) The convex surface of any cylinder is equal to the product of its axis, and the perimeter of a section which is perpendicular to the axis *sch.* 175
(*h*) The solid content of every cylinder is equal to the product of its base and altitude, $= \pi R^2 A$. 170 and *sch.* 175
(*i*) Every cylinder is equal to a prism which has an equal base and an equal altitude . *cor.* 170 and *sch.* 175.
(*k*) Cylinders which have equal altitudes are to one another as their bases; and cylinders, which have equal bases, as their altitudes; also any two cylinders are to one another in the ratio which is compounded of the ratios of their bases and altitudes *cor.* 170 and *sch.* 175
(*l*) The surfaces of similar cylinders are in the duplicate ratio (or as the squares) of their axes; and their solid contents are in the triplicate ratio (or as the cubes) of their axes 170 and *sch.* 175
(*m*) If a cylinder be cut by a plane which is parallel to its base, the section is a circle, having its centre in the axis of the cylinder, and its radius equal to the radius of the base 231
(*n*) The subcontrary section of an oblique cylinder is a circle, having its centre in the axis of the cylinder, and its radius equal to the radius of the base 232
(*o*) If a cylinder, whether right or oblique, be cut by a plane, which is neither parallel to the axis, nor parallel to the base, nor subcontrary, the section is an ellipse . . 231

Data (Lat. things given), in geometrical problems, limits of . . *sch.* 27, 124
Decagon (figure of ten sides) *regular.* *See* "Circle" and "Regular Polygon."
Developable, a term applied to the surfaces of the cone and cylinder.—[All coniform and cylindrical surfaces are developable.] *sch.* 175
Diagonals of a quadrilateral . *def.* 3
See "Quadrilateral," "Parallelogram," and "Circle."
[The diagonals of any rectilineal figure are the straight lines, which join any two angles not immediately adjacent to one another.]
Diagonals of a polyhedron . *def.* 126
See "Parallelopiped."
Diameter of a circle 3, of a sphere 127, of a conic section 219.
Dihedral angle . . . *def.* 125
When said to be right, when acute, when obtuse 125
(*a*) If a dihedral angle is cut by any two parallel planes, it intercepts equal angles in those planes . *cor.* 135
(*b*) A dihedral angle is measured by the rectilineal angle of the perpendiculars to the common section which are drawn in its two planes from any the same point of the common section *sch.* 136
(*c*) Or by the angle of two perpendiculars to its planes, which are drawn from any the same point *sch.* 136
(*d*) To bisect a given dihedral angle 155
(*e*) If a dihedral angle is bisected, every point in the bisecting plane is at equal distances from the planes of the dihedral angle . *cor.* 155
See "Plane"
Direction of projection, in orthographic projection . . . . *def.* 208
Directrix of a conic section *sch.* 229
Distance between two points 9, from a point to a straight line 10, from a point to a plane 131, from one point to another over the surface of a sphere 189.
Diverging lines, projection by. *See* "Projection."
Dividendo, a rule in proportion. *See* "Proportion."
Dodecahedron (solid contained by twelve planes) *regular.* *See* "Regular Polyhedrons."
Duplicate ratio, . . . *def.* 34

Edge, of a polyhedron . . *def.* 126
See "Polyhedron."
Elementary course of Geometry, in the present Treatise . . . *p.* iv
Enneagon (figure of nine sides) *regular,* to inscribe in a circle, very nearly . 121
Enunciation, of a geometrical proposition 3
Equal, circles said to be . *def.* 79
——— ratios said to be . *def.* 32
Equality and inequality, axioms of . 4
Equations of two unknown quantities, solved geometrically . . . *sch.* 124
Equiangular, triangles are said to be, when the angles of the one are equal to the angles of the other, each to each *see* 59
Equilateral triangle . . *def.* 2
Is equiangular, and conversely. *See*

also "Circle" and "Regular Polygon."

Equimultiples of two (or more) magnitudes *def.* 31
- (*a*) Equimultiples of equal magnitudes are equal; and conversely: also the equimultiple of the greater is greater than the equimultiple of the less; and conversely . . *ax.* 34
- (*b*) If two magnitudes, A, A′, are equimultiples of other two B, B′, which again are equimultiples of other two C, C′, the first two shall be equimultiples of the last two . . 34
- (*c*) If two magnitudes A, A′ are equimultiples of other two B, B′, and again of other two C, C′, and if B be a multiple of C, B′ shall be the same multiple of C′. . . *cor.* 34

Ex æquali in proportione directâ, or *ex æquali*, or *ex æquo*, a rule in proportion. *See* "Proportion."

Ex æquali in proportione perturbatâ, or *ex æquo perturbato*, a rule in proportion. *See* "Proportion."

Excube, extetrahedron, exoctahedron, exdodecahedron, exicosahedron, solids formed from the regular solids . *see sch.* 162

Explementary, an angle said to be *note* 85

Exterior angle. *See* "Straight Line," "Triangle," "Rectilineal Figure," "Spherical Geometry."

Extreme and mean ratio, a straight line said to be divided in. (*See* "Straight line.") 71

Extremes of a proportion, are the first and last terms . . . . *def.* 33

Faces of a polyhedron . . *def.* 126
- May be of any number greater than three. *See* "Polyhedron."

Figure, plane, *def.* 2, plane rectilineal, *def.* 2 *See* "Rectilineal Figure," and "Circle."

Focus, of a conic section. *See* "Conic Section."

Foot of a perpendicular, is the point in which it meets the line (or plane) to which it is perpendicular.

Fourth proportional . . *def.* 33
- May be found to any three given magnitudes . . *note* 43, 54

Frustum of a pyramid, *def.* 127, of a cone 167. *See* "Pyramid," and "Cone."

Geometrical Mean, *def.* 33. *See* "Arithmetical Mean," and "Straight line."

Geometrical Analysis. *See* "Analysis."

Geometrical Progression, magnitudes said to be in 34
- (*a*) The differences of magnitudes, which are in geometrical progression form a geometrical progression having the same common ratio . *cor.* 42
- (*b*) Magnitudes A, B, C, &c., which are in geometrical progression, are the differences of other magnitudes A′, B′, C′, D′, &c. in a geometrical progression having the same common ratio and in which the first term A′ is to A as A to A ∼ B. . . *cor.* 42
- [(*c*) And the sum of any number of the magnitudes A, B, C, &c. in succession, is equal to the difference of two of the magnitudes A′, B′, C′, D′, &c.: thus, A is the difference of A′ and B′, A + B of A′ and C′, A + B + C of A′ and D′, and so on.]
- [(*d*) Hence, if A is greater than B, the sum of the whole series A, B, C, D, &c. continued without end is equal to A′; *i. e.* the sum of a finite number of terms is less than A′, but by the continued addition of new terms may be made to approach to it by less than any given difference.]

Geometry, its subject . . . 1
- Geometry is distinguished by the epithets of "plane," "solid," and "spherical," according as it treats of plane figures and lines in one plane, or of solid figures and lines in different planes, or of figures and lines upon the surface of a sphere.

General theory of proportion . 48

*General** *properties of the conic sections* 214

Generated, meaning of the word as applied to solids of revolution . . 127
- *See* also "Cone," "Cylinder," "Plane," "Sphere."

Great Circle of a sphere . *def.* 184
- *See* "Spherical Geometry."

Harmonical mean . . *def.* 67
- (*a*) The harmonical mean between two numbers, m and n, is $\frac{2mn}{m+n}$ *sch.* 97
- (*b*) If five harmonical means are taken between 1 and $\frac{1}{2}$, we obtain the progression 1, $\frac{6}{7}$, $\frac{3}{4}$, $\frac{2}{3}$, $\frac{3}{5}$, $\frac{6}{11}$, $\frac{1}{2}$; if seven, the progression 1, $\frac{8}{9}$, $\frac{4}{5}$, $\frac{8}{11}$, $\frac{2}{3}$, $\frac{8}{13}$, $\frac{4}{7}$, $\frac{8}{15}$, $\frac{1}{2}$; from which progressions, rejecting such fractions as have 7, 11, and 13 in the denominator, those remaining, viz. 1, $\frac{8}{9}$, $\frac{4}{5}$, $\frac{3}{4}$, $\frac{2}{3}$, $\frac{3}{5}$, $\frac{8}{15}$, $\frac{1}{2}$, will represent the lengths of strings producing with the same thickness and tension the sounds denoted by C, D, E, F, G, A, B, *c.* *note* 67

Harmonically divided, a straight line said to be 68

Harmonicals, four straight lines when said to be 68

Harmonical progression, magnitudes said to be in 68
- to what observation the name is owing *note*, 67

Hemisphere, the half of a sphere, [is equal to

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* By "general properties," are here meant such properties as may be declared in the same words, or nearly so, (*i. e.* with some slight difference arising from their difference of form,) to pertain to the three conic sections. Those in the Appendix are moreover only such as admit of being derived from the circle by perspective projection. See "Conic Sections."

two-thirds of a cylinder upon the same base and of the same altitude, *cor.* 179].

Hendecagon (figure of 11 sides) *regular*, to inscribe in a circle, very nearly . 121

Heptagon (figure of 7 sides) *regular*, to inscribe in a circle, very nearly . 121

Hexagon (figure of 6 sides) *regular*. *See* "Regular Polygon" and "Circle."

Hexahedron (solid contained by 6 planes) *regular*. *See* "Regular Polyhedrons."

Homologous edges of similar polyhedrons, are the homologous sides of their similar faces.

Homologous sides of similar figures *def.* 57

Homologous terms of a proportion *def.* 33

Hyperbola, one of the conic sections *def.* 215 *See* "Conic Section."

Hypotenuse of a right-angled triangle *def.* 3 *See* "Triangle."

[The side opposite to the right angle is called the Hypotenuse also in right-angled spherical triangles.]

Icosa-dodecahedron, a solid derivable either from the icosahedron, or from the dodecahedron 163

Icosahedron (solid contained by 20 planes) *regular*. *See* "Regular Polyhedrons."

Inclination of a straight line to a straight line, is the acute angle which the former makes with the latter.

——— of a plane to a plane, is the acute dihedral angle, which the former makes with the latter.

——— of a straight line to a plane *def.* 125

Incommensurable, magnitudes said to be 32

(*a*) If one magnitude is incommensurable with another, it is incommensurable with every magnitude which is commensurable with that other *cor.* 37

(*b*) Although the ratio of two incommensurable magnitudes can never be exactly expressed by numbers, yet two numbers may be obtained which shall express it to any required degree of exactness 48

(*c*) Magnitudes are incommensurable, when the process for finding the greatest common measure leads to no conclusion, but has an unlimited number of steps . . *cor.* 35

(*d*) If P and Q are two magnitudes of the same kind, and if Q be contained in P any number of times with a remainder R, which is to Q as Q to P, P and Q are incommensurable *sch.* 73

(*e*) The same being supposed, if n be the number of times that Q is contained in P, the ratio of Q to P shall lie between the ratios of any two successive terms of the series 1, n, a, b, c, &c., where a, the third term, is equal to $n \times n + 1$, b, the fourth term $= na + n$, c, the fifth term $= nb + a$, and so on, every successive term being equal to n times the last, together with the last but one [*] . *sch.* 73

(*f*) The parts of a line divided in medial ratio are incommensurables of this class, and their ratio may be approximated to by the series 1, 1, 2, 3, 5, 8, &c.: the side and the sum of the side and diagonal of a square are incommensurables of the same class, and their ratio may be approximated to by the series 1, 2, 5, 12, 29, &c. *sch.* 73

Infinite arc is an arc of unlimited extent; such as, for example, occurs in perspective projection, when any point of the original curve lies in the vertical plane. 211

The hyperbola affords an example of infinite arcs with asymptotes, the parabola of infinite arcs without asymptotes.

Infinite extent. Straight lines and arcs of infinite extent may have finite projections, and *vice versâ* . . 209, 210

Inscribed in a circle, a rectilineal figure said to be 79

——— in a rectilineal figure, a circle said to be 79

——— in a cone or cylinder, a pyramid or prism said to be . . 167

——— in a sphere, a solid of revolution said to be . . . 167

Interior, a term applied to any of the angles of a triangle, in contradistinction to the exterior angle formed by producing one of the sides; also to certain angles made by parallels with a straight line cutting them 13

Invertendo, a rule in proportion. (*See* "Proportion.")

Joined, two points are said to be,

(*a*) In space, when the straight line between them is drawn.

(*b*) Upon the surface of a sphere, when the spherical arc between them is drawn. . . . 184

Isosceles[†] *triangle*. (*See* "Triangle.") *def.* 2

Isosceles spherical triangle. *See* "Spherical Geometry."

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[*] It is observed in p. 74, that this series may be derived from the doctrine of continued fractions: For, every successive quotient in the division of P by Q, of Q by R, of R by the next remainder, and so on, being n, the ratio of Q to P is represented by the continued fraction $\cfrac{1}{n+\cfrac{1}{n+\cfrac{1}{n+\&c.}}}$; therefore the fractions $\frac{1}{n}$, $\cfrac{1}{n+\cfrac{1}{n}}$, $\cfrac{1}{n+\cfrac{1}{n+\cfrac{1}{n}}}$, &c., i. e. $\frac{1}{n}$, $\frac{n}{n \times n+1}$ $\left(\frac{n}{a}\text{ suppose}\right)$, $\frac{a}{na+n}$ $\left(\frac{a}{b}\text{ suppose}\right)$, $\frac{b}{nb+a}$, &c. will represent the ratio, each more nearly than the preceding.

[†] "Having equal legs," from the Greek

Lateral surface of a prism or pyramid *def.* 127
Leg, of an angle, *def.* 1. *See* "Angle."
Lemma (Gr.), is an auxiliary proposition.
Length of a line is the number of times it contains the unit of length . *sch.* 18
Like parts of two (or more) magnitudes, *def.* 31
See "equimultiples" and "ratio."
Limits.
(*a*) Two fixed magnitudes A, B, are said to be limits of two others P, Q, when P and Q, by increasing together, or by diminishing together, may be made to approach more nearly to A and B respectively, than by any the same given difference, but can never become equal to, much less pass A and B: *e.g.* two circles are the limits of the similar inscribed or circumscribed regular polygons 46
(*b*) If two magnitudes A and B are the limits of two others P and Q, and if P is always to Q in the same constant ratio, A is to B in the same ratio [*] 46
For examples of this theorem see 56, 96, 146, 171, 173.
Limits of a geometrical problem, are of frequent occurrence, and are commonly indicated by the construction: loci are useful in determining them . 27, 106, 124
Line, (also "straight line" "curved line.") *def.* 1
See "Curve," "Straight Line," and "Projection."
Locus (Lat., place), . . *def.* 106
[More generally, a locus is any part of space, every point of which, and none else, satisfies certain conditions.]
When said to be a simple locus, when a plane locus, when of higher dimensions . . . 107
How serviceable in the solution of problems . . . 106
Is the limit between excess and defect 106
(A) *Examples of Simple Loci.*
(*a*) All points which are equidistant from two given points . 107
(*b*) All points which are equidistant from two given straight lines 107
[(*c*) The extremities of all equal parallels whose other extremities lie in the same given straight line, I. 16. *cor.*]
(*d*) All points which divide lines falling from a given point to a given straight line in the same given ratio . 108
(*e*) The vertices of all triangles upon the same base, which have the side terminated in one extremity of the base greater than the side terminated
or between two parallel planes, or such
in the other extremity, and the difference of their squares equal to a given square . . . 108
(*f*) The points which divide the chords of a circle passing through any the same given point harmonically 112
(*g*) The points in which the two tangents, drawn at the extremities of all chords pasing through a given point, meet one another . . 112

(B) *Examples of Plane Loci.*
(*a*) The vertices of all triangles upon the same base and the same side of it, which have equal vertical angles 108
(*b*) The vertices of all triangles upon the same base, which have the sum of the squares of their sides equal to a square 109
(*c*) The vertices of all triangles upon the same base, which have the side terminated in one extremity greater than the side terminated in the other extremity, and the sides (or which is the same thing, the squares of the sides) in a given ratio . 109
(*d*) The vertices of all triangles upon the same base, which have the square of one side in a given ratio to the square of the other side diminished by a given square . *sch.* 110
(*e*) The vertices of all triangles upon the same base, which have the square of one side diminished by a given square in a given ratio to the square of the other side diminished by another given square . *sch.* 110

(C) *Examples of Loci satisfying conditions in Solid Geometry.*
[(*a*) All the points in space which are equidistant from two given points lie in the plane which bisects the distance between them at right angles.]
[(*b*) All the points in a given plane, which are equidistant from two given points without the plane, lie in the common section with the given plane of the plane which bisects at right angles the distance between the two points.]
(*c*) All the points which are equidistant from three given points, not lying in the same straight line, lie in the straight line which is drawn perpendicular to their plane from the centre of the circle which passes through them. *cor.* 151
[(*d*) All the points equidistant from a straight line and plane, or from two given planes, lie in the straight line or plane which bisects their angle of inclination.]
(*e*) The extremities of all equal parallels whose other extremities lie in one and the same plane, lie in a plane parallel to it. . . . *cor.* 134
(*f*) The points which divide all straight lines drawn from a point to a plane,

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[*] There are some very important errors in the latter part of the demonstration of this theorem, for the correction of which the reader is desired to consult the errata.

straight lines produced in a given ratio, lie in another parallel plane *cor.* 135

(D) *Examples of Loci satisfying conditions in Spherical Geometry.*

[(*a*) All points upon the surface of a sphere, which are equidistant from two given points of that surface, lie in the circumference which bisects their distance at right angles.]

[(*b*) All points which are equidistant from the circumferences of two given great circles, lie in the circumference which bisects the angle between them.]

(*c*) The vertices of all equal spherical triangles upon the same base and upon the same side of it, lie in the circumference of a small circle, such that one equal and parallel to it may be drawn through the extremities of the base . . *cor.* 199

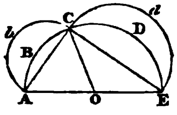
(*d*) The vertices of all spherical triangles upon the same base, which have the vertical angle equal to the sum of the other two, lie in the circumference of a small circle, whose pole is the middle point of the base, and its polar distance half the base *sch.* 201

Lowest terms of the ratio of two magnitudes, *See* "Numerical ratio."

Lune, (Lat., moon) spherical . *def.* 180 *See* "Sphere" and "Spherical Geometry."

Lunes (*contained by circular arcs in the same plane*) *quadrature of.*

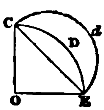
[(*a*) If a semicircumference A B C D E be divided into any two arcs, A B C, C D E, and if upon the chords of these arcs semicircles are described, as in the adjoined figure, the lunes A B C *b*, C D E *d* shall be together equal to the triangle A C E.



For, semicircles (III. 33.) being as the squares of their diameters, the semicircles upon A C and C E are together equal to the semicircle upon A E; therefore, taking away the segments A B C and C D E, the lunes which remain are together equal to the triangle A C E.]

[(*b*) The lune which is included by a semicircumference C *d* E and a quadrant C D E, is equal to the triangle C O E, whose vertex O is the centre of the quadrant.

For if the arcs A C, C E, in the former figure, are equal to one another, each of them will be a quadrant, and the two lunes will be equal to one another; and the two triangles, C O A, C O E are also equal; therefore, since the halves of equals are equal, the lune C D E *d* is equal to the triangle C O E.]



Maximum (Lat., greatest) is a name given to the greatest among all magnitudes of the same kind which are subject to the same given conditions: as *minimum* (Lat., least) is, on the other hand, the name given to the least.

For examples of maxima and minima on a plane *See* III. § 5, p. 99.
on a spherical surface, *See* 199 and *sch.* 201.

Mean. *See* "Arithmetical mean," "Geometrical mean," "Harmonical mean,"

Mean proportional, one or more *def.* 33, 34 *See* "Geometrical mean."

Means of a proportion, are the two middle terms.

Measure, one magnitude said to be of another . . . 31

Measure of a rectilineal angle. *See* "Angle."
——— of a dihedral angle. *See* "Dihedral angle."
——— of a solid angle. *See* "Solid angle."
——— of a spherical angle. *See* "Spherical angle."
——— of the surface of a spherical triangle or polygon. *See* "Spherical Geometry."

Medial ratio is the same with "extreme and mean ratio" . . . 71 *See* "Straight line," and "Incommensurable."

Medially divided, a line said to be . 71

Minimum, see "Maximum."

Multiple, one magnitude said to be of another 31

Nonagon. *See* "Enneagon."

Numbers, common factor of two or more. *See* "Common factor."

Numerical ratio of two magnitudes *def.* 32
Its terms, antecedent and consequent *def.* 32
(*a*) The lowest terms are those which are determined by the greatest common measure; and are, therefore, prime to one another . *cor.* 36
(*b*) Any other terms are equimultiples of the lowest terms . *cor.* 36
(*c*) The lowest terms may be found from any terms by dividing them by their greatest common factor: they serve when found to determine whether two numerical ratios are different, or only different forms of the same ratio *cor.* 36
(*d*) A numerical ratio, which is compounded of any number of ratios, has for its antecedent the product of their

antecedents, and for its consequent the product of their consequents. *sch.* 45

Oblique, a term applied to angles (whether rectilineal, dihedral, or spherical) which are not right angles.

Oblique cone, def. (*See* "Cone.") . 167

Oblique cylinder, def. (*See* "Cylinder.") 166

Obtuse (Lat. blunted), a term applied to angles (whether rectilineal, dihedral, or spherical) which are greater than right angles.

Octagon, (figure of eight sides) *regular*. (*See* "Regular Polygon") . . *sch.* 121

Octahedron (solid contained by eight planes) *regular*. *See* "Regular Polyhedron."

Opposite cone, def. (also "opposite conical surface") . . . 214

Opposite points, on the surface of a sphere, *def.* (*See* "Spherical Geometry.") 186

Orb, spherical, *def.* (*See* "Sphere.") 180

Ordinate, in a conic section . *def.* 220
See "Conic Section."

Original point, line or plane, in projection *def.* 208

Orthographic projection. *See* "Projection."

Parallel[*] *circles*, of the sphere *def.* 184
See "Sphere."

Parallel planes, def. (*See* "Plane.") 125

Parallel ruler, an instrument for drawing parallel lines . . . . *note*, 24

Parallel straight lines. (*See* "Straight line.") *def.* 2

Parallel to a plane, a straight line said to be 125

Parallel to a straight line, a plane said to be 125

Parallelogram . . . . *def.* 2
- (*a*) Its opposite sides and angles are equal to one another . 15
- (*b*) Its diagonals bisect one another 15
- (*c*) It is bisected by each of its diagonals . . . . . *cor.* 16
- (*d*) The squares of its diagonals are, together, equal to the squares of its four sides . . . *cor.* 24
- (*e*) If a quadrilateral figure has its opposite sides equal to one another, or its opposite angles equal, or if its diagonals bisect one another, or if it is bisected by each of its diagonals, or if the squares of its diagonals are, together, equal to the squares of the four sides, the quadrilateral is a parallelogram 15, *cor.* 17, and *cor.* 24
- (*f*) The complements of parallelograms about the diagonal of a parallelogram are equal to one another . 16
- (*g*) If one angle of a parallelogram is a right angle, all its angles are right angles . . . *cor.* 16
- (*h*) Every parallelogram is equal to a rectangle having the same base and the same altitude, *i. e.* to the product of its base and altitude . *cor.* 16
- (*i*) Parallelograms upon the same or equal bases, and between the same parallels, are equal to one another 16
- (*k*) If a parallelogram and a triangle stand upon the same base, and between the same parallels, the parallelogram is double of the triangle . 17
- (*l*) Parallelograms having the same altitude are to one another as their bases; parallelograms having equal bases are to one another as their altitudes; and any two are to one another in the ratio which is compounded of the ratios of their bases and altitudes *cor.* 62, 63
- (*m*) Equiangular parallelograms are to one another in the ratio which is compounded of the ratios of their sides *cor.* 63
- (*n*) Equiangular parallelograms, which have their sides reciprocally proportional, are equal; and, conversely, equiangular parallelograms, which are equal, have their sides reciprocally proportional . . *cor.* 64
- (*o*) To describe a parallelogram
  1. When two sides and the included angle are given . . 28
  2. Upon a given base, and with a given angle, which shall be equal to a given triangle, or rectilineal figure . . *cor.* 30

Rectangular. *See* "Rectangle."

Parallelopiped . . . *def.* 127
- (*a*) Its opposite faces are similar and equal parallelograms, and its opposite edges are [equal and] parallel . 139
- (*b*) Its four diagonals pass through the same point, and bisect one another *sch.* 140
- (*c*) It is bisected by each of the diagonal planes[*] . . 139
- (*d*) The squares of its four diagonals are together equal to the squares of the twelve edges . . *sch.* 140
- (*e*) The complements of the parallelopipeds about the diagonal plane of a parallelopiped are equal to one another *cor.* 140
- [(*f*) If one face is at right angles to each of its adjoining faces, every face is at right angles to each of its adjoining faces.]
- (*g*) Every parallelopiped is equal to a rectangular parallelopiped, having the same base and the same altitude; *i. e.* to the product of its base and altitude *cor.* 142
- (*h*) Parallelopipeds upon the same, or upon equal bases, and between the

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[*] From two Greek words, signifying "along one another."

[*] The prisms, into which a parallelopiped is divided by its diagonal plane, are symmetrical; and, therefore, except in the case of the rectangular parallelopiped, their equality cannot be established by coincidence; a circumstance which has been overlooked in Book XI. prop. 28, of Simson's Euclid. *See* "Polyhedron."

same parallel planes, are equal to one another . . 140, 141

(*i*) If a parallelopiped and a tetrahedron have the same three edges, the parallelopiped is equal to six times the tetrahedron . . . *cor.* 147

(*k*) Parallelopipeds having the same altitude are to one another as their bases; parallelopipeds having equal bases are to one another as their altitudes; and any two are to one another in the ratio which is compounded of the ratios of their bases and altitudes 143

(*l*) Parallelopipeds which are equiangular (so that a solid angle of the one may be made to coincide with a solid angle of the other) are to one another in the ratio which is compounded of the ratios of their edges *cor.* 144

Rectangular. *See* "Rectangular Parallelopiped."

Part, or measure . . *def.* 31

Pentagon (figure of five sides) *regular*. *See* "Regular polygon" and "Circle."

Pentedecagon (figure of 15 sides) *regular*. *See* "Regular polygon" and "Circle."

Perimeter of a plane figure . . *def.* 2

Minimum of. *See* "Circle."

Periphery, (Gr.) another name for "circumference."

Perpendicular to a straight line, a straight line said to be, 2; a plane said to be, 125

To a plane, a straight line said to be, 125; a plane said to be, 125.

To a spherical arc, a spherical arc said to be, 185.

See "Straight line," "Plane," and "Spherical Geometry."

Plane . . . . *def.* 1

When said to be parallel or perpendicular to a plane . . 125

When said to be parallel or perpendicular to a straight line . . 125

(A) (*a*) A plane, and one only, may be made to pass through a straight line and a point without it, or three given points not in the same straight line, or the sides of a given rectilineal angle, or two given parallels . . 127

(*b*) Any number of parallels through which the same straight line passes, are in one and the same plane *cor.* 128

(*c*) Any number of planes may be made to pass through the same straight line *cor.* 128

(*d*) The common section of two planes is a straight line . . 128

(*e*) If there be three planes, and if the common section of two of the planes be not parallel to the third plane, the three planes shall pass through the same point; viz., the point in which the common section of two of the planes meets the third plane *sch.* 156

(*f*) A plane surface is less than any other surface having the same contour *lem.* 167

(B) *Of a plane and perpendicular straight line.*

A straight line when said to be perpendicular to a plane . . 125

(*a*) If a straight line stand at right angles to each of two other straight lines at their point of intersection, it shall be at right angles to the plane in which they are . . 128

(*b*) Any number of straight lines, which are drawn at right angles to the same straight line from the same point of it, lie all of them in the plane which is perpendicular to the straight line at that point . . . *cor.* 129

(*c*) If the plane of a right angle be made to revolve about one of its legs, the other leg will describe a plane at right angles to the first leg *cor.* 129

(*d*) If a straight line be perpendicular to a plane, and if from its foot a perpendicular be drawn to a straight line taken in the plane, any straight line, which is drawn from a point in the former perpendicular to meet the foot of the latter perpendicular, shall likewise be perpendicular to the straight line taken in the plane . 129

(*e*) If a straight line be perpendicular to a plane, and if from any point of it a perpendicular be drawn to a straight line taken in the plane, the straight line which joins the feet of the perpendiculars shall likewise be perpendicular to the straight line taken in the plane . . . 129

(*f*) Straight lines which are perpendicular to the same plane are parallel; and conversely, if there be two parallel straight lines, and if one of them be perpendicular to a plane, the other shall be perpendicular to the same plane . . . 129

(*g*) Perpendiculars to the same plane, which are drawn to it from points of the same straight line, lie in one and the same plane . . *cor.* 130

(*h*) A straight line may be drawn perpendicular to a plane of indefinite extent, from any given point, whether the given point be without or in the plane; but from the same point there cannot be drawn more than one perpendicular to the same plane 130

(*i*) From a point to a plane the perpendicular is the shortest distance; and of other straight lines which are drawn from the point to the plane, such as are equal to one another, cut the plane at equal distances from the foot of the perpendicular; and such as are unequal, cut the plane at unequal distances from the foot, the greater being always further from the perpendicular, and conversely . . 131

(*k*) If from any point taken without a plane, a sphere be described with a radius less than the perpendicular, it will not meet the plane; if with a radius equal to the perpendicular, it will meet it in one point only, viz. the foot of the perpendicular; and, if with a radius greater than the perpendicular, its surface will cut the plane in the circumference of a circle, having for its centre the foot of the perpendicular . . . *cor.* 131

(*l*) To draw a plane perpendicular to a given straight line—
  (1) From a given point in the straight line.
  (2) From a given point without it . 153

(*m*) To draw a straight line perpendicular to a given plane—
  (1) From a given point in the plane.
  (2) From a given point without it 151

(C) *Of a plane and inclined straight line.*

Angle of inclination of a straight line to a plane . . . *def.* 125

(*a*) If a straight line be inclined to a plane; of all the angles which it makes with straight lines meeting it in that plane, the least is the angle of inclination; and, with respect to every other of these angles, a second angle may always be drawn which shall be equal to it—viz., upon the other side of the angle of inclination; but there cannot be drawn in the plane more than two straight lines with which the inclined straight line shall make equal angles, one upon each side of the angle of inclination . . . 132

[(*b*) If a straight line cuts a plane, every straight line which is parallel to it shall cut, and be equally inclined to the same plane.]

(*c*) Through a given point in a given plane, to draw a straight line at right angles to a straight line which is inclined to the plane at that point 153

(D) *Of a plane and parallel straight line.*

(*a*) If one straight line is parallel to another, it is parallel to every plane which passes through that other 132

(*b*) If a straight line is parallel to a plane, it is parallel to the common section of every plane which passes through it with that plane . . 132

[(*c*) If a straight line is parallel to a plane, every straight line, which is parallel to it, is parallel to the same plane, *see* app. prop. 7.]

(*d*) If two straight lines are parallel, the common section of any two planes passing through them is parallel to either of them . . *cor.* 132

[(*e*) If there be three planes, and if the common section of two of the planes is parallel to the third plane, the common sections of the three planes are parallel.]

(*f*) To draw through a given point—
  1. In a given plane, a straight line which shall be parallel to another given plane . . 153
  2. A straight line which shall be parallel to each of two given planes . . *cor.* 153

(*g*) To draw through a given straight line a plane which shall be parallel to a given straight line . . 154

(E) *Of parallel, inclined, and perpendicular planes.*

(*a*) If two straight lines, which cut one another, are parallel each of them to the same plane, the plane of the two straight lines is parallel to that plane *cor.* 133

(*b*) Planes, to which the same straight line is perpendicular, are parallel; and, conversely, if two planes are parallel, and if one of them is perpendicular to a straight line, the other is perpendicular to the same straight line 133

(*c*) Through any given point a plane may be drawn, and one only, which shall be parallel to a given plane . *cor.* 134

(*d*) Planes, which are parallel to the same plane, are parallel to one another . . . *cor.* 134

(*e*) If parallel planes are cut by the same plane, their common sections with it are parallel . . 134

(*f*) If two planes, which cut one another, are parallel to other two which cut one another, each to each, the common sections of the first two and of the second two are parallels *cor.* 134

(*g*) If two planes are parallel, a straight line which cuts one of them may be produced to cut the other likewise *cor.* 209

[(*h*) If two planes are parallel, a straight line, which is parallel, or perpendicular, or inclined to one of them, shall be parallel or perpendicular, or equally inclined to the other.]

(*i*) If parallel straight lines are cut by parallel planes, the parts of the straight lines, which are intercepted between the planes, are equal to one another . . . 134

(*k*) Parallel planes are everywhere equidistant . . . *cor.* 134

(*l*) If any two straight lines are cut by three parallel planes, the parts of the straight lines which are intercepted by the planes shall be to one another in the same ratio . 134

(*m*) If two straight lines, which meet one another, are parallel respectively to other two which meet one another, the contained angles shall be equal, and their planes parallel . 135

(*n*) The dihedral angle made by any two planes which meet one another is measured by the rectilineal angle of the perpendiculars to the common section which are drawn in the two planes from any the same point of the common section . 136

(o) Dihedral angles, which have the sides of the one parallel, or perpendicular, or equally inclined to the sides of the other, and in the same order, are equal to one another *sch.* 136

(p) If one plane is at right angles to another, the perpendiculars to the common section, which are drawn in the two planes from any the same point of the common section, are at right angles to one another; and conversely . . . *cor.* 136

(q) If one plane is perpendicular to another, any straight line, which is drawn in the first plane at right angles to their common section, is perpendicular to the other plane 136

(r) If a straight line is perpendicular to a plane, every plane which passes through it is perpendicular to the same plane 136

(s) If two planes which cut one another are each of them perpendicular to a third plane, their common section is perpendicular to the same plane . . *cor.* 137

(t) If through the same point there pass any number of planes perpendicular to the same plane, they all of them pass through the same straight line, viz., the perpendicular which is drawn from the point to the plane . *cor.* 137

(u) If two planes are parallel, a plane which is parallel, or perpendicular, or inclined to one of them, shall be parallel or perpendicular, or equally inclined to the other . *sch.* 136

(v) Through a given point, to draw a plane which shall be parallel to a given plane . . . 154

(x) Through a given point, to draw a plane perpendicular to each of two given planes . . . 154

(y) Through a given straight line to draw a plane perpendicular to a given plane 153

Plane Geometry is that part of Geometry which treats of plane figures and lines in one plane.

What constructions are said to be practicable by Plane Geometry *sch.* 26

Plane Locus. (*See* "Locus.") *def.* 107

Plane Section of a solid, is any section made by a plane.

———— of a cone, cylinder, or sphere. *See* "Cone," "Cylinder," or "Sphere."

Planes, angle contained by. *See* "Dihedral angle."

Point . . . . *def.* 1

Point of contact . . *def.* 79

(a) If a straight line touches a circle, or if a plane touches a sphere, the foot of the perpendicular which is drawn from its centre to the straight line or plane is the point of contact 80, 129

(b) If a circle touches a circle, or a sphere a sphere, the straight line joining their centres, or that straight line produced, passes through the point of contact . . 82, 151

Point of intersection, of three planes. (*See* "Plane.") . . . *sch.* 156

————, of three spherical surfaces . . . *sch.* 151

Pole of a circle of the sphere. (*See* "Spherical Geometry.") . . *def.* 184

Polar distance of a circle of the sphere · *def.* 184

Polar triangle. (*See* "Spherical Geometry.") *def.* 185

Polygon. (*See* "Rectilineal Figure," "Regular Polygon.") . . *def.* 2

Polygon, spherical. (*See* "Spherical Geometry.") . . . . *def.* 186

Polyhedron. (*Also* "Diagonals of a Polyhedron.") . . . *def.* 126

When said to be regular . 126

When two polyhedrons are said to be similar. (*See* Note upon this *def.*) 126

[Two polyhedrons are said to be symmetrical, when a face of the one may be made to coincide with a face of the other, and, these being made to coincide, the straight lines which join the solid angles of the one with the corresponding solid angles of the other are perpendicular to, and bisected by, the plane of the faces.]

(a) If S, E, and F represent respectively the number of solid angles, the number of edges, and the number of faces of a polyhedron, $S - E \times F = 2$ *sch.* 197

(b) If S represents the number of solid angles, the sum of all the plane angles of the faces is equal to $(S - 2)$ times 4 right angles . . *sch.* 197

(c) The solidity of a polyhedron may be obtained by dividing it into pyramids, having for their common vertex one of the vertices of the polyhedron, or some point within it . *sch.* 147

(d) Similar polyhedrons are divided into the same number of similar pyramids, by drawing diagonals from any two corresponding angles, and planes along those diagonals . 150

(e) Similar polyhedrons are to one another in the triplicate ratio (or as the cubes) of their homologous edges; and their convex surfaces are in the duplicate ratio (or as the squares) of those edges . . 150

[(f) If four straight lines are proportionals, any two similar polyhedrons which have the first and second for homologous edges, and any two which have the third and fourth, are proportionals. (IV. 27. *cor.* 3.]

[(g) Symmetrical polyhedrons may be divided into the same number of

symmetrical pyramids, by drawing diagonals from any two corresponding angles, and planes along those diagonals.]
[(*h*) Symmetrical polyhedrons are equal to one another.]
Postulates (Lat., things required) .. 3
——— in the constructions of theorems in plane geometry . . . 3
——— in the constructions of problems in plane geometry . . . 24
———, in the constructions of problems in solid geometry . . 150
——— in the constructions of problems in spherical geometry . . 203
Prime, numbers said to be (*See* "Common Factor.") . . . *sch.* 36
Principal edges of a prism or pyramid, *def.* 127
Principal diameter of a conic section *def.* 217
Principal vertices, or vertex, of a conic section *def.* 217
Prism, def. (also its bases, sides, principal edges, lateral or convex surface) 127
When said to be right, when oblique 127
When said to be regular (its axis) 127
(*a*) Triangular prisms, which have two sides and the included angle of the one equal to two sides and the included angle of the other, are equal to one another . . . 138
(*b*) If the upper part of a triangular prism be cut away by any plane, the remaining solid is equal to the sum of three pyramids, having the same base with the prism, and for their vertices the upper extremities of the diminished edges . *sch.* 147
(*c*) Every prism is equal to a rectangular parallelopiped, having an equal base and altitude; *i. e.* to the product of its base and altitude . 144
(*d*) Prisms which have equal altitudes are to one another as their bases; and prisms which have equal bases as their altitudes; also, any two prisms are to one another in the ratio which is compounded of the ratios of their bases and altitudes . *cor.* 145
(*e*) The solid content of any prism is equal to the product of the principal edge, and the area of a plane section perpendicular to it; and the convex surface is equal to the product of the principal edge, and the perimeter of the same section. In the right prism, this section is the same with the base *sch.* 145
[(*f*) Prisms which have for their bases similar polygons, and the principal edges drawn from two corresponding angles of those polygons, making equal angles with each of the homologous adjoining sides, and in the same ratio as those sides, are similar.]
[(*g*) Similar prisms are as the cubes of their homologous edges. (IV. 35.)]
[(*h*) Symmetrical prisms are equal to one another.]
(*i*) If the convex surface of a prism be produced to any extent, the sections of it by parallel planes will be similar and equal polygons . *sch.* 145
Problem (Gr., a thing put forth or proposed) 3
Product, strict meaning of the word when used in geometry . *sch.* 18, 142
And hence it is used synonimously with rectangle or rectangular parallelopiped, these figures being measured by the products of their bases and altitudes, or of their respective dimensions.
Progression, arithmetical. *See* "Arithmetical Progression."
——— geometrical. *See* "Geometrical Progression."
——— harmonical. *See* "Harmonical Progression."
Projection, orthographic and perspective 208

(A) *Of Orthographic Projection.*
Orthographic projection of a point, line, or figure . . . *def.* 208
Plane of projection, direction of projection, original . . *def.* 208
(*a*) The orthographic projection of every given point may be found upon the plane of projection . . 212
(*b*) The orthographic projection of a straight line (not parallel to the direction of projection) is a straight line—viz., the common section of the plane of projection, and a plane drawn through the original, parallel to the direction of projection . 213
(*c*) If the original straight line be parallel to the plane of projection, the orthographic projection of such straight line is a point . *cor.* 213
(*d*) The orthographic projection of a straight line, which is parallel to the plane of projection, is parallel, and equal to its original . *cor.* 213
(*e*) The orthographic projections of any parallel straight lines are parallels, and have the same ratio to their respective originals . . 213
(*f*) The orthographic projection of a curve (the plane of which is not parallel to the direction of projection) is a curve; and the orthographic projection of a straight line, touching the original curve, is a straight line touching the projection of that curve 214
(*g*) If the plane of the original curve be parallel to the direction of projection, the projection of a straight line, and the projection of the tangent coincides with it, or, in one case, is a point of it *cor.* 214
(*h*) The orthographic projection of a straight line cutting a curve, is a straight line cutting the projection of that curve, except always as in (*g*) *cor.* 214
[(*i*) The orthographic projection of a

circle is a circle, or an ellipse (*app*. 25, 26, 27,)]

(B) *Of Perspective Projection.*

Perspective projection of a point, line, or figure . . . *def*. 208

Vertex, vertical plane, plane of projection, original . . *def*. 208

(*a*) The perspective projection may be found of every point which is without the vertical plane, but of no point in that plane . . . 208

(*b*) The perspective projection of a straight line (not passing through the vertex) is a straight line—viz. the common section of the plane of projection, and the plane which passes through the vertex and original straight line . . 209

(*c*) If the original straight line passes through the vertex, the projection of such straight line is a point *cor*. 209

(*d*) The perspective projection of a straight line parallel to the plane of projection, is parallel to its original *cor*. 209

(*e*) The perspective projection of a straight line, which is terminated by the vertical plane, is a straight line of unlimited extent in one direction; and the projection of a straight line, which cuts that plane, is the whole of a straight line of unlimited extent, in both directions, excepting only the part which lies between the projections of the extreme points . . 209

[(*f*) The perspective projection of a straight line, which is of unlimited extent in one direction, but does not meet the vertical plane (and is not parallel to that plane), is finite.]

(*g*) The perspective projection of a straight line, which is not parallel to the plane of projection, passes, if produced, through the point in which a parallel to the original, drawn through the vertex, cuts the plane of projection *cor*. 208

(*h*) The perspective projections of parallel straight lines, which are likewise parallel to the plane of projection, are parallels 209

(*i*) The perspective projections of parallel straight lines, which cut the plane of projection, are not parallels, but pass, when produced, all of them through the same point—viz., the point in which a parallel to the originals, drawn through the vertex, cuts the plane of projection . . 210

(*k*) The perspective projection of a curve (the plane of which does not pass through the vertex) is a curve; and, if any point of the original curve lies in the vertical plane, the projection shall have an arc of unlimited extent corresponding to the arc of the original curve, which is terminated in that point 210

(*l*) The perspective projection of a straight line touching any curve is a straight line touching the projection of that curve, if the original point of contact be without the vertical plane; but if it be in that plane, the projection of the tangent is an asymptote to the projection of the curve . 211

(*m*) If, however, the tangent at the original point be also in the vertical plane, it has no projection, and the projection of the original curve has no asymptote . . . *cor*. 212

(*n*) The perspective projection of a straight line, which cuts a curve in a point without the vertical plane, is a straight line which cuts the projection of that curve . . *cor*. 212

[(*o*) The perspective projection of a circle (except as in (*k*)) is a circle, or a conic section (app. 22, 23, 34).]

Proportion, and how denoted (also its antecedents and consequents) . *def*. 33

——— the first and second terms must be of the same kind, and also the third and fourth 33

Of commensurable magnitudes . 37

——— general theory of . . 48

——— when four magnitudes A, B, C, D, are said to be proportionals; *def*. [7.] p. 33; and *def*. 7 p. 49

Proportionals according to *def*. [7.] are also proportionals according to *def*. 7. 38, 49

(A) *Comparative magnitude of the terms, in a proportion*, A : B : : C : D *.

(*a*) If A $>$ or $=$ or $<$ B, then C $>$ $=$ or $<$ D; and conversely . 52

(*b*) If A $>$ or $=$ or $<$ C, then B $>$ or $=$ or $<$ D; and conversely 53

(*c*) If A $>$ or $=$ or $< \frac{m}{n}$B, then C $>$ or $=$ or $< \frac{m}{n}$ D; and conversely *cor*. 50

(*d*) If nA $>$ or $=$ or $<$ mB, then nC $>$ or $=$ or $<$ mD; and conversely. *cor*. 50

(*e*) If there are four magnitudes A, B, C, D, and if when nA $>$ or $=$ or $<$ mB, nC $>$ or $=$ or $<$ mD, for all values of n and m, then A, B, C, D are proportionals . . *cor*. 51

(*f*) If four magnitudes are proportionals, the greatest and least together are greater than the other two together . . . 53

(B) *Rules for transforming a single proportion*, A : B : : C : D.

(*a*) C : D : : B : A, by merely placing the second ratio first . . 33

(*b*) B : A : : D : C, *invertendo* 52

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* For the sign $<$ in this division read, "is less than," and for $>$ "is greater than."

(*c*) $A : C :: B : D$, *alternando* (when the four are of the same kind) 53
(*d*) $A \sim B : B :: C \sim D : D$, *dividendo* 53
(*e*) $A + B : B :: C + D : D$, *componendo* 53
(*f*) $A : A \pm B :: C : C \pm D$, *convertendo* . . . *cor.* 53
(*g*) $A + B : A \sim B :: C + D : C \sim D$ *miscendo* . . . *cor.* 54

(C) *Rules for combining two or more proportions.*
(*a*) If $A : C :: A' : C'$, and $B : C :: B' : C'$, then $A \pm B : C :: A' \pm B' : C'$; and so, if any number of proportions have the same consequents 54
(*b*) If $A : B :: A' : B'$, and $B : C :: B' : C'$, then, *ex æquali*, $A : C :: A' : C'$; and so, if there are any number of magnitudes A, B, C, D, and A', B', C', D', the ratios of which are the same respectively in the same order. 54
(*c*) If $A : B :: B' : C'$, and $B : C :: A' : B'$, then *ex æquo perturbato*, $A : C :: A' : C'$; and so, if there are any number of magnitudes A, B, C, D, and A', B', C', D', the ratios of which are the same respectively in a cross order. . . 55

(D) *Rules of Arithmetical Proportion.*
(*a*) If four magnitudes are proportionals, and if A, B, C, D represent those magnitudes numerically, then $\frac{A}{B} = \frac{C}{D}$; and, conversely, if $\frac{A}{B} = \frac{C}{D}$, the four are proportionals* . *sch.* 47
(*b*) The same being supposed, $A \times D = B \times C$, and conversely . *sch.* 47
(*c*) The same being supposed, $A \times M : B \times M :: C : D$; and conversely *sch.* 47
(*d*) The same being supposed $A \times M : B :: C \times M : D$, and conversely *sch.* 47
(*e*) If four magnitudes, numerically represented by A, B, C, D, are proportionals; and four others, represented by A', B', C', D', are likewise proportionals; then $A \times A' : B \times B' :: C \times C' : D \times D'$; a rule which is called "compounding the proportions," and is applicable to any number of proportions . . *sch.* 48

Proportional. See "Third Proportional," "Fourth Proportional," "Mean Proportional."
Proportional Straight Lines. See "Straight Line."
Pyramid, (also its vertex, base, sides, principal edges, lateral or convex surface, frustum, or truncated pyramid) . *def.* 127
When said to be regular (its axis) 127
When two pyramids are said to be similar 126
When symmetrical. *See* "Polyhedron."
(*a*) Triangular pyramids which have equal bases and altitudes are equal to one another . . . 145
(*b*) Every pyramid (triangular or other-

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* This is true, not only when the magnitudes are commensurable proportionals (as is demonstrated in p. 47), but also when four magnitudes are proportionals according to *def.* 7, and as such, is meant to be asserted by placing it among the properties of proportionals. It is necessary, therefore, to add the following demonstration of the general case:—

Let A, B, C, D, be proportionals according to *def.* 7, p. 49; let B be divided into any number, b, of equal parts, and let A contain a of those parts; that is, a certain number exactly together with a fraction, which number and fraction are, together, represented by a; also let D be divided into any number, d, of equal parts, and let C contain c of those parts, c being a whole number and fraction as before; $\frac{a}{b}$ shall be equal to $\frac{c}{d}$.

Take m and n any equimultiples of b and d, so that $\frac{m}{n} = \frac{b}{d}$, and consequently $md = nb$; and let B and D be divided, each, into md, or, which is the same, nb equal parts; then, because A contains $\frac{a}{b}$ths of B, it contains $\frac{na}{nb}$ths of B, and in like manner C contains $\frac{mc}{md}$ths of D; but because A, B, C, D are proportionals according to *def.* 7, and that nq is equal to md, the units (p suppose) in na are equal to the units (q suppose) in mc; and therefore $\frac{p}{nb} = \frac{q}{md}$: and this will be the case, how great soever the numbers m and n may be taken; that is, how great soever the numbers nb and md may be. But because $\frac{p}{nb}$ differs from $\frac{na}{nb}$ by less than $\frac{1}{nb}$ and $\frac{q}{md}$ from $\frac{mc}{md}$ by less than $\frac{1}{md}$, by increasing m and n, $\frac{p}{nb}$ and $\frac{q}{md}$ may be made to differ from $\frac{na}{nb}$ and $\frac{mc}{md}$ by less than any the same given difference. Therefore, because $\frac{p}{nb}$ is always equal to $\frac{q}{md}$, $\frac{na}{nb} = \frac{mc}{md}$ (II. 28) and consequently $\frac{a}{b} = \frac{c}{d}$.

Also, conversely, if $\frac{a}{b} = \frac{c}{d}$, A, B, C, D shall be proportionals according to *def.* 7. For let B and D be divided, each, into any number, n, of equal parts; then, because A contains $\frac{a}{b}$ths of B, it contains $\frac{1}{n}$th of B as often as the number n is contained in $\frac{a}{b}$, and in like manner C contains $\frac{1}{n}$th of D as often as n is contained in $\frac{c}{d}$; but, because $\frac{a}{b} = \frac{c}{d}$, n is contained in $\frac{a}{b}$ with a remainder, as often as it is contained in $\frac{c}{d}$ with a remainder; therefore A, B, C, D are proportionals.

wise) is equal to the third part of a prism which has the same base and altitude . . . 146, 147
(*c*) The solid content of a pyramid is equal to one-third of the product of its base and altitude . . *cor.* 147
(*d*) The frustum of a pyramid is equal to the sum of three pyramids which have the same altitude with the frustum, and for their bases the sum of its two bases and a mean proportional between them . . . 147
(*e*) Pyramids which have equal altitudes are to one another as their bases, and pyramids which have equal bases, as their altitudes; also, any two pyramids are to one another in the ratio which is compounded of the ratios of their bases and altitudes . *cor.* 147
[(*f*) Pyramids which have for their bases similar polygons, and the principal edges drawn from two corresponding angles of those polygons making equal angles with each of the homologous adjoining sides, and in the same ratio as those sides, are similar.]
(*g*) Similar pyramids are to one another in the triplicate ratio (or as the cubes) of their homologous edges . 149
[(*h*) Symmetrical pyramids are equal to one another; for they have equal bases and altitudes.]
(*i*) If the convex surface of a pyramid be produced to any extent, the sections of it by parallel planes will be similar polygons . . . *sch.* 145
Quadrature of the circle; an exact quadrature is impracticable in the way of calculation, *sch.* 99. *See* "Circle" and "Lunes."
Quadrilateral plane figure . *def.* 2
(*a*) Its angles are together equal to four right angles. . . *cor.* 15
(*b*) If any other than a parallelogram, the squares of its diagonals are together greater the squares of the four sides by four times the straight line which joins the middle point of the diagonals. . . . 23
(*c*) If two quadrilaterals have three sides of the one equal to three sides of the other, each to each, and the angles of the one lying in the circumference of a circle of which the fourth side is diameter, but the angles of the other not so lying, the first quadrilateral shall be greater than the other * 103
(*d*) If two quadrilaterals have the sides of the one equal to the sides of the other each to each, and the angles of the one lying in the circumference of a circle, but the angles of the other not so lying, the first quadrilateral shall be greater than the other 104
(*e*) To construct a quadrilateral which shall have its sides equal to four given straight lines (of which every three are greater than the fourth), and its angles lying in the circumference of a circle . . . *cor.* 91
In what cases a quadrilateral is a parallelogram, and the properties of such a figure. *See* "Parallelogram."
In what cases it has its angles in the circumference of a circle, and the properties of such a figure. *See* "Circle."
Quindecagon. *See* "Pentedecagon."

Radius of a circle, or sphere *def.* 3, 127
Ratio, meaning of the word . . 31
Is independent of the kind of magnitudes compared, but requires that the two shall be of the same kind 31
Of commensurable magnitudes, how expressed in numbers . 32
Of incommensurable magnitudes . 48
Ratio of equality, inverse or reciprocal ratio . . *def.* 32
Equal ratios, greater ratio, less ratio *def.* 32, 49
Duplicate ratio, triplicate ratio, &c. (*Also* Sub-duplicate, Sub-triplicate, &c.) . . . *def.* 34
Compound ratio; *i. e.*, a ratio which is compounded of any number of ratios *def.* 34
When one ratio is said to be taken away from another . . . 45

(A) *Of equal Ratios* *.
(*a*) If $A = B$, then $A : C = B : C$; and conversely . . . 51
(*b*) If $A : B = C : D$ and $C : D = E : F$, then $A : B = E : F$. 51
(*c*) If $A : B = A' : B'$, then $A : B = A \pm A' : B \pm B'$. 53, 54
(*d*) If $A : B = A' : B' = A'' : B''$, then $A : B = A + A' + A' : B + B' + B''$; and so of any number of equal ratios . . . 54
(*e*) Magnitudes are to one another as their equimultiples . . 52
(*f*) If $A : B = C : D$, $mA : mB = nC : nD$; and conversely *cor.* 52
(*g*) If $A : B = C : D$, $mA : nC = mC : nD$; and conversely . 52
(*h*) If there are any number of magnitudes of the same kind A, B, C, D, and as many others A′, B′, C′, D′, and if the ratios of the first to the second, of the second to the third, of the third to the fourth, and so on, are the same respectively in the two series, any two combinations by sum and difference of the magnitudes in the first series shall be to one another as two similar combinations of the magnitudes in the second series . *cor.* 55

(B) *Of unequal Ratios.*
(*a*) If $A >$ or $< B$, then $A : C >$ or $< B : C$; and conversely . 51

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* These propositions (*c*) and (*d*) are true, whether the equal sides lie in the same, or in different orders.

* For the signs $>$ and $<$ which are here used, read "is greater than" for $>$, and "is less than" for $<$.

(b) If $A > B$, then $C : B > C : A$; and conversely 51
(c) If $A : B = C : D$ and $C : D >$ or $< E : F$, then $A : B >$ or $< E : F$ . . . cor. 52
(d) If $A : B >$ or $< C : D$ and $C : D = E : F$, then $A : B >$ or $< E : F$. . . . cor. 52
(e) If $A : B >$ or $< A' : B'$, then $A : B >$ or $< A + A' : B + B'$ cor. 54
(f) If $A : B$ not $= A' : B'$, then $A + A' : B + B'$ lies between $A : B$ and $A' : B'$. . . . cor. 54

(C) *Of compound Ratios.*
(a) Ratios which are compounded of the same ratios, in whatsoever orders, are equal to one another . . 55
(b) Ratios which are the duplicates, triplicates, &c. of equal ratios, are equal to one another . . . cor. 56
(c) If equal ratios are taken from equal ratios, the remaining ratios are equal cor. 56

(D) *Of reciprocal Ratios.*
(a) If a ratio, which is compounded of two ratios be a ratio of equality, one of these is the reciprocal of the other; and conversely . . cor. 51
(b) The reciprocals of equal ratios are equal 40
(c) In the composition (*i. e.*, compounding) of ratios, any two which are reciprocals may be neglected without affecting the resulting compound ratio cor. 56
See also "Numerical Ratio" and "Proportion."
Reciprocal, one ratio said to be of another 32
Reciprocally proportional; two magnitudes and other two are said to be reciprocally proportional, when the four constitute a proportion, in which one pair are extremes and the other means.
Rectangle . . . . . *def.* 2
[(a) Its diagonals are equal to one another, and all its angles are right angles (I. 4.).]
(b) If the adjoining sides of a rectangle contain any the same straight line M, the one *a* times and the other *b* times, the rectangle shall contain the square of M, $a \times b$ times* . . 18
(c) Hence, a rectangle is measured by, and said to be equal to, the product of its adjoining sides . *sch.* 18
(d) Rectangles, having the same base, are to one another as their altitudes 62
(e) Rectangles are to one another in the ratio which is compounded of the ratios of their sides . . 62
(f) Rectangles, which have their sides reciprocally proportional, are equal to one another; and, conversely, rectangles, which are equal to one another, have their sides reciprocally proportional 63
(g) The equality of two rectangles is convertible into a proportion of four straight lines; and *vice versâ* *sch.* 64
See "Straight Line" and "Parallelogram."
Rectangular Parallelopiped . *def.* 126
[(a) Its diagonals are equal to one another, and every face is at right angles to the two adjoining faces.]
(b) If the conterminous edges of a rectangular parallelopiped contain any the same straight line M, *a*, *b*, and *c* times respectively, the rectangular parallelopiped shall contain the cube of M, $a \times b \times c$ times† . 142
(c) Hence, a rectangular parallelopiped is measured by, and said to be equal to, the product of its three conterminous edges . . *sch.* 142
(d) Rectangular parallelopipeds, having the same base, are to one another as their altitudes . . . 143
(e) Rectangular parallelopipeds are to one another in the ratio which is compounded of the ratios of their edges 143
See "Straight Line" and "Parallelopiped."
Rectilineal figure, or polygon . *def.* 2
When said to be a regular polygon 91

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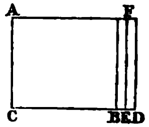
* This (although only demonstrated in p. 18, in the case of whole numbers) is true generally, *i. e.* when *a* and *b* are any numbers whatever.

Of the general case we add the following demonstration:—Let C A, C B be the sides of the rectangle, containing M *a* and *b* times respectively; and, first, let *a* be supposed a whole number. Let the square of M be contained in the rectangle $a \times d$ times; and, first, let *d* be supposed greater than *b*, so that if C D be taken equal to *d* times M, it will be greater than C B, which is only *b* times M. Let any part of M, which is less than B D, say the half, be taken, and let C E be the first multiple of this part, which is greater than C B; then E lies between B and D. Complete the rectangle A C E F; and, because half M is contained in C A $2a$ times, and in C D $2d$ times, and in C E an exact number of times (say *p*) less than $2d$, its square is contained in the rectangle A C E F $2a \times p$ times, that is, less than $2a \times 2d$ times; but its square is contained in the rectangle A B $2a \times 2d$ times, for the square of M is supposed to be contained in A B $a \times d$ times; therefore the rectangle A B is greater than A E C F, which is impossible—consequently *d* cannot be greater than *b*. And in the same manner it may be shown that it cannot be less than *b*, that is, it is equal to it. And this being shown in the case in which one of the numbers, as *a*, is whole, the like demonstration, resting upon this, may be applied to the case in which neither of them is a whole number.



The case of the rectangular parallelopiped may be demonstrated in the same manner.

† This is true, whatever the numbers *a*, *b*, and *c* are—see the last note.

When said to be inscribed in a circle or circumscribed about a circle . 79
When two rectilineal figures are said to be similar, and what are homologous sides. (*See* the Note upon this *def.*) 57
When straight lines are said to be similarly placed in similar rectilineal figures . . . 57

(A) (*a*) Any one of the sides is less than the sum of all the others . . *cor.* 8
(*b*) The exterior angles of a rectilineal figure (formed by producing its sides) are together equal to four right angles; and the sum of its interior angles, together with four right angles, is equal to twice as many right angles as the figure has sides . . . 15
(*c*) The area of a rectilineal figure may be obtained by dividing it into triangles, having for their common vertex one of the angular points of the figure, or some point within it . *sch.* 18

(B) *Of similar rectilineal figures.*
(*a*) Rectilineal figures of more than three sides which have their several angles equal in order, each to each, are not for that reason necessarily similar . . *sch.* 59
(*b*) Rectilineal figures which have all their angles but two, equal each to each in order, and the sides about the equal angles proportionals, are similar *cor.* 60
(*c*) Any two which are similar to the same rectilineal figure are similar to one another . . . *def.* 57
(*d*) Similar rectilineal figures are divided into the same number of similar triangles by the diagonals which are drawn from any two corresponding angles . . . *cor.* 60
(*e*) Lines, which are similarly placed in similar figures, cut the homologous sides at equal angles, and are to one another as those sides . *cor.* 60
(*f*) Similar rectilineal figures are to one another in the duplicate ratio (or as the squares) of their homologous sides; [and their perimeters are as those sides] . . . 67
(*g*) If four straight lines are proportionals, any two similar rectilineal figures which have the first and second for homologous sides, and any two which have the third and fourth, are proportionals . . . *cor.* 67
(*h*) If similar rectilineal figures are similarly described upon the hypotenuse and sides of a right angled triangle, the figure upon the hypotenuse is equal to the sum of the figures upon the two sides . . 67

(C) *Of rectilineal figures which are not similar.*
(*a*) If a rectilineal figure have not all its sides equal, and all its angles equal, a greater may be found having the same number of sides, and the same perimeter . . . 100
(*b*) Of regular polygons which have the same perimeter, that which has the greatest number of sides is greatest . . 101
(*c*) A regular polygon is greater than any other rectilineal figure which has the same (or a less) number of sides, and the same perimeter *cor.* 101, 102
(*d*) A regular polygon has a less perimeter than any rectilineal figure which has the same (or a less) number of sides and the same area . *cor.* 101
(*e*) If a rectilineal figure have not all its angles in the circumference of a circle, a greater may be found having the same sides . . *cor.* 105
(*f*) A rectilineal figure, which has all its angles lying in the circumference of a circle, is greater than any other having the same sides, whether in the same or in different orders *cor.* 105
(*g*) If two rectilineal figures have all their sides but one equal each to each (in the same, or in different orders), and the angles of the one lying in the circumference of a circle of which the excepted side is the diameter; but the angles of the other not so lying, the first figure shall be greater than the other . . . . *note* 103

(D) *Problems relating to Rectilineal Figures.*
(*a*) To describe a triangle which shall be equal to a given rectilineal figure, and which shall have a side and angle adjoining to it, the same with a side and adjoining angle of the figure 28
(*b*) To divide a given rectilineal figure into two (or more) equal parts by a line (or lines) drawn from a given angle, or from a given point in one of its sides 29
(*c*) To divide a given rectilineal figure in a given ratio (or into any number of parts having given ratios) by a line (or lines) drawn from one of its angles, or from a given point in one of its sides 77
(*d*) To describe a rectilineal figure upon a given straight line, similar to a given rectilineal figure; the given line being homologous with a given side of the figure 75
(*e*) To describe a rectilineal figure, which shall be similar to a given rectilineal figure, and have a given perimeter . . . 76
(*f*) To describe a rectilineal figure, which shall be similar to one given rectilineal figure, and equal to another 76

Regular Polygon, *def.* (also its apothem and centre) 91

Regular polygons of the same name are similar 91
To describe any regular polygon upon a given straight line . *cor.* 120
See "Circle," and "Rectilineal figure."
Re-entering, an angle said to be . 2
Regular Polyhedron . . *def.* 126
Its centre . . . *cor.* 161
[The *apothem* of a regular polyhedron is a perpendicular drawn from the centre to any one of the faces, and is equal to the radius of the inscribed sphere.]

(A) (*a*) There are only five regular polyhedrons, three of which (the tetrahedron, octahedron, and icosahedron) are contained by equilateral triangles, one (the hexahedron, or cube) by squares, and one (the dodecahedron) by pentagons *cor.* 138
(*b*) A sphere may be inscribed in any of the regular polyhedrons, and a sphere, concentric with the former, circumscribed about it . *cor.* 161
[(*c*) The common centre of which spheres (*i. e.* the centre of the polyhedron) may be found by bisecting any three of the dihedral angles, or by bisecting any three of the edges by planes at right angles to them (IV. 50. *cor.* 2.)]
(*d*) The solid content of a regular polyhedron is equal to one third of the product of its convex surface and apothem (*i. e.* the radius of the inscribed sphere) . . . *cor.* 165
(*e*) Regular polyhedrons of the same name are similar . *cor.* 161
(*f*) The regular polyhedrons of 6, 8, 12, and 20 faces have for every face a face opposite and parallel to it; and the opposite edges of those faces likewise parallel; also the straight line which joins two opposite angles passes through the centre of the polyhedron *cor.* 161
(*g*) Any one of the regular polyhedrons may be inscribed in a regular polyhedron, which has a greater number of faces, by taking for its vertices certain of the vertices of the latter, or of the centres of its faces, or of the middle points of its edges . *note* 162
(*h*) To find the inclination of the adjoining faces of any of the regular polyhedrons . . . 163
(*i*) The edge of any regular polyhedron being given, to find the radii of the inscribed and circumscribed spheres 164
(*k*) The edge of any regular polyhedron being given, to describe the polyhedron 158

(B) *Of the regular Tetrahedron.*
[(*a*) The tetrahedron has 4 faces (equilateral triangles), 4 solid angles, and 6 edges.]
(*b*) The centres of its faces are the vertices of an inscribed regular tetrahedron . . . *sch.* 162
(*c*) The inclination of its adjoining faces is measured by the greater acute angle of a right-angled triangle, whose hypotenuse is equal to three times one of its sides . . . 163
(*d*) The diameter of the circumscribed sphere is to the edge, as the hypotenuse to the greater side of a right-angled triangle, whose sides are as the side and diagonal of a square 165
(*e*) The diameter of the inscribed sphere is one-third that of the circumscribed sphere . . . *sch.* 166

(C) *Of the regular Hexahedron, or Cube.*
[(*a*) The cube has 6 faces (squares), 8 solid angles, and 12 edges.]
(*b*) The centres of its faces are the vertices of an inscribed regular octahedron . . . *sch.* 162
(*c*) Four of its vertices are the vertices of an inscribed octahedron, as the side and diagonal of a square *note* 162
(*d*) Its adjoining faces are at right angles to one another . . 163
(*e*) The diameter of the circumscribed sphere is to the edge, as the hypotenuse to the lesser side of a right-angled triangle, whose sides are as the side and diagonal of a square . 165
(*f*) The diameter of the inscribed sphere is equal to the edge of the cube 165

(D) *Of the regular Octahedron.*
[(*a*) The octahedron has 8 faces (equilateral triangles), 6 solid angles, and 12 edges.]
(*b*) The centres of its faces are the vertices of an inscribed cube *sch.* 162
(*c*) The inclination of its adjoining faces is measured by twice the greater acute angle of a right-angled triangle, whose sides are as the side and diagonal of a square 163
(*d*) The diameter of the circumscribed sphere is to the edge, as the diagonal of a square to its side . 165
(*e*) The ratio of the diameters of the inscribed and circumscribed spheres is the same as in the cube, viz. 1 : $\sqrt{3}$, *sch.* 166

(E) *Of the regular Dodecahedron.*
[(*a*) The dodecahedron has 12 faces (pentagons), 20 solid angles, and 30 edges.]
(*b*) The centres of its faces are the vertices of an inscribed icosahedron *sch.* 162
(*c*) Eight of its vertices are the vertices of an inscribed cube; and four of an inscribed tetrahedron . *note* 162
(*d*) The inclination of its adjoining faces is measured by twice the greater acute angle of a right-angled triangle whose sides are in the medial ratio 163
(*e*) The diameter of the circumscribed sphere is to the edge as the hypotenuse

to the lesser side of a right-angled triangle, whose sides are in the duplicate of the medial ratio . 165
(*f*) The ratio of the diameters of the inscribed and circumscribed spheres is the same as in the icosahedron *sch.* 166

(F) *Of the regular Icosahedron.*
[(*a*) The icosahedron has 20 faces (equilateral triangles), 12 solid angles, and 30 edges.]
(*b*) The centres of its faces are the vertices of an inscribed dodecahedron *sch.* 162
(*c*) The inclination of its adjoining faces is measured by twice the greater acute angle of a right-angled triangle, whose sides are in the duplicate of the medial ratio . . 163
(*d*) The diameter of the circumscribed sphere is to the edge, as the hypotenuse to the lesser side of a right-angled triangle, whose sides are in the medial ratio . . 165
(*e*) The ratio of the diameters of the inscribed and circumscribed spheres is the same as in the dodecahedron, [viz. as the greater side to the hypotenuse of a right-angled triangle, the lesser side of which is equal to twice the lesser segment of the greater side medially divided; or (if *r*, greater than unity, represent the medial ratio) in the subduplicate ratio of r^4 to $r^4 + 4$.] . . *sch.* 166
Regular Prism, def. (also its axis) 127
See "Prism" and "Cylinder."
Regular pyramid, def. (also its axis) 127
See "Pyramid" and "Cone."
Reverse angle. *See* "Re-entering."
Revolution, figures of. *See* "Solid of Revolution."
Rhombus . . . . . *def.* 2
(*a*) Has all its sides equal, and its diagonals bisect one another at right angles . . . *cor.* 16
[(*b*) If a quadrilateral has all its sides equal, or if its diagonals bisect one another at right angles, it is a rhombus.]
(*c*) To describe a rhombus with a given side and angle . . 28
Right, a term applied to any angle (whether rectilineal, dihedral, or spherical) which is equal to the adjacent angle, formed by producing either of its sides.
(*a*) A right angle is measured by a quadrant, or fourth part of the circumference, which is described about its angular point with a given radius *sch.* 85
(*b*) All the angles about a point in a plane, are together equal to four right angles . . . *cor.* 5
See "Angle."
Right solid angle . . . *sch.* 203
(*a*) A right solid angle is measured by an octant, or eighth part of the spherical surface, which is described about its angular point with a given radius *sch.* 203
(*b*) All the solid angles about any point are together equal to eight solid right angles . . *sch.* 203
See "Solid Angle."
Right, a cone, or cylinder said to be. (*See* "Cone" and "Cylinder.") 166, 167
——, a prism said to be. (*See* "Prism.") 127
Right-angled triangle, def. (*See* "Triangle.") 2
Right-angled spherical triangle. *See* "Spherical Geometry."
Round bodies, the sphere, right cone, and right cylinder, so called . . 167

Scalene (Gr. limping), a triangle said to be 2
Scholium (Gr.), an observation . 3
Section (Lat., division). *See* "Plane Section."
Sector of a circle, *def.* (*See* "Circle.") 79
——— of a sphere, *def.* (*See* "Sphere.") 179
Segment, of a circle, *def.* (*See* "Circle.") 79
——— of a sphere, *def.* (*See* "Sphere.") 179
Semicircle, def. 79, (also "Semicircumference," 78, "Semidiameter" 3.)
Sides, of a triangle, 2; isosceles triangle, 2; rightangled triangle, 2.
—— homologous, of similar figures 127
—— of a prism or pyramid . 127
Similar, rectilineal figures said to be 57
——— arcs, circular sectors and segments said to be 91
——— annuli said to be. *See* "Annulus."
——— polyhedrons said to be . 126
——— cones, cylinders said to be . 167
——— spherical segments and sectors, orbs, ungulæ, said to be . . 184
——— zones, lunes, spherical triangles, said to be 184, 207

[In all these figures there is one general property, and which it would not be difficult to demonstrate in each separately, viz., that any two that are similar, may be so placed, as that every straight line which is drawn from a certain point, to cut the perimeters or surfaces, shall be divided by them in the same ratio. Generally, therefore, let the term similar be applied to all figures which answer this description; *i. e.,* let it be the general test of similarity, that two figures, which are said to be similar, may be so placed as that every straight line which is drawn from a certain point (similarly placed with regard to the two) to cut the perimeters or surfaces, shall be divided by them in the same ratio.

According to this definition, plane figures will be similar, when a point may be found in each, and a straight line drawn from each of these points, such that every two straight lines which are drawn from the same points at equal angles to these, and terminated by the figure, shall be to one another in the same ratio. Of such figures it may be demonstrated—

1. That if any rectilineal figure whatever be inscribed in, or circumscribed about the one, a similar rectilineal figure may be inscribed in, or circumscribed about the other. (II. 32.)
2. That their areas are in the duplicate ratios (or as the squares) of any two homologous lines, and their perimeters as those lines. (II. 28.)

Again, according to this definition, solid figures will be similar, when a point may be found in each, and two straight lines drawn from each of those points, in the same direction from one another, such, that any two straight lines which are drawn from the same points at equal angles to these, and terminated by the figure, shall be to one another in the same ratio; and of such figures it may be demonstrated—

1. That if any polyhedron whatever is inscribed in, or circumscribed about the one, a similar polyhedron may be inscribed in, or circumscribed about the other.
2. That their solid contents are in the triplicate ratio (or as the cubes) of any two homologous lines, and their convex surfaces in the duplicate ratio (or as the squares) of those lines. (II. 28.)]

Similarly divided, straight lines said to be 57

Similarly placed, straight lines said to be, in similar figures 57

Simple locus. (*See* "Locus.") . *def.* 107

Small arc, in spherical geometry, is the arc of a small circle.

Small circle, of the sphere, *def.* (*See* "Spherical Geometry.") . . . 184

Solid . . . . . *def.* 1

Solid Geometry, is that part of geometry which treats of solid figures and lines in different planes. *See* "Polyhedron," &c.

Solid angle, *def.* (also its "edges") . 125
  When said to be a right angle *sch.* 203
  (*a*) If a solid angle is contained by three plane angles, any two of them are together greater than the third; and any one is greater than the difference of the other two . 137
  (*b*) The plane angles, which contain any solid angle, are together less than four right angles . . 137
  (*c*) If two solid angles are each of them contained by three plane angles, and if two of these and the included dihedral angle in the one, be equal to two and the included dihedral angle in the other, each to each; their other dihedral angles are equal, each to each, and the third plane angle of the one is equal to the third plane angle of the other . . *cor.* 157
  (*d*) If two solid angles are each of them contained by three plane angles, and if the three containing the one are equal to the three containing the other, each to each, their dihedral angles are also equal, each to each—viz. those which are opposite to the equal plane angles . . . *cor.* 157
  (*e*) Given the three plane angles which contain a solid angle, to find any of the dihedral angles by a plane construction . . 156
  (*f*) Given two of three plane angles containing a solid angle, and the dihedral angle which they contain, to find the third plane angle, and the other two dihedral angles by a plane construction . . . *cor.* 157
  (*g*) Every solid angle is measured by the spherical surface described about its angular point with a given radius, and included between its planes *sch.* 203
  (*h*) The properties of solid angles are analogous to those of spherical triangles and polygons . *sch.* 203
  See "Right Solid Angle."

Solid content. *See* "Content."

Solid of revolution, is any solid which is generated by the revolution of a plane figure about an axis in the same plane.
  (*a*) If an isosceles triangle revolve about an axis, which passes through its vertex, the convex surface generated by the base is equal to the product of that portion of the axis which is intercepted by perpendiculars drawn to it from the extremities of the base, and the circumference which has for its radius the perpendicular drawn from the vertex to the base; and the solid generated by the triangle is equal to one-third of the product of this surface by the same perpendicular . 175
  (*b*) If any triangle revolve about an axis, which passes through its vertex, the solid generated by the triangle is equal to one-third of the product of the surface generated by the base, and the perpendicular drawn from the vertex to the base . . *cor.* 176
  (*c*) If the half of any regular polygon of an even number of sides revolve about the diagonal, the whole convex surface is equal to the product of the diagonal and the circumference of a circle which has the apothem for its radius; and the whole solid content is equal to one-third of the product of this surface and the apothem 176
  See also "Cylinder," "Cone," "Sphere."

Sphere, *def.* (also its "radius," and "diameter") 127
  Is generated by the revolution of a semicircle about its diameter 127

(A) *Of the whole Spherical Surface and Content.*
  (*a*) A sphere is greater, both as to sur-

face and content, than any inscribed solid of revolution, and less than any circumscribed solid . . 177

(*b*) Any sphere being given, a solid of revolution may be inscribed in it or circumscribed about it, which shall differ from the sphere, in convex surface, or in solid content, by less than any given difference . *cor.* 177

(*c*) The whole convex surface of a sphere,
= the product of the circumference and diameter of the generating circle . . . 178
= four times the area of the generating circle . *cor.* 178
= πD^2, if D is the diameter of the sphere . . . *cor.* 178
= the convex surface (or two-thirds of the whole surface) of the circumscribed cylinder *cor.* 178

(*d*) The whole solid content
= one-third of the product of the radius and surface . . 178
= one-third of the product of the radius and four times the area of the generating circle . *cor.* 179
= $\frac{1}{6}\pi D^3$, if D is the diameter of the sphere . . *cor.* 179
= two-thirds of the content of the circumscribed cylinder. *cor.* 179

(*e*) If a polyhedron be circumscribed about a sphere, the contents of the sphere and polyhedron will be to one another as their convex surfaces *cor.* 179

(*f*) The surfaces of any two spheres are as the squares of the radii, and their solid contents are as the cubes of the radii . . . 179

(B) *Of certain portions of the Sphere.*
Spherical segment, sector, orb, ungula, zone, lune . . *def.* 179, 180

These portions are generated by the revolution of certain portions of a circle 180

(*a*) If a semicircle revolve about its diameter, the zone, which is generated by any arc of the semicircle, shall be greater than the surface generated by the chord of that arc, and less than the surface generated by the tangent of the same arc which is drawn parallel to the chord, and terminated by the radii passing through its extremities 180

(*b*) Every spherical zone is equal to the product of the circumference of the generating circle and that portion of the axis which is intercepted between its convex surface and base, or (if it be double-based) between its two bases . . . 181

(*c*) If a cylinder having the same axis be circumscribed about the sphere, any zone generated about that axis is equal to that part of the convex surface of the cylinder which is intercepted between the base of the cylinder and the plane of the base of the zone, or (if double-based) between the planes of its two bases . *cor.* 181

(*d*) In the same, or in equal spheres, any two zones are to one another as the parts of the axis, or axes, which are intercepted between their respective bases . . . *cor.* 181

(*e*) Every spherical sector is equal to one-third of the product of its base and the radius of the sphere; and the same may be said of every spherical pyramid* . . . 181

[(*f*) In the same, or in equal spheres, any two sectors are to one another as their bases; and the same may be said of any two spherical pyramids]

(*g*) Every spherical segment upon a single base is equal to the half of a cylinder having the same base and the same altitude, together with a sphere of which that altitude is the diameter 182

(*h*) Every double-based spherical segment is equal to the half of a cylinder having the same altitude with the segment, and a base equal to the sum of its two bases, together with a sphere of which that altitude is the diameter . . . 182

(*i*) The solid generated by the revolution of any circular segment about a diameter of the semicircle is equal to $\frac{1}{6}\pi \times GK \times FH^2$; GK being that portion of the diameter which is intercepted between two perpendiculars drawn to it from the extremities of the segment, and FH the chord which is the base of the segment . *cor.* 182

(*k*) Every spherical orb is equal to the sum of three pyramids, having their common altitude equal to the thickness of the orb, and their bases equal to its exterior and interior surfaces, and a mean proportional between them 183

(*l*) Every spherical ungula is to the whole sphere as the angle between its planes to four right angles, and its lune, or convex surface is to the surface of the whole sphere in the same ratio 183

(*m*) Every spherical ungula is equal to one-third of the product of the radius by its lune or convex surface *cor.* 184

(*n*) In the same, or in equal spheres, any two ungulas are to one another as the angles between their respective planes; and the same may be said of any two lunes . . *cor.* 184

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* A spherical pyramid is the solid figure which is contained by the surface of a spherical triangle or polygon, and the planes which pass through the sides of the triangle or polygon; the polygon is called its base, and the centre of the sphere its vertex. Two spherical pyramids are said to be similar when their bases are similar triangles or polygons.

(*o*) Similar zones and lunes, and also the triangular or polygonal surfaces which are the bases of similar pyramids, are as the squares of the radii; and similar sectors, pyramids, segments, orbs, and ungulas, are as the cubes of the radii . *sch.* 184, 207

Spheres, intersection and contact of *sch.* 151

Spherical Geometry, is that part of geometry which treats of figures and lines upon the surface of a sphere.

Great and small circles, axis, poles, and polar distance of a circle, parallel circles, opposite points . *def.* 184

Spherical arc, spherical angle (right, acute, and obtuse), spherical triangle, polar triangle, spherical polygon *def.* 184, 185, 186

When two points are said to be joined on the surface of a sphere *def.* 184

When two triangles are said to be symmetrical . . . *def.* 185

When two triangles are said to be similar *sch.* 207

(A) *Of great and small Circles.*

(*a*) Every plane section of a sphere is a circle, the centre of which is either the centre of the sphere, or the foot of the perpendicular which is drawn to its plane from the centre of the sphere 186

(*b*) The radius of a great circle is the same with the radius of the sphere; and the radius-square of a small circle is less than the radius-square of the sphere by the square of the perpendicular which is drawn to its plane from the centre of the sphere *cor.* 186

(*c*) Either pole of a circle of the sphere is equally distant from all points in the circumference of that circle, whether the direct or the spherical distance be understood . . . 186

(*d*) Any circle of a sphere may be conceived to be described round either of its poles as a centre, with the spherical distance of that pole as a radius *cor.* 186

(*e*) The distances of any circle from its two poles are together equal to a semicircumference . . *cor.* 187

(*f*) Equal circles have equal polar distances; and conversely . 187

(*g*) Circles whose polar distances are together equal to a semicircumference are equal to one another *cor.* 187

(*h*) If a point be taken within a circle which is not its pole, of all the spherical arcs which can be drawn from it to the circumference, the greatest is that in which the pole is, and the other part of that arc produced is the least; and, of any others, that which is nearer to the greatest is always greater than the more remote; and from the same point there can be drawn only two arcs that are equal to one another, which two make equal angles upon either side with the shortest arc . . . 193

(*i*) If there be taken a point from which there fall more than two equal spherical arcs to the circumference of a circle, that point is one of the poles of the circle . . . *cor.* 193

(*k*) A great circle is equally distant from its two poles; a small circle not so . . . *cor.* 187

(*l*) Great circles bisect one another: [but no two small circles bisect one another; neither can a small circle bisect a great circle] . . 187

(*m*) The spherical arc which is drawn from the pole of a great circle to any point in its circumference, is a quadrant, and is at right angles to the circumference . . 187

(*n*) Great circles which are at right angles to one another, pass each of them through the poles of the other *cor.* 188

[(*o*) If the perpendicular drawn from any point to a great circle be a quadrant, that point is a pole of the great circle.]

(*p*) If, from a point to a great circle, there fall two spherical arcs, each of which is a quadrant, that point is a pole of the great circle . *cor.* 188

(B) *Of Spherical Angles.*

(*a*) Any two spherical arcs may be produced to meet one another in two points which are opposite to one another . . . *cor.* 187

(*b*) If any number of angles are formed at the same point, their sides, being produced, shall pass through the opposite point, [and there form as many angles equal to the former respectively, each to each] . . *cor.* 187

(*c*) A spherical angle has the same measure with the dihedral angle of its planes, and with the plane angle of the tangents at the angular point *def.* 185

(*d*) Every spherical angle is measured by the spherical arc which is described about the angular point as a pole, and included between its legs; or, which is equal to it, by the distance between the poles of its legs . . 188

(*e*) The spherical angles which one arc makes with another upon one side of it, are either two right angles or are together equal to two right angles 189

(*f*) If two spherical arcs cut one another, the vertical or opposite angles are equal . . *cor.* 189

(*g*) All the spherical angles about any the same point are together equal to four right angles . . *cor.* 189

(C) *Of Spherical Triangles.*

(*a*) If one triangle be the polar triangle of another, the latter shall likewise be the polar triangle of the first; and the sides of either triangle shall be the supplements of the arcs which measure the opposite angles of the other 188

(*b*) In a spherical triangle, any two of the sides are together greater than the third side; and any side is greater than the difference of the other two . 189

(*c*) The three sides of a spherical triangle (and, generally, all the sides of any spherical polygon) are together less than the circumference of a great circle . . *cor.* 189

(*d*) In every spherical triangle, the sum of the angles is greater than two and less than six right angles . 189

(*e*) A spherical triangle may have two or three right angles, or two or three obtuse angles . . *cor.* 189

(*f*) If a spherical triangle has two of its sides equal to one another, the opposite angles are likewise equal, and conversely . . . 190

(*g*) If one side of a spherical triangle be greater than another, the opposite angle is greater than the angle opposite to that other; and conversely 190

(*h*) If one side of a spherical triangle is produced, the exterior angle is less than the sum of the two interior and opposite angles; and the exterior angle is equal to or greater than, or less than either of the interior and opposite angles, according as the sum of the two sides not common to them is equal to, or less than, or greater than a semicircumference . *cor.* 190

(*i*) Two spherical triangles have all their sides and angles equal, each to each, when they have,

1. Two sides, and the included angle equal:

or 2. Two angles and the interjacent side:

or 3. The three sides:

or 4. The three angles:

Whether the equal parts be in the same, or in reverse orders . . 191, 192

(*k*) Of spherical triangles which have the two sides of the one equal to the sides of the other, each to each, that which has the greater vertical angle has the greater base; and conversely 193

(*l*) In a right-angled spherical triangle, either of the two sides is of the same affection with the opposite angle; and the hypotenuse is equal to, or greater than, or less than a quadrant, according as one of the adjoining angles is a right angle, or these two angles are of the same or of different affections 194

(*m*) The spherical triangle, whose three sides are quadrants, coincides with its polar triangle, and is equal to an eighth part of the surface of the sphere . . . *sch.* 189

(D) *Of the surface of Spherical Triangles.*

(*a*) Spherical triangles which have the three sides of the one equal to the three sides of the other, each to each, have equal surfaces; whether the triangles are symmetrical or otherwise 194

(*b*) Every spherical triangle is equal to half the difference between the hemispherical surface and the sum of three lunes which have their angles respectively equal to the three angles of the triangle 195

(*c*) Every spherical triangle is measured by the excess of the sum of its angles above two right angles . *cor.* 196

(*d*) A spherical triangle, whose angles are A, B, and C, is equal to a lune whose angle is $\frac{A+B+C}{2}-R$, R being a right angle *cor.* 196

(*e*) Every spherical polygon is measured by the excess of the sum of its angles together with four right angles above twice as many right angles as the figure has sides . . *cor.* 196

(*f*) Spherical triangles which stand upon the same base, and between the same equal and parallel small circles, are equal to one another; and conversely, equal spherical triangles which stand upon the same base, and upon the same side of it, are between the same equal and parallel small circles 198

(*g*) Of equal spherical triangles upon the same base, the isosceles has the least perimeter; and of spherical triangles upon the same base, and having the same perimeter, the isosceles has the greatest area 199, 200

(*h*) Of all spherical triangles which have the same two sides, the greatest is that in which the angle included by them is equal to the sum of the other two angles . . 200

Of which triangle it may be remarked:—

1. The two sides are by less than a semicircumference, and the angle included by them is greater than a right angle.

2. The difference between the lune which has the same angle, and the triangle, has always the same surface, viz., a fourth of the surface of the sphere *sch.* 201

(*i*) A spherical polygon, which has all its sides equal, and all its angles equal, is greater than any other which has the same number of sides and the same perimeter . . *sch.* 201

(*k*) A spherical polygon, which has all its angles lying in the circumference of a circle, is greater than any other which has the same sides *sch.* 201

(*l*) A small circle of a sphere is greater than any spherical polygon having the same perimeter . *sch.* 201
(*m*) The lunular surface, included by a small arc and a spherical arc, is greater than any other surface, having the same perimeter, of which the same spherical arc is part . *sch.* 201

(E) *Problems upon the Surface of a Sphere.*
(*a*) To find the diameter of a given sphere; and, hence the quadrant of a great circle . . . 203
(*b*) Any point being given, to find the opposite point . . . 204
(*c*) To join two given points . 204
(*d*) Any spherical arc being given, to produce it, and complete the great circle, of which it is a part *cor.* 204
(*e*) To bisect a given spherical arc 204
(*f*) To draw a perpendicular to a given spherical arc—
1. From a given point in the arc 204
2. From a given point without it 205
(*g*) To bisect a given spherical angle 205
(*h*) At a given point, in a given arc, to make a spherical angle equal to a given spherical angle . 205
(*i*) To circumscribe a circle about a given spherical triangle, or (which is the same) to describe a circle through three given points . . 205
(*k*) To find the poles of a given circle *cor.* 205
(*l*) Through two given points and a third (not lying in the circumference of a great circle) to describe two equal and parallel small circles . 205
(*m*) To describe a spherical triangle which shall be equal to a given spherical polygon, and shall have a side and adjacent angle the same with a given side and adjacent angle of the polygon 206
(*n*) Two spherical arcs being given, which are together less than a semicircumference; to place them so, that with a third not given they may contain the greatest surface possible 206
(*o*) Through a given point to describe a great circle which shall touch two given equal and parallel small circles 206
(*p*) To inscribe a circle in a given spherical triangle, or (which is the same) to describe a circle touching three given spherical arcs . . 207

Square, *def.* (also "Square of a straight line A B.") 2
(*a*) A square has all its sides equal and all its angles right angles *cor.* 16
[(*b*) Its diagonals are equal and bisect one another at right angles.]
(*c*) The squares of equal lines are equal; and conversely . . *cor.* 17
(*d*) The square of 2 M is 4 M^2, of 3 M is 9 M^2, &c. . . . . *cor.* 18
(*e*) The square of 5 M is equal to the square of 4 M together with the square of 3 M . . . *cor.* 18
(*f*) Squares are to one another in the duplicate ratio of their sides *cor.* 63
(*g*) The squares of proportional straight lines are proportionals . *cor.* 63
(*h*) To describe a square,
1. Upon a given straight line 28
2. Which shall be equal to a given rectangle, or other rectilineal figure . . . 30
3. Which shall be equal to the difference of two given squares, or other rectilineal figures . 30
4. Which shall be equal to the sum of two or more given squares, or other rectilineal figures 30
See "Rectangle," "Straight Line," and "Circle."

Straight Line . . . *def.* 1
When said to be perpendicular to another straight line . . . 2
When parallel . . . 2

(A) (*a*) Straight lines which pass through the same two points lie in the same straight line . . . *ax.* 4
(*b*) If at a point in a straight line two other straight lines upon opposite sides of it make the adjacent angles together equal to two right angles, these two straight lines shall be in one and the same straight line 5
(*c*) A straight line is the shortest distance between two points . *sch.* 8
(*d*) A perpendicular may be drawn to a given straight line from any given point, whether without or in the straight line; but from the same point there cannot be drawn more than one perpendicular to the same straight line 9
(*e*) From a point to a straight line, the perpendicular is the shortest distance, and of other straight lines which are drawn from the same point, such as are equal to one another are at equal distances from the foot of the perpendicular, and conversely; and the greater is always further from the perpendicular, and conversely . *cor.* 10

(B) *Of Angles made by Straight Lines.*
(*a*) The adjacent angles, made by one straight line with another upon one side of it, are, together, equal to two right angles . . 5
(*b*) If two straight lines cut one another, the vertical or opposite angles are equal . . 5
(*c*) All the angles which are made upon one side of a straight line at the same point of it, are, together, equal to two right angles . *cor.* 5
All the angles about the same point

are, together, equal to four right angles . . *cor.* 5

(*e*) Angles, which have the sides of the one parallel, or perpendicular, or equally inclined to the sides of the other, in the same order, are equal 14

(C) *Of parallel Straight Lines.*

Difficulty in the theory . *sch.* 11

(*a*) Straight lines which are perpendicular to the same straight line are parallel; and conversely . 11

(*b*) A parallel to a given straight line may be drawn through any given point without it; but through the same point there cannot be drawn more than one parallel to the same given straight line . *cor.* 11

(*c*) Straight lines which make equal angles with the same straight line towards the same parts are parallel; and conversely . . . 12

(*d*) If a straight line falls upon two other straight lines, so as to make the alternate angles equal to one another, or the exterior angle equal to the interior and opposite upon the same side, or the two interior angles upon the same side together equal to two right angles, those two straight lines are parallel . . . *cor.* 13

(*e*) And conversely, if a straight line falls upon two parallel straight lines, it makes the alternate angles equal to one another, the exterior angle equal to the interior and opposite upon the same side, and the two interior angles upon the same side together equal to two right angles . . *cor.* 13

(*f*) If a straight line falls upon two other straight lines, so as to make the two interior angles upon the same side together less than two right angles, those two straight lines are not parallel, but may be produced to meet one another upon that side *cor.* 13

(*g*) Straight lines, to which the same straight line is parallel, are parallel to one another . . . 13

(*h*) Parallel straight lines are every where equidistant . . 13

(*i*) Parallel straight lines intercept equal parts of parallel straight lines *cor.* 13

(*k*) The straight lines which join the extremities of equal and parallel straight lines towards the same parts, are equal and parallel . . 15

(*l*) If two straight lines, which pass through the same point, are cut by two parallel straight lines, their parts terminated in that point shall be proportional 57

(*m*) If two (or more) straight lines which pass through the same point, are cut by any number of parallel straight lines, they shall be similarly divided by them . . *cor.* 58

(*n*) If two (or more) parallels are cut by any number of straight lines, which pass through the same point, they shall be similarly divided; and, if two parallels are similarly divided, the straight lines which join the corresponding points of division, pass, or may be produced to pass, all of them through the same point . 58

(*o*) If the legs of an angle cut two parallel straight lines, the intercepted parts of the parallels shall be to one another as the parts which they cut off from either of the legs . *cor.* 59

(C) *Of Straight Lines, which are not in the same plane.*

(*a*) Straight lines, to which the same straight line is parallel, although not in one plane with it, are parallel to one another . . . 130

(*b*) The shortest distance of two straight lines, which are not in the same plane, is a straight line, which is at right angles to each of them, and is equal to the perpendicular which is drawn from the vertex to the hypotenuse of a right-angled triangle, whose sides are the perpendiculars drawn to one of the straight lines from the two points in which the other is cut by any two planes passing through the first at right angles to one another *cor.* 155

(D) *Rectangles under the parts of divided Lines*, II. § 5 . . . 19, 20

The theorems of which section are briefly expressed as follows:—

(*a*) $A \times (B + C + D) = AB + AC + AD$.

(*b*) $(A + B) \times B = AB + B^2$.

(*c*) $(A + B)^2 = A^2 + B^2 + 2AB$.

(*d*) $(A - B)^2 = A^2 + B^2 - 2AB$.

(*e*) $(A + B)^2 - (A - B)^2 = 4AB$.

(*f*) $(A + B)^2 + (A - B)^2 = 2(A^2 + B^2)$.

(*g*) $(A + B) \times (A - B) = A^2 - B^2$.

(E) *Of Straight Lines which are proportionals.*

(*a*) If four straight lines are proportionals, the rectangle under the extremes is equal to the rectangle under the means; and conversely, if two rectangles are equal to one another, their sides are proportionals, the sides of the one being extremes, and the sides of the other means . 63

(*b*) If A, B are two straight lines, and A′, B′ other two, the rectangle $A \times A'$ shall be to the rectangle $B \times B'$ in the ratio which is compounded of the ratios of A to B and A′ to B′ . 63

(*c*) If A, B and A′, B′ are proportionals, $A \times A'$ is to $B \times B'$ as A^2 to B^2 *cor.* 63

(*d*) If A, B, C are proportionals, A is to C as A^2 to B^2 . . *cor.* 63

(*e*) If A, B, C, D, and also A′, B′, C′,

D', are proportionals, the rectangles $A \times A'$, $B \times B'$, $C \times C'$, and $D \times D'$ are also proportionals . *cor.* 63
- (*f*) The squares of proportional straight lines are proportionals; and conversely . . . . *cor.* 63
- (*g*) If A, B are two straight lines, A', B' other two, and A'', B'' other two, the rectangular parallelopipeds $A \times A' \times A''$, $B \times B' \times B''$ shall be to one another in the ratio which is compounded of the ratios of A to B, A' to B', and A'' to B'' 143
- [(*h*) If the ratios of A to B, A' to B', and A'' to B'' are all equal, $A \times A' \times A''$ is to $B \times B' \times B''$ as A^3 to B^3.]
- [(*i*) If A, B, C, D are in continued proportion, A is to D as A^3 to B^3.]
- [(*k*) If A, B, C, D, and also A', B', C', D', and also A'', B'', C'', D'', are proportionals, the rectangular parallelopipeds $A \times A' \times A''$, $B \times B' \times B''$, $C \times C' \times C''$, $D \times D' \times D''$, are also proportionals.]
- (*l*) The cubes of proportional straight lines are proportionals; and conversely . . . 144

(F) *Of Straight Lines harmonically divided.*
- (*a*) If A B, A C, A D are harmonical progressionals in the same straight line, and A B the least, D C, D B, D A shall likewise be harmonical progressionals 68
- (*b*) The same being supposed, and the mean A C being bisected in K, K B, K C, K D are proportionals; and, conversely, if K B, K C, K D are proportionals, and if K A is taken in the opposite direction equal to K C, A B, A C, A D shall be in harmonical progression 68
- (*c*) The same being supposed, D A, D K, D B, D C are proportionals . 69
- (*d*) The harmonical mean between two straight lines is a third proportional to the arithmetical and geometrical means . . . . *cor.* 69
- (*e*) If any four straight lines pass through the same point, and lie in the same plane, to whichsoever of the four a parallel is drawn, its parts intercepted by the other three shall be to one another in the same ratio 70
- (*f*) If a parallel is drawn to any one of four harmonicals, equal parts of such parallel are intercepted by the other three; and, conversely, if four straight lines pass through the same point, and if a parallel drawn to one of them has equal parts of it intercepted by the other three, the four are harmonicals . . . . *cor.* 70
- (*g*) Harmonicals divide every straight line, which is cut by them, harmonically 70
- (*h*) The two sides of a triangle, the straight line which is drawn from the vertex through the middle point of the base, and the straight line which is drawn through the vertex parallel to the base are harmonicals *cor.* 70
- (*i*) The two sides of a triangle, and the lines which bisect the vertical and exterior vertical angles, are harmonicals . . . *cor.* 71

(G) *Problems relating to Straight Lines.*
- (*a*) Through a given point to draw a straight line—
  1. Perpendicular to a given straight line; several methods 24, 25
  2. Parallel to a given straight line 26
     When many parallels are to be drawn, *See* II. 49.
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  2. Sum of squares, and difference of squares.
  3. Sum, and sum of squares.
  4. Difference, and sum of squares.
  5. Sum, and difference of squares.
  6. Difference, and difference of squares.
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  8. Sum, and ratio.
  9. Difference, and ratio.
  10. Sum, and rectangle.
  11. Difference, and rectangle.
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Surface of revolution, is the surface of a solid of revolution. *See* "Solid of Revolution."
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[*Symmetrical.* Any two solids are said to be symmetrical, when they can be placed upon the two sides of a plane, so that for every point in the surface of the one, there is a point in the surface of the other, in the same perpendicular to the plane, and at the same distance from it; also, the solids, when so placed, are said to be *symmetrically situated*, with regard to the plane[*].
Of such solids it may be demonstrated—
  1. That if any polyhedron whatever be inscribed in, or circumscribed about the one, a polyhedron symmetrical with it may be inscribed in, or circumscribed about the other.
  2. That they are equal to one another, and have equal surfaces.]

Symmetrically divided; a plane (or solid) figure is said to be symmetrically divided by a straight line (or plane), when, for every point in the contour of the figure which is upon one side of that line (or plane), there is another point in the contour upon the other side, in the same perpendicular to the line (or plane), and at the same distance from it.
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Tangent of a conic section. *See* "Conic Section."
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[*Touch*; (*a*) a straight line is said to touch a curve in any point, when it meets, but does not cut the curve, in that point[*]. Such a line is called a *tangent*.
  (*b*) One curve is said to touch another in any point when they have the same tangent at that point.
  (*c*) A plane is said to touch a curved surface in any point or line, when it meets, but does not cut, the surface in that point or line: such a plane is called a *tangent-plane*, and the point or line in which it touches the surface, is called the point or line of contact.
  (*d*) One curved surface is said to touch another in any point or line, when they have the same tangent plane at that point or at every point of that line.
  (*e*) The following examples may be given of the contact of surfaces—
    1. A plane touches
       The surface of a sphere in a point, the convex surface of a cylinder in a straight line which is parallel to the axis, the convex surface of a cone in a slant side.
    2. A spherical surface touches
       A spherical surface, whether externally or internally, in a point; the convex surface of a cylinder or

---

[*] According to a similar description, it is evident that two plane figures may be said to be symmetrical, and symmetrically situated with regard to a straight line which is in the same plane with them; but such figures are also similar, and may be made to coincide, which is not the case with symmetrical solids.

[*] Some curves are of a winding or serpentine form, having the curvature or bending now towards the one side, now towards the other. In such curves, there are always one or more points of contrary flexure, *i. e.* points where the curvature which had before been towards one side changes to the other side. Through such points no straight line can be drawn so as not to cut the curve: every other point, however, admits of a straight line being drawn, which meets and does not cut the curve in that point.

cone externally in a point, internally in one or two points or in the circumference of a circle.]
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[(*b*) Half the sum of the parallel sides of a trapezoid is equal to the straight line which joins the middle points of the other two sides.]
Triangle, rectilineal (also its "Vertex," "Sides," "Base") . . . *def.* 2
When said to be equilateral, isosceles, scalene 2
When said to be right-angled, oblique-angled, obtuse-angled, acute-angled 2

(A) *Of the first Properties, and of Triangles which are equal in all respects.*
(*a*) Any one of the angles of a triangle is less than two right angles *cor.* 5
(*b*) Triangles which have two sides and the included angle of the one equal to two sides, and the included angle of the other, each to each, are equal in all respects . . . 5
(*c*) Triangles which have two angles and the interjacent side of the one equal to two angles and the interjacent side of the other, each to each, are equal in all respects . . 6
(*d*) If one side of a triangle be equal to another, the opposite angle is likewise equal to the angle opposite to the other; and conversely . . 6
(*e*) In an isosceles triangle,
1. The angles at the base are equal to one another; and if the equal sides are produced, the angles upon the other side of the base are likewise equal . *cor.* 6
2. The following lines, viz. the line which bisects the vertical angle, the line which is drawn from the vertex to the middle point of the base, and the line which is drawn from the vertex perpendicular to the base, coincide with one another . . . *cor.* 7
3. The straight line which bisects the base at right angles passes through the vertex, and bisects the vertical angle . *cor.* 7
(*f*) Triangles which have the three sides of the one equal to the three sides of the other, each to each, are equal in all respects . . 7
(*g*) Any two angles of a triangle are together less than two right angles 7
(*h*) A triangle cannot have more than one right angle, or more than one obtuse angle . . *cor.* 7
(*i*) If one side of a triangle be produced, the exterior angle is greater than either of the interior and opposite angles . . . *cor.* 7
(*k*) If one side of a triangle be greater than another, the opposite angle shall likewise be greater than the angle opposite to that other; and conversely 7
(*l*) Any two of the sides are together greater than the third side, and any side is greater than the difference of the other two . . 8
(*m*) If two triangles stand upon the same base, and if the vertex of the one falls within or upon a side of the other, that triangle has a less perimeter than the other . . . *cor.* 8
(*n*) Of triangles which have the two sides of the one equal to the two sides of the other, each to each, that which has the greater vertical angle has the greater base; and conversely . 9
(*o*) The three angles of a triangle are together equal to two right angles; and if one of the sides be produced, the exterior angle is equal to the two interior and opposite angles . 14
(*p*) In a right angled triangle—
1. One of the angles, viz. the right angle, is equal to the sum of the other two . . *cor.* 14
2. The straight line which joins one of the angles, viz. the right angle, with the middle point of the opposite side is equal to half that side; and conversely, if a triangle has either of these properties, it is a right-angled triangle *cor.* 14
[(*q*) Any two triangles are equal to one another in all respects, when they have
1. Two sides and the included angle of the one equal to two sides and included angle of the other, each to each (*b*).
2. Two angles and the interjacent side (*c*).
3. The three sides (*f*).
4. Two angles, and a side opposite to one of them . *cor.* 14
5. An angle of the one equal to an angle of the other, and the sides about two other angles, each to each, and the remaining angles of the same affection, or one of them a right angle (see II. 33.)]
(*r*) Two right-angled triangles are equal to one another in all respects, when they have
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3. The two sides (I. 4.)
4. A side and the adjoining acute angle (I. 5.)
5. A side, and the [illegible] angle (I. 16. and [illegible]
(*s*) If in two right-[illegible] hypotenuse of the [illegible]

hypotenuse of the other, but a side of the one greater than a side of the other; the angle which is opposite to the side of the first shall be greater than the angle which is opposite to the side of the other . *lem.* 131

(B) *Of the mutual Relations of the Sides.*

(*a*) In a right-angled triangle, the square of the hypotenuse is equal to the squares of the two sides . 21

(*b*) In every triangle, the square of the side which is opposite to a given angle is greater or less than the squares of the sides containing that angle, by twice the rectangle contained by either of these sides, and that part of it which is intercepted between the perpendicular, let fall upon it from the opposite angle, and the acute angle; greater if the given angle is greater than a right angle, and less if it is less . . . 22

(*c*) Any angle of a triangle is equal to or greater or less than a right angle, according as the square of the opposite side is equal to or greater or less than the squares of the containing sides *cor.* 21, 22

(*d*) In every triangle, if a perpendicular be drawn from the vertex to the base, the difference of the squares of the sides is equal to the difference of the squares of the segments of the base, *i. e.* the base is to the sum of the sides as the difference of the sides to the difference of the segments of the base or sum of the segments of the base produced . . 22

(*e*) In a right-angled triangle, if a perpendicular be drawn from the right angle to the hypotenuse, the square of the perpendicular is equal to the rectangle under the segments of the hypotenuse, and the square of either side is equal to the rectangle under the hypotenuse and the segment adjoining to it; *i. e.* the perpendicular is a mean proportional between the segments of the hypotenuse, and either side is a mean proportional between the hypotenuse and the segment adjoining to it . *cor.* 21 and 61

(*f*) In an isosceles triangle, if a straight line is drawn from the vertex to any point in the base, or in the base produced, the square of that straight line shall be less or greater than the square of the side by the rectangle under the segments of the base, or of the base produced . . . 23

(*g*) In every triangle, the squares of the [illegible] sides are together double of the [illegible] of half the base, and of the [illegible] line which is drawn from the [illegible] to the middle point of the base 23

[illegible] every triangle, if the vertical or exterior vertical angle be bisected by a straight line which cuts the base, or the base produced, the base or base produced shall be divided in the ratio of the sides: also the square of the bisecting line shall be equal to the difference of the rectangles under the sides and the segments of the base, or base produced . . 70, 89

(*i*) In every triangle, if the base is equally produced both ways, so that the base produced is a third proportional to the base and sum of the sides, the sides of the triangle are to one another as the corresponding segments of the base produced *sch.* 64

(*k*) In every triangle, if the base is equally reduced both ways, so that the base reduced is a third proportional to the base and the difference of the sides, the sides of the triangle are to one another as the corresponding segments of the base reduced *sch.* 65

(*l*) In an isosceles triangle, if the equal angles are each of them double of the vertical angle, the sides and base are in extreme and mean ratio. *See* "Errata."

(C) *Of the Area of a Triangle, and of Triangles which are not equal in all respects.*

(*a*) If a triangle and a parallelogram stand upon the same base, and between the same parallels, the parallelogram is double of the triangle . 17

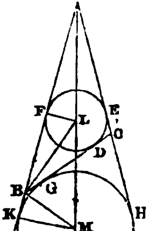
(*b*) Every triangle is equal to the half of a rectangle which has the same base and the same altitude; *i. e.* to half the product of its base and altitude * *cor.* 17

---

* When the sides of a triangle are given, its area may be directly computed from the following theorem:—

Every triangle is a mean proportional between two rectangles, the sides of which are equal to the semiperimeter of the triangle and the excesses of the semiperimeter above the three sides.

The demonstration of this is briefly as follows:—Let a circle be described within the triangle A B C, and a circle upon the other side of B C, the one touching B C, A C, and A B. in the points D, E, F, and the other B C and A C, A B produced in the points G, H, K; then the centres L, M of these circles lie in the straight line which bisects the angle at A, and if B L, B M be joined, the angle L B M will be a right angle, because it is half the sum of the angles A B C, C B K; also, because A K and A H are together equal to A B, B G and A C, C G, that is to the perimeter of the triangle A B C, each of them is equal to the semiperimeter. Again, because F B and B K (*i. e.* B D and B G) are together equal to E C and C H (*i. e.* to C D and C G), taking away G D from each, 2 B G is equal to 2 C D, and consequently B G to C D; therefore F B is equal to C H, and B K to E C. A K, therefore, is the semiperimeter, B K is the excess above the side A B, B F the excess above the side A C, and A F the excess above the side B C.



(*c*) Triangles which stand upon the same or upon equal bases, and between the same parallels, are equal to one another 17

(*d*) Triangles which have equal altitudes are to one another as their bases, and triangles which have equal bases, as their altitudes; also any two triangles are to one another in the ratio, which is compounded of the ratios of their bases and altitudes . . 65, 66

(*e*) Triangles which have one angle of the one equal to one angle of the other are to another in the ratio, which is compounded of the ratios of the sides about the equal angles, or as the rectangles under those sides . 66

(*f*) Triangles which have one angle of the one equal to one angle of the other, and the sides about the equal angles reciprocally proportional, are equal to one another; and, conversely, equal triangles, which have one angle of the one equal to one angle of the other, have the sides about the equal angles reciprocally proportional . . 66

(*g*) Two triangles are similar, when they have
  1. The three angles of the one equal to the three angles of the other, each to each . . 59
  or 2. The three sides of the one proportional to the three sides of the other . . . 59
  or 3. One angle of the one equal to one angle of the other, and the sides about the equal angles proportionals . . . 60
  or 4. One angle of the one equal to one angle of the other, and the sides about two other angles proportionals, and the remaining angles of the same affection, or one of them a right angle 61

(*h*) Similar triangles are to one another in the duplicate ratio (or as the squares) of their homologous sides 67

(*i*) Of all triangles having the same two sides, that one has the greatest area, in which the angle contained by the two sides is a right angle . 103

(*k*) Of triangles which have equal bases, and equal areas, the isosceles has the least perimeter; and of triangles having equal bases and equal perimeters, the isosceles has the greatest area 99

(D) *Problems relating to the Triangle.*
  (*a*) To describe a triangle, when there are given—
    1. Two sides and the included angle.
    2. Two sides, and an angle opposite to one of them.
    3. Two angles and the interjacent side.
    4. Two angles and a side opposite to one of them.
    5. The three sides . . 27
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Undecagon. *See* "Hendecagon."
Ungula. *See* "Sphere."
Unit of length, or linear unit, is any arbitrary straight line, as an inch, a foot.
  —of surface, is the square of the linear unit . . . . *sch.* 18
  —of content, is the cube of the linear unit . *sch.* 142

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Vertical angle, of a triangle. *See* "Triangle."
Volume. *See* "Content."

Wedge, spherical. *See* "Sphere."

Zone, spherical. *See* "Sphere."

Now the area of the triangle A B C is equal to L F × A K, for it is equal to half the rectangle under the radius L F of the inscribed circle, and the sum 2 A K of the three sides (I. 26. cor.); also B F × B K is equal to L F × M K, because, L B M being a right angle, the right-angled triangles B F L, M K B are similar: but A K × A F : A K × F L : : A F : F L, *i. e.* : : A K : K M, *i. e.* : : A K × F L : K M × F L : therefore A K × F L is a mean proportional between A K × A F and K M × F L or B F × B K, that is (if a, b, c represent the three sides opposite to the angles A, B, C respectively and S the half of $(a+b+c)$ the area of the triangle is a mean proportional between $S \times (S-a)$ and $(S-b) \times (S-c)$.

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A TREATISE

ON

ALGEBRAICAL GEOMETRY.

BY THE

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$$y-y_1=\frac{\alpha\sin.\omega-\beta(1+\alpha\cos.\omega)}{\sin.\omega+\beta(\alpha+\cos.\omega)}(x-x_1).$$

The length of the perpendicular from a given point on a given line is

$$p=\pm\frac{(y_1-\alpha x_1-b)\sin.\omega}{\sqrt{\{1+2\alpha\cos.\omega+\alpha^2\}}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad 31$$

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$$x=\frac{X\sin.(\omega-\theta)+Y\sin.(\omega-\theta')}{\sin.\omega}=\{X\sin.XAy+Y\sin.YAy\}\frac{1}{\sin.xAy}. \quad . \quad 34$$

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57. If the original axes be rectangular, and the new oblique,

$$y = \mathrm{X} \sin. \theta + \mathrm{Y} \sin. \theta',$$
$$x = \mathrm{X} \cos. \theta + \mathrm{Y} \cos. \theta'. \quad . \quad . \quad . \quad . \quad . \quad . \quad 34$$

58. Let both systems be rectangular,

$$y = \mathrm{X} \sin. \theta + \mathrm{Y} \cos. \theta = \mathrm{X} \cos. \mathrm{X\,A}\,y + \mathrm{Y} \cos. \mathrm{Y\,A}\,y,$$
$$x = \mathrm{X} \cos. \theta - \mathrm{Y} \sin. \theta = \mathrm{X} \cos. \mathrm{X\,A}\,x + \mathrm{Y} \cos. \mathrm{Y\,A}\,x. \quad . \quad . \quad . \quad 34$$

60. To transform an equation between co-ordinates x and y, into another between polar co-ordinates r and θ.

$$y = b + \frac{r \sin. (\theta + \varphi)}{\sin. \omega},$$
$$x = a + \frac{r \sin. \{\omega - (\theta + \varphi)\}}{\sin. \omega}. \quad . \quad . \quad . \quad . \quad . \quad . \quad 35$$

61. If the original axes be rectangular,

$$y = b + r \sin. \theta,$$
$$x = a + r \cos. \theta. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 35$$

62. To express r and θ in terms of x and y,

$$\theta + \varphi = \tan.^{-1} \left\{ \frac{(y - b) \sin. \omega}{x - a + (y - b) \cos. \omega} \right\}$$
$$r^2 = (x - a)^2 + (y - b)^2 + 2\,(x - a)\,(y - b) \cos. \omega. \quad . \quad . \quad . \quad 35$$

63. If the original axes be rectangular, and the pole at the origin,

$$\theta = \tan.^{-1} \frac{y}{x} = \sin^{-1} \frac{y}{\sqrt{y^2 + x^2}} = \cos.^{-1} \frac{x}{\sqrt{y^2 + x^2}}$$
$$r^2 = x^2 + y^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 36$$

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$$(y - b)^2 + (x - a)^2 = r^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 37$$

If the origin is at the centre,

$$y^2 + x^2 = r^2.$$

If the origin is at the extremity of that diameter which is the axis of x,

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or, generally, $(y - b)\,(y' - b) + (x - a)\,(x' - a) = r^2$ 39

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72. If the axes are oblique, the equation to the circle is

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73, 74. The Polar equation between u and θ is

$$u^2 - 2cu\cos.(\theta-\alpha) + c^2 - r^2 = 0,$$

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77. $b^2 - 4ac$ positive; the Locus is an Hyperbola if x_1 and x_2 are real and unequal, or are imaginary; but consists of two straight lines if x_1 and x_2 are real and equal. Examples 46

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$$a' = \frac{1}{2}\left\{a + c \pm \sqrt{(a-c)^2 + b^2}\right\}$$

$$c' = \frac{1}{2}\left\{a + c \mp \sqrt{(a-c)^2 + b^2}\right\}$$

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ALGEBRAICAL GEOMETRY.

PART I.

APPLICATION OF ALGEBRA TO PLANE GEOMETRY.

CHAPTER I.

INTRODUCTION.

1. The object of the present Treatise is the Investigation of Geometrical Theorems and Problems by means of Algebra.

Soon after the introduction of algebra into Europe, many problems in plane geometry were solved by putting letters for straight lines, and then working the questions algebraically; this process, although of use, did not much extend the boundaries of geometry, for each problem, as heretofore, required its own peculiar method of solution, and therefore could give but little aid towards the investigation of other questions.

It is to Descartes that we owe the first general application of algebra to geometry, and, in consequence, the first real progress in modern mathematical knowledge; in the discussion of a problem of considerable antiquity, and which admitted of an infinite number of solutions, he employed two variable quantities x and y for certain unknown lines, and then showed that the resulting equation, involving both these quantities, belonged to a series of points of which these variable quantities were the co-ordinates, that is, belonged to a curve, the assemblage of all the solutions, and hence called "the Locus of the Equation."

It is not necessary to enter into further details here, much less to point out the immense advantages of the system thus founded. However, in the course of this work we shall have many opportunities of explaining the method of Descartes; and we hope that the following pages will, in some degree, exhibit the advantages of his system.

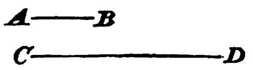
2. In applying algebra to geometry, it is obvious that we must understand the sense in which algebraical symbols are used.

In speaking of a yard or a foot, we have only an idea of these lengths by comparing them with some known length; this known or standard length is called a unit. The unit may be any length whatever: thus, if it is an inch, a foot is considered as the sum of twelve of these units, and may therefore be represented by the number 12; if the unit is a yard, a mile may be represented by the number 1760.

But any straight line A B fig. (1) may be taken to represent the unit of length, and if another straight line C D contains the line A B an exact number (a) of times, C D is equal to (a) linear units, and omitting the words "linear units," C D is equal to (a).

In fig. (1) C D = 3 times A B, or C D = 3.

1



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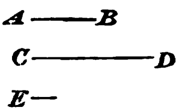


If C D does not contain A B an exact number of times, they may have a common measure E, fig. (2); let, then, C D = m times E = m E, and A B = n E, then C D has to A B the same ratio that m E has to n E, or that m has to n, or that $\frac{m}{n}$ has to unity; hence C D = $\frac{m}{n}$ times A B $= \frac{m}{n} = b$.

In fig. (2) C D $= \frac{5}{3}$ of A B $= \frac{5}{3}$

If the lines A B and C D have no common measure, we must recur to considerations analogous to those upon which the theory of incommensurable quantities in arithmetic is founded.

We cannot express a number like $\sqrt{2}$ by integers or fractions consisting of commensurable quantities, but we have a distinct idea of the magnitude expressed by $\sqrt{2}$, since we can at once tell whether it be greater or less than any proposed magnitude expressed by common quantities; and we can use the symbol $\sqrt{2}$ in calculation, by means of reasoning founded on its being *a limit* to which we can approach, as nearly as we please, by common quantities.

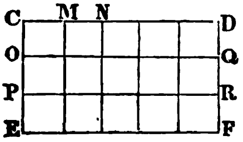
Now suppose E to be a line contained an exact number of times in A B, fig. (2), but not an exact number of times in C D, and take m a whole number, such that m E is less than C D, and $(m + 1)$ E greater than C D. Then the smaller E is, the nearer m E and $(m + 1)$ E will be to C D; because the former falls short of, and the latter exceeds, C D, by a quantity less than E. Also E may be made as small as we please; for if any line measure A B, its half, its quarter, and so on, *ad infinitum*, will measure A B. Hence we may consider C D as a quantity which, though not expressible precisely by means of any unit which is a measure of A B, may be approached as nearly as we please by such expressions. Hence C D is *a limit* between quantities commensurable with E, exactly as $\sqrt{2}$ is *a limit* between quantities commensurable with unity.

We conclude, then, that any line C D may be represented by some one of the letters a, b, c, &c., these letters themselves being the representatives of numbers either integral, fractional, or incommensurable.

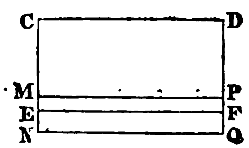
3. If upon the linear unit we describe a square, that figure is called the square unit.

Let C D F E, fig. (1), be a rectangle, having the side C D containing (a) linear units C M, M N, &c., and the side C E containing (b) linear units C O, O P, &c., divide the rectangle into square units by drawing lines parallel to C E through the points M, N, &c., and to C D through the points O, P, &c. Then in the upper row C O Q D there are (a) square units, in the second row O P R Q the same, and there are as many rows as there are units in C E, therefore altogether there are $(b \times a)$ square units in the figure, that is, C F contains $(a\,b)$ square units, or is equal in magnitude to ($a\,b$) square units; suppressing the words "square units," the rectangle C F is equal to $a\,b$.

If C D = 5 feet and C E = 3 feet, the area C F contains 15 square feet.



(1.)



(2.)

The above proof applies only to cases where the two lines containing the rectangle can be exactly measured by a common linear unit.

Suppose C D to be measurable by any linear unit, but C E (fig. 2) not to be commensurable with C D; then, as has been shown, we may find lines C M, C N commensurable with C D approaching in magnitude as nearly as we please to C E.

Completing the rectangles C P and C Q, we see, that as C M and C N approach to C E, the rectangles C P and C Q approach to the rectangle C F, that is, the rectangle C E, C D is *the limit* of the rectangle C M, C D, just as C E is *the limit* of C M. Let therefore a and b be respectively the commensurable numbers representing C D and C M, and let c be the incommensurable number expressing C E, then the rectangle C E, C D = the limit of the rectangle C M, M P = the limit of the number $a\,b$, by the first part of this article, = the product of the respective limits of a and b = $a\,c$.*

Hence, generally, the algebraical representative of the area of a rectangle is equal to the product of those of two of its adjacent sides.

If $b = a$, the figure C F becomes the square upon C D, hence the square upon C D is equal to ($a \times a$) times the square unit = a^2.

We are now able to represent all plane rectilineal figures, for such figures can be resolved into triangles, and the area of a triangle is equal to half the rectangle on the same base, and between the same parallel lines.

4. To represent a solid figure, it will be sufficient to show how a solid rectangular parallelopiped may be represented.

Let a, b, c, be, respectively, the number of linear units in the three adjacent edges of the parallelopiped; then, dividing the solid by planes parallel to its sides, we may prove, as in the last article, that the number of solid units in the figure is $a \times b \times c$, and, consequently, the parallelopiped equal to $a \times b \times c$.

The proof might be extended to the case where the edges of the parallelopiped are fractional, or incommensurable with the linear unit.

If $b = c = a$, the solid becomes a cube, and is equal to $a \times a \times a$, or a^3.

5. We proceed, conversely, to explain the sense in which algebraic expressions may be interpreted consistently with the preceding observations.

---

* That "the product of the limits of two incommensurable numbers is the limit of their product," may be thus shown. Let v and w be incommensurable numbers, and let $v = m + m'$ and $w = n + n'$, m and n being commensurable numbers, and m' and n' diminishable without limit; that is, v and w are the respective limits of m and n, then $v w = m n + m n' + n m' + m' n'$, the right-hand side of this equation ultimately becomes $m n$, and the left-hand side of the equation is the product of the limits.

Algebraic expressions may be classed most simply under the form of homogeneous equations, as follows:—

$$x = a$$
$$x^2 + ax = bc$$
$$x^3 + ax^2 + bcx = def$$
$$x^4 + ax^3 + bcx^2 + defx = ghkl$$

. . . .

$$x^m + ax^{m-1} + bcx^{m-2} + \&c \ldots\ldots = pqrs \ldots \text{ to } m \text{ terms.}$$

In the first place, each equation may be understood as referring to linear units; thus, if L be put instead of the words 'the linear units,' the equations may be written

x times L $=$ a times L,

x^2 times L $+$ ax times L, or $(x^2 + ax)$ times L $=$ bc times L,

$(x^4 + ax^3 + bcx^2 + defx)$ times L $=$ $ghkl$ times L, and so on. The solution of each equation gives x times L in terms of $(a, b, c, \ldots)$ times L; and thus the letters $a, b, c, \ldots x$ are merely numbers, having reference to lines, but not to figures.

This will be equally true if L is not expressed, but understood; and it is in this sense that we shall interpret all equations beyond those of the third order.

The same reasoning would equally apply if we assumed L to represent the square or cubic unit, only it would lead to confusion in the algebraic representation of a line.

6. Again, these equations may, to a certain extent, have an additional interpretation.

For if we consider the letters in each term to be the representatives of lines drawn *perpendicular* to each other, the second equation refers to areas, and then signifies that the sum of two particular rectangles is equal to a third rectangle; the third equation refers to solid figures, and signifies, that the sum of three parallelopipeds is equal to a fourth solid.

Moreover we can pass from an equation referring to areas to another referring to lines, without any violation of principle; for, considering the second equation as referring to areas, the rectangles can be exhibited in the form of squares; and if the squares upon two lines be equal, the lines themselves are equal, or the equation is true for linear units.

7. It follows as a consequence of the additional interpretation, that every equation of the second and third order will refer to some geometrical theorem, respecting plane or solid figures; for example, the second equation, when in the form $x^2 = a\,(a - x)$ is the representation of the well-known problem of the division of a line into extreme and mean ratio.

By omitting the second and third terms of the third equation, and giving the values of $2a$, a, and a to d, e and f, respectively, we obtain the algebraic representation of the ancient problem of the duplication of the cube.

8. The solution of equations leads to various values of the unknown quantity, and there are then two methods of exhibiting these values; first, by giving to a, b, c, &c., their numerical values, and then performing any operation indicated by the algebraic symbols.

Thus, if $a = 4$, $b = 5$, and $c = 9$,

we may have $x = a + c - b = 8$ times the linear unit.

$$x = \frac{ab}{c} = \frac{20}{9} = 2\frac{2}{9} \text{ of the linear unit.}$$

$$x = \sqrt{ac} = \sqrt{36} = 6 \text{ times the linear unit.}$$

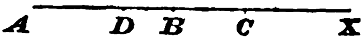
We can then draw the line corresponding to the particular value of x. This is the most practical method.

Again, we may obtain the required line from the algebraical result, by means of geometrical theorems; this method is called 'the Construction of Quantities'; it is often elegant, and is, moreover, useful to those who wish to obtain a complete knowledge of Algebraical Geometry.

THE CONSTRUCTION OF QUANTITIES.

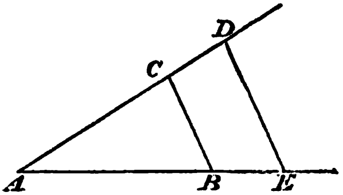
9. Let $x = a + b$.

In the straight line A X, let A be the point from whence the value of x is to be measured; take A B $= a$, and B C $= b$, then A C $=$ A B $+$ B C $= a + b$ is the value of x.



Let $x = a - b$, in B A take B D $= b$, then A D $=$ A B $-$ B D $= a - b$.

Let $x = \frac{ab}{c}$, then $x : a :: b : c$, and x is a fourth proportional to the three given quantities c, b, and a; hence the line whose length is expressed by x, is a fourth proportional to three lines, whose respective lengths are c, b, and a. From A draw two lines A C D, A B E, forming any angle at A; take A B $= c$, B E $= a$, and A C $= b$, join B C, and draw D E parallel to B C; then, A B : A C :: B E : C D, or $c : b :: a :$ C D $\therefore$ C D is the required value of x.



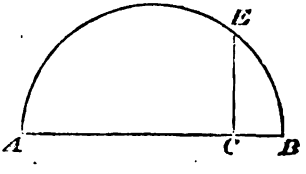
$$\text{Let } x = \frac{abc}{de}; \text{ construct } y = \frac{bc}{e}, \text{ and then } x = \frac{ay}{d};$$

$$\text{similarly for } x = \frac{abc}{d^2}, \text{ or } \frac{ab^2}{d^2}, \text{ or } \frac{a^3}{d^2}, \text{ or } \frac{abcd}{efg}.$$

Let $x = \frac{abc + def}{gh} = \frac{abc}{gh} + \frac{def}{gh}$, construct each term separately, and then the sum of the terms.

10. Let $x = \sqrt{ab}$.

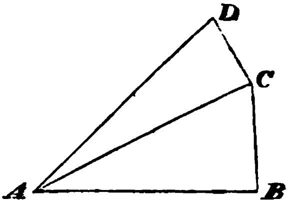
Since $x^2 = ab$, x is a mean proportional between a and b. In the straight line A B take A C $= a$, and C B $= b$; upon A B describe a semicircle, from C draw C E perpendicular to A B, and meeting the circle in E; then C E is a mean proportional to A C and C B, (Euclid, vi. 13, or Geometry, ii. 51,) and therefore C E is the required value of x.



The same property of right-angled triangles may be advantageously employed in the construction of the equation $x = \frac{a^2}{b}$; for, take AC $= b$, and draw C E perpendicular to A C and equal to a, join A E and draw E B perpendicular to A E; then C B $= \frac{a^2}{b}$.

Let $x = \sqrt{ab + cd}$, $x^2 = ab + cd = a\left(b + \frac{cd}{a}\right) = ay$ by substitution; construct y, and then $x = \sqrt{ay}$.

Again, x is a line, the square upon which is equal to the sum of the rectangles ab, cd. This sum may be reduced to a single rectangle, and the rectangle converted into a square, the base of which is the required value of x.—Euclid, i. 45, and ii. 14; or Geometry, i. 57, 58.



Let $x = \sqrt{a^2 + b^2}$; take a straight line A B $= a$, from B draw B C ($= b$) perpendicular to A B; A C is the value of x.

Let $x = \sqrt{a^2 + b^2 + c^2}$, from C draw C D ($= c$) perpendicular to A C, A D is the required value of x.

Let $x = \sqrt{a^2 - b^2} = \sqrt{(a+b)(a-b)}$; x is a mean proportional between $a + b$ and $a - b$; or by taking (in the last figure but one) A B $= a$, and A E $= b$, we have B E $= \sqrt{a^2 - b^2}$.

Let $x = \sqrt{\{a^2 + b^2 - c^2 - d^2\}}$, find $y^2 = a^2 + b^2$ and $z^2 = c^2 + d^2$ and then x.

Let $x = \sqrt{\frac{a^4 + bc^3}{b^2 - c^2}}$, find $y^2 = a^2 + \frac{bc^3}{a^2}$, and $z^2 = b^2 - c^2$, and then $x = \frac{ay}{z}$.

11. Of course the preceding methods will equally apply, when instead of the letters we have the original numbers, the linear unit being understood as usual.

Thus $x = \sqrt{12} = \sqrt{3.4}$ is a mean proportional between 3 and 4; hence (see last figure but one) take A C equal four times the unit, and C B equal three times the unit, C E is the value of x; or since $\sqrt{12} = \sqrt{16 - 4} = \sqrt{4^2 - 2^2}$, by constructing a right-angled triangle of which the hypothenuse is four times the linear unit, and one side twice that unit: the remaining side $= \sqrt{12}$.

Similarly $x = \sqrt{7} = \sqrt{4 + 4 - 1} = \sqrt{2^2 + 2^2 - 1^2}$, which is of the form $\sqrt{a^2 + b^2 - c^2}$.

Let $x = \sqrt{5} = \sqrt{2^2 + 1}$. In the last figure let A B, B C, and C D each be equal to the linear unit, then A D $= \sqrt{5}$

Let $x = \sqrt{23} = \sqrt{5^2 - 1^2 - 1^2}$.

Let $x = \frac{1}{\sqrt{2}} = \sqrt{\frac{1}{2}} = \sqrt{\frac{1}{4} + \frac{1}{4}} = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$,

then x is the hypothenuse of a right-angled triangle, each of whose sides is half the unit.

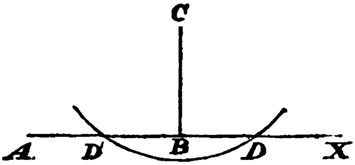
Let $x = \sqrt{\frac{3}{4}}$; this may be constructed as the last.

Let $x = \sqrt{\frac{1}{3}} = \sqrt{\frac{3}{9}}$; and so on for all numbers, since any finite number can be decomposed into a series of numbers representing the squares upon lines.

If the letter a be prefixed to any of the above quantities, it must be introduced under the root.

12. In constructing compound quantities, it is best to unite the several parts of the construction in one figure.

Thus if $x = a \pm \sqrt{a^2 - b^2}$, in the line A X take A B $= a$, from B draw B C ($= b$) perpendicular to A B; with centre C and radius a describe a circle cutting A X in D and D′; A D and A D′ are the values required:



$$\text{for A D} = \text{A B} + \text{B D} = a + \sqrt{a^2 - b^2}.$$

$$\text{A D}' = \text{A B} - \text{B D}' = a - \sqrt{a^2 - b^2}.$$

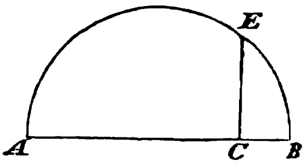
This construction fails when b is greater than a, for then the circle never cuts the line A X; this is inferred also from the impossibility of the roots.

13. Since theorems in geometry relate either to lines, areas, or solids, the corresponding equations must in each case be homogeneous, and will remain so through all the algebraic operations. If, however, one of the lines in a figure be taken as the linear unit and be therefore represented by unity, we shall find resulting expressions, such as $x = \frac{a}{b}$, $x = \sqrt{a}$, $x = \sqrt{a^2 + b}$,&c., in which, prior to construction, the numerical unit must be expressed; thus these quantities must be written $\frac{a}{b} \times 1$, $\sqrt{a \times 1}$, $\sqrt{a^2 + b \times 1}$, and then constructed as above.

CHAPTER II.

DETERMINATE PROBLEMS.

14. Geometrical Problems may be divided into two classes, Determinate and Indeterminate, according as they admit of a finite or an infinite number of solutions.



If A B be the diameter of the semicircle A E B, and it be required to find a point C in A B such, that drawing C E perpendicular to A B to meet the circumference in E, C E shall be equal to half the radius of the circle, this is a determinate problem, because there are only *two* such points in A B, each at an equal distance from the centre. Again, if it be required to find a point E out of the line A B such, that joining E A, E B, the included angle A E B shall be a right angle, this is an indeterminate problem, for there are an infinite number of such points, all lying in the circumference A E B.

The determinate class is by no means so important as the indeterminate, but the investigation of a few of the former will lead us to the easier comprehension of the latter; and therefore we proceed to the discussion of determinate problems.

15. In the consideration of a problem, the following rules are useful.

1. Draw a figure representing the conditions of the question.
2. Draw other lines, if necessary, generally parallel or perpendicular to those of the figure.
3. Call the known lines by the letters $a, b, c,$ &c., and some of the unknown lines by the letters $x, y, z,$ &c.
4. Consider all the lines in the figure as equally known, and from the geometrical properties of figures deduce one, two, or more equations, each containing unknown and given quantities.
5. From these equations find the value of the unknown quantities.
6. Construct these values, and endeavour to unite the construction to the original figure.

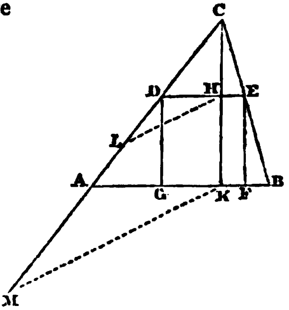
16. To describe a square in a given triangle A B C.

Let D E F G be the required square
C H K the altitude of the triangle.

The question is resolved into finding the point H, because then the position of D E, and therefore of the square, is determined.

Let $CK = a$, $AB = b$, $CH = x$;
then by the question, $DE = HK$,
and $DE : AB :: CH : CK$,
or $DE : b :: x : a$,

$$\therefore DE = \frac{bx}{a}, \text{ and } HK = a - x$$



$$\therefore \frac{bx}{a} = a - x$$

$$\therefore x = \frac{a^2}{a+b}.$$

Thus x is a third proportional to the quantities $(a+b)$ and a.

In C A take C L = a, produce C A to M so that L M = b, join M K, and draw L H parallel to M K; C H is the required value of x.

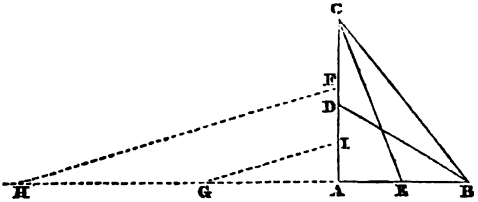
17. In a right-angled triangle the lines drawn from the acute angles to the points of bisection of the opposite sides are given, to find the triangle.

Let C E = a, B D = b, A D = C D = x, A E = E B = y.

Then the square upon C E = square upon C A + square upon A E,

$$\text{or } a^2 = 4x^2 + y^2$$

$$\text{similarly } b^2 = x^2 + 4y^2$$



whence $y = \pm \sqrt{\frac{4b^2 - a^2}{15}}$. Make any right angle A, and on one of the sides take A F = $\frac{a}{2}$, with centre F and radii b and 2 a, describe circles cutting the other side produced in G and H, respectively; draw G I parallel to F H; then 2 A I is the required value of y. Hence A D, and therefore A C and A B are found, and the triangle is determined.

18. To divide a straight line, so that the rectangle contained by the two parts may be equal to the square upon a given line b.

Let A B = a

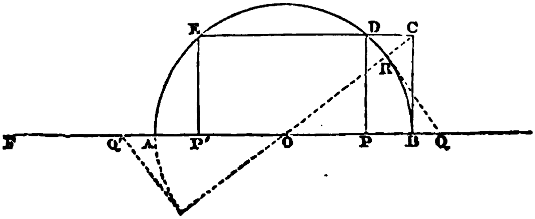
A P = x

Then the rectangle A P, P B = b^2

$$\text{or } x(a - x) = b^2$$

$$\therefore x^2 - ax = -b^2$$

$$\therefore x = \frac{a}{2} \pm \sqrt{\frac{a^2}{4} - b^2}$$



Upon A B describe a semicircle, draw B C ($= b$) perpendicular to A B, through C draw C D E parallel to A B, from D and E draw D P, E P$'$, perpendicular to A B; P and P$'$ are the required points.

If b is greater than $\frac{a}{2}$, the value of x is irrational, and therefore the problem is impossible; but then a point Q may be found in A B produced, such that, the rectangle A Q, Q B $= b^2$.

Let A Q $= x$,

$$\therefore x(x-a) = b^2$$

$$\therefore x = \frac{a}{2} \pm \sqrt{\frac{a^2}{4} + b^2}.$$

From the centre O draw the line O C cutting the circle in R, from R draw R Q perpendicular to O R, then Q is the required point; for

O Q $=$ O C $= \sqrt{\left(\frac{a^2}{4} + b^2\right)}$, and therefore A Q $= \frac{a}{2} + \sqrt{\frac{a^2}{4} + b^2}$.

Let us examine the other root $\frac{a}{2} - \sqrt{\frac{a^2}{4} + b^2}$, which is negative, and may be written in the form $-\left\{\sqrt{\frac{a^2}{4} + b^2} - \frac{a}{2}\right\}$; the magnitude of this quantity, independent of the negative sign, or its *absolute* magnitude, is evidently B Q or A Q$'$.

Now if the problem had been "to find a point Q in either A B produced, or B A produced, such, that the rectangle A Q, Q B $= b^2$", we might have commenced the solution by assuming the point Q to be in B A produced as at Q$'$; thus letting A Q$' = x$, we should have $x(a + x) = b^2$, and $x = -\frac{a}{2} \pm \sqrt{\frac{a^2}{4} + b^2}$, of which two roots the first or $-\frac{a}{2} + \sqrt{\frac{a^2}{4} + b^2} = -\left\{\frac{a}{2} - \sqrt{\frac{a^2}{4} + b^2}\right\}$ is the absolute value of the negative root in the last question; hence the negative root of the last question is a real solution of the problem expressed in a more general form, the negative sign merely pointing out the *position* of the second point Q$'$. Both roots may be exhibited in a positive form by measuring x not from A, but from a point F, A F being greater than b; for letting FA $= c$, and F Q or F Q$' = x$, we find

$$x = c + \frac{a}{2} \pm \sqrt{b^2 + \frac{a^2}{4}}.$$

The celebrated problem of dividing a given straight line in extreme and mean ratio, is solved in the same manner; letting A P $= x$ we have the rectangle A B, B P $=$ the square upon A P, or $a(a - x) = x^2$, whence $x = -\frac{a}{2} \pm \sqrt{a^2 + \frac{a^2}{4}}$; here the negative root, which gives a point to the left of A, is a solution of the problem enunciated more generally *.

19. Through a point M equidistant from two straight lines A A′ and B B′ at right angles to each other, to draw a straight line P M Q, so that the part P Q intersected by A A′ and B B′ may be of a given length b.

From M draw the perpendicular lines M C, M D.

Let $MD = a$, $DQ = x$, $CP = y$,

$$\text{then } PQ = PM + MQ,$$

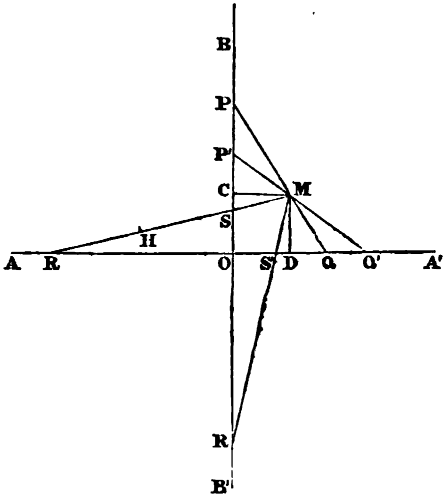
$$\text{or } b = \sqrt{a^2 + y^2} + \sqrt{a^2 + x^2},$$

$$\text{and } \frac{x}{a} = \frac{a}{y} \text{ from the similar triangles } PCM,\ MDQ.$$

$$\therefore b = \sqrt{a^2 + \frac{a^4}{x^2}} + \sqrt{a^2 + x^2}$$

$$= \sqrt{a^2 + x^2}\,(1 + \frac{a}{x});$$

$$\text{whence } x^4 + 2ax^3 + (2a^2 - b^2)\,x^2 + 2a^3 \cdot x + a^4 = 0.$$



We might solve this recurring equation, and then construct the four roots, as in the last problems; but since the roots of an equation of four dimensions are not easily obtained, we must, in general, endeavour to avoid such an equation, and rather retrace our steps than attempt its solution. Let us consider the problem again, and examine what kind of a result we may expect.

---

* Lucas de Borgo, who wrote a book on the application of this problem to architecture and polygonal figures, was so delighted with this division of a line, that he called it the Divine Proportion.

Since, in general, four lines P M Q, P′ M Q′, R S M, R′ S′ M, may be drawn fulfilling the conditions of the question, the two former, in all cases, though not always the two latter, we may conclude that there will be four solutions; but since the point M is similarly situated with respect to the two lines A A′, B B′, we may also expect that the resulting lines will be similarly situated with regard to A A′ and B B′. Thus, if there be one line P M Q, there will be another P′ M Q′ such that O Q′ = OP, and O P′ = O Q.

Again O S will be equal to O S′, and O R to O R′. Hence, if we take the perpendicular from O upon the line S R for the unknown quantity (y), we can have only two different values of this line, one referring to the lines S R and S′ R′, the other to P Q and P′ Q′; hence the resulting equation will be of *two* dimensions only. In this case the equation is

$$by^2 + 2a^2y - ba^2 = 0.$$

Again, since M R = M R′ we may take M H, H being the point of bisection of the line S R, for the unknown quantity, and then also we may expect an equation, either itself of two dimensions, or else reducible to one of that order.

Let $\text{MH} = x$; $\therefore \text{MR} = x + \frac{b}{2}$, $\text{MS} = x - \frac{b}{2}$,

$$\text{and MR : MD :: RSS : O} = \frac{ab}{x + \frac{b}{2}}$$

$$\text{MS : OD :: RS : RO} = \frac{ab}{x - \frac{b}{2}};$$

but the square upon R S = square upon R O + square upon S O,

$$\therefore b^2 = \left(\frac{ab}{x - \frac{b}{2}}\right)^2 + \left(\frac{ab}{x + \frac{b}{2}}\right)^2$$

$$\therefore x^4 - \left(2a^2 + \frac{b^2}{2}\right)x^2 + \frac{b^4}{16} - \frac{a^2b^2}{2} = 0$$

$$\therefore x = \pm\sqrt{\left\{a^2 + \frac{b^2}{4} \pm a\sqrt{a^2 + b^2}\right\}}$$

an expression of easy construction; the negative value of x is useless: of the remaining two values that with the positive sign is always real, and refers to the lines M S R, M S′ R′; the other, when real, gives the lines P M Q, P′ M Q′; it is imaginary if b^2 is less than $8a^2$, that is, joining O M and drawing P M Q perpendicular to O M, if b is less than P M Q.

This question is taken from Newton's Universal Arithmetic, and is given by him to show how much the judicious selection of the unknown quantity facilitates the solution of problems. The principal point to be attended to in such questions is, to choose that line for the unknown quantity which must be liable to the least number of variations.

20. Through the point M in the last figure to draw P M Q so that the sum of the squares upon P M and M Q shall be equal to the square upon a given line b.

Making the same substitutions as in the former part of the last article, we shall obtain the equations

$$x^2 + a^2 + y^2 + a^2 = b^2,\ xy = a^2,$$

$$\therefore\ x^2 + y^2 + 2xy = b^2, \text{ and } x + y = \pm b,$$

$$\text{or } x + \frac{a^2}{x} = \pm b, \text{ whence } x = \pm\frac{b}{2} \pm \sqrt{\frac{b^2}{4} - a^2}.$$

To construct these four values describe a circle with centre M and radius $\frac{b}{2}$, cutting A A′ in two points L, L′; with centres L, L′ and radius $\frac{b}{2}$ describe two other circles cutting A A′ again in four points: these are the required points.

21. To find a triangle A B C such that its sides A C, C B, B A, and perpendicular B D, are in continued geometrical progression.

Take any line A B $= a$ for one side, let B C $= x$,

A C : C B :: C B : B A :: B A : B D;

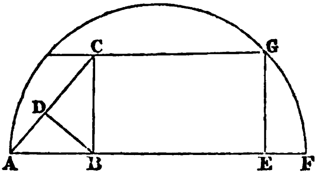
hence the triangles A C B, A D B, are equiangular, (Eucl. vi. 7, or Geometry, ii. 33,) and the angle A B C is a right angle; also A C $= \frac{x^2}{a}$, then

the square upon A C = the square upon B C + the square upon A B;

$$\therefore\ \frac{x^4}{a^2} = x^2 + a^2, \text{ or } x^4 - a^2x^2 - a^4 = 0,$$

$$\text{whence } x = \pm\sqrt{\frac{a^2 \pm \sqrt{5a^4}}{2}};$$

of these roots two are impossible, since $a^2\sqrt{5}$ is greater than a^2; and of the remaining two the negative one is useless.



In A B produced take B E $= a\sqrt{5}$ (11), and E F $= \frac{a}{2}$; upon AF describe a semicircle, and draw the perpendicular E G; then E G $= \sqrt{\left\{\frac{a}{2}(a + a\sqrt{5})\right\}} = \sqrt{\frac{a^2 + a^2\sqrt{5}}{2}}$ is the required value of x.

CHAPTER III.

THE POINT AND STRAIGHT LINE.

22. Determinate problems, although sometimes curious, yet, as they lead to nothing important, are unworthy of much attention. It was, however, to this branch of geometry that algebra was solely applied for some time after its introduction into Europe. Descartes, a celebrated French

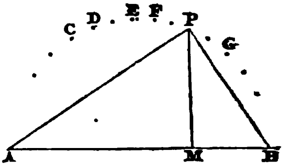
philosopher, who lived in the early part of the seventeenth century, was the first to extend the connexion. He applied algebra to the consideration of curved lines, and thus, as it were, invented a new science.

Perhaps the best way of explaining his method will be by taking a simple example. Suppose that it is required to find a point P without a given line A B, so that the sum of the squares on A P and P B shall be equal to the square upon A B.

Let P be the required point, and let fall the perpendicular P M on A B. Let A M $= x$, M P $= y$, and A B $= a$; then by the question, we have

The square on A B = the square on A P + the square on P B.
= the squares on A M, M P + the squares on P M, M B,

or $a^2 = (x^2 + y^2) + y^2 + (a - x)^2$
$= 2y^2 + 2x^2 - 2ax + a^2$
$\therefore y^2 = ax - x^2$.



Now this equation admits of an infinite number of solutions, for giving to x or A M any value, such as $\frac{a}{2}$, $\frac{a}{3}$, $\frac{a}{4}$, &c., we may, from the equation, find corresponding values of y or M P, each of them determining a separate point P which satisfies the condition of the problem.

Let C, D, E, F, &c., be the points thus determined. The number of the values of y may be increased by taking values of x between those above-mentioned and this to an infinite extent, thus we shall have an infinite number of points C, D, E, F, &c., indefinitely near to each other, so that these points ultimately form a line which geometrically represents the assemblage of all the solutions of the equation. This line A C D E F, whether curved or straight, is called the locus of the equation.

In this manner all indeterminate problems resolve themselves into investigations of loci; and it is this branch of the subject which is by far the most important, and which leads to a boundless field for research*.

23. For the better investigation of loci, equations have been divided into two classes, algebraical and transcendental.

An algebraical equation between two variables x and y is one which can be reduced to a finite number of terms involving only integral powers of x, y, and constant quantities: and it is called complete when it contains all the possible combinations of the variables together with a constant term, the sum of the indices of these variables in no term exceeding the degree of the equation; thus of the equations

$$ay + bx + c = 0$$
$$ay^2 + bxy + cx^2 + dy + ex + f = 0$$

the first is a complete equation of the first order, and the next is a complete equation of the second order, and so on.

Those equations which cannot be put into a finite and rational algebraical form with respect to the variables are called transcendental, for they can only be expanded into an infinite series of terms in which the power of the variable increases without limit, and thus the order of the equation is infinitely great, or transcends all finite orders.

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* For the definition and examples of Loci, see Geometry, iii. § 6; and the Index, article Locus.

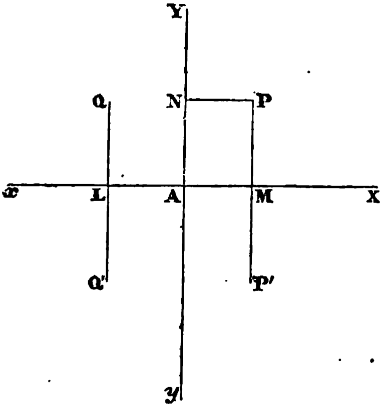
$y = \sin.\ x$, and $y = a^x$, are transcendental equations.

24. The loci of equations are named after their equations, thus the locus of an equation of the first order is a line of the first order; the locus of an equation of the second order is a line of the second order; the locus of a transcendental equation is a transcendental line or curve.

Algebraical equations have not corresponding loci in all cases, for the equation may be such as not to admit of any real values of both x and y; the equation $y^2 + x^2 + a^2 = 0$ is an example of this kind, where, whatever real value we give to x, we cannot have a real value of y: there is therefore no locus whatever corresponding to such an equation.

THE POSITION OF A POINT IN A PLANE.

25. The position of a point in a plane is determined by finding its situation relatively to some fixed objects in that plane; for this purpose suppose the plane of the paper to be the given plane, and let us consider as known the intersection A of two lines x X and y Y of unlimited length, and also the angle between them; from any point P, in this plane, draw P M parallel to A Y, and P N parallel to A X, then the position of the point P is evidently known if A M and A N are known. For it may be easily shown, *ex absurdo*, that there is but one point within the angle Y A X such that its distance from the lines A Y and A X is P N and P M respectively.



A M is called the abscissa of the point P; A N, or its equal M P, is called the ordinate; A M and M P are together the co-ordinates of P; X x is called the axis of abscissas, Y y the axis of ordinates. The point A where the axes meet is termed the origin.

The axes are called oblique or rectangular, according as Y A X is an oblique or a right angle. In this treatise rectangular axes as the most simple will generally be employed.

Let the abscissa A M $= x$, and the ordinate M P $= y$, then if on measuring these lengths A M and M P we find the first equal to a and the second equal to b, we have, to determine the position of this point P, the two equations

$$x = a,\ y = b$$

and as they are sufficient for this object, we call them, when taken together, the equations to this point.

The same point may also be defined by the equation

$$(y - b)^2 + (x - a)^2 = 0$$

for this equation can only be satisfied by the values $x = a$ and $y = b$.

And in general any equation which can only be satisfied by a single real value of each variable quantity x and y, refers to a point whose situation is determined by the co-ordinates corresponding to these values.

26. In this manner the position of any point in the angle Y A X can be determined, but in order to express the positions of points in the angle Y A x, some further considerations are necessary.

In the solution of the problem, article (18), we observed that negative quantities may be geometrically represented by lines drawn in a certain direction. An extension of this idea leads to the following reasoning.

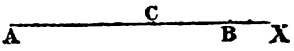
When we affix a negative sign to any quantity, we do not signify any change in its magnitude, but merely the way in which the quantity is to be used, or the operation to be performed on it. Thus the absolute magnitude of -5 is just as great as that of $+5$; but -5 means that 5 is to be subtracted, and $+5$ that it is to be added. As the sign $+$ is applied to quantities variously estimated, the sign $-$ will have in each of these various cases a corresponding meaning, necessarily following from that of the sign $+$. Whatever $+$ means, we must always have $-a+a=0$. Hence we may define $-a$ to be a quantity estimated in such a manner that the altering it by the operation indicated by $+a$ reduces the result to nothing. This is properly the meaning of the sign $-$; it depends entirely on that of the sign $+$ in every case.

The symbol of positive quantity is used in a variety of ways; but in every instance the above principle shows in what way the negative quantity must, as a necessary consequence of the meaning of the positive quantity, be used.

Thus, if we placed a mark on a pole stuck vertically into the ground, at some point in the pole which was bare at low water and covered at high water, and scored upwards the inches from that mark, we might express the height of the surface by the number of inches above the mark, positively, when the surface was above the mark; but at low water when the surface is below the mark, 11 inches for instance, we should call the height -11; because when 11 inches were added to the height, (that is, when the surface of the water was advanced 11 inches upwards, which is the direction in which the positive quantities are supposed to be reckoned,) the surface would be just at the mark, and would be *no inches* in height reckoning from the mark.

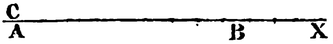
Suppose a man to advance directly from a given point p miles in the first 6 hours of a day, and to go back in the next 6 hours q miles; at the end of the 12 hours his advance from the given point would be $(p-q)$ miles. Thus, suppose $p=10$, and $q=6$, he will advance $(10-6)$ or 4 miles. But suppose he recedes 10 miles, then his advance will in the 12 hours be $(10-10)$ or 0: he will be just where he was at first. Suppose he recedes 15 miles, at the end of the 12 hours he will be 5 miles *behind* the original point. Here we say *behind*, because the movement in the direction of the original advance was considered to be *forward*. And it is clear that in this case, from an advance of 10 miles, and a recess of 15, the advance is -5; that is, it requires a further advance of 5 miles to make the man exactly as forward as he was at starting.

Now let us consider a fixed point A, and a line measured from it by positive quantities in the direction A X. Suppose the line to be described by the motion of a point from A along A X; and after the point has been carried *forward* (that is, towards X) m linear units, as to B, let it be carried

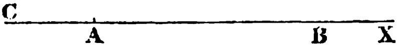


back n linear units, as to C; then altogether the advance of the point or the length of A C will be $(m - n)$ linear units.

Again, suppose n to become $= m$; that is, let the point be carried back exactly to A; then the advance of the point along A X will still be measured by $(m - n) = m - m = 0$.



Once more, let n exceed m; that is, let B C exceed A B; the advance of the point will be expressed by $(m - n)$ still; but this will now be a negative number, showing by how many linear units the point must be advanced in order to bring it forward to the original starting point A. Now any line A C may be considered to be determined by the motion of a point either simply along A C, or along first A B and then B C. We see, therefore, if we begin by reckoning distances from A in the direction A X as positive quantities, we are compelled to consider distances from A in the opposite direction as negative quantities.



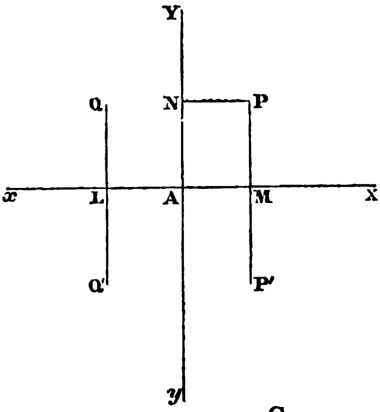
Conversely again, having designated positive quantities by lines in one direction from a given point, suppose the calculation produces a negative result, what meaning are we to assign to it? The negative result shows how much positive quantity is required to bring the whole result to nothing. Now positive quantity, by the hypothesis, is distance measured in the original direction; therefore the negative quantity shows how much distance measured in the original direction is required to bring the result to nothing. But if there be a distance from A, such that a linear units in the original direction must be subjoined to bring the result to nothing, (that is, to reduce to nothing the distance from A,) it is clear that this distance must be that of a linear units measured in a direction from A opposite to the original direction. That is, the negative quantity must be represented by lines drawn in the direction opposite to that in which the lines representing the positive quantities are drawn.

It is immaterial in which direction the line is drawn which we consider positive: but when chosen, negative quantities of the same kind must be taken in the opposite direction.

27. We are now able to express the position of points in the remaining angles formed by the axes, by considering all lines in the direction A X to be positive and those in A x to be negative: and similarly all those drawn in the direction A Y will be considered positive, and therefore those in A y will be negative.

We have then the following table of co-ordinates.

P in the angle X A Y, $+ x$, $+ y$,
Q in the angle Y A x, $- x$, $+ y$,
Q′ in the angle x A y, $- x$, $- y$,
P′ in the angle X A y, $+ x$, $- y$.



Hence the equations to a given point P are $x = a,\ y = b$

. . . . Q . . $x = -a,\ y = b$

. . . . Q′ . . $x = -a,\ y = -b$

. . . . P′ . . $x = a,\ y = -b$

28. If, the abscissa AM remaining the same, the ordinate MP diminishes, the point P approaches to the axis AX; and when MP is nothing, P is situated on that axis; in this case the equations to the point P are

$$x = a,\ y = 0: \text{ or } y^2 + (x - a)^2 = 0.$$

Similarly when the point P is situated on the axis AY, its equations are

$$x = 0,\ y = b: \text{ or } (y - b)^2 + x^2 = 0.$$

If both AM and MP vanish, we have the equations to the origin A,

$$x = 0,\ y = 0: \text{ or } y^2 + x^2 = 0.$$

Ex. 1. The point whose equations are $x = 4$, $y = -2$, is situated in the angle X A y, at a distance AM = 4 times the linear unit from the axis of y, and MP′ = twice that unit from the axis of x.

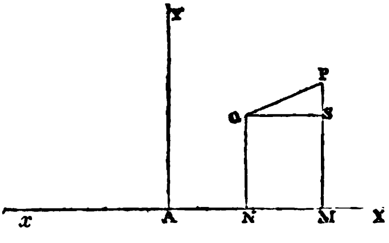
Ex. 2. The point whose equation is $(y + 3)^2 + (x + 2)^2 = 0$ is situated in the angle x A y, at distances AL = 2, LQ′ = 3, from the axes.

Ex. 3. The point whose equations are $x = 0$, $y = -3$ is in the line A y, at a distance = 3 times the linear unit.

Ex. 4. The point whose equation is $y^2 + (x + a)^2 = 0$, is in A x, at a distance a from the origin.

The preceding articles are true if the co-ordinate axes be oblique.

29. To find an expression for the distance D between two points P and Q.



Let the axes be rectangular and let the equations to

P be $x = a,\ y = b$

Q $x = a',\ y = b'$;

or in other words, let the co-ordinates of P be AM = a, MP = b, and those of Q be AN = a', NQ = b', draw QS parallel to AX.

Then the square upon QP = the square upon QS + the square upon PS;

$$\text{and } QS = NM = AM - AN = a - a'$$

$$\text{also } PS = PM - QN = b - b'$$

$$\therefore D^2 = (a - a')^2 + (b - b')^2$$

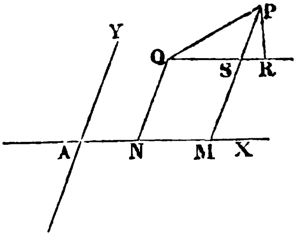
If Q be in the angle Y A x we have AN = $-a'$,

$$\therefore D^2 = (a + a')^2 + (b - b')^2.$$

If Q be at the origin we have $a' = 0$ and $b' = 0$

$$\therefore D^2 = a^2 + b^2, \text{ or } D = \sqrt{a^2 + b^2}$$

30. If the angle between the axes be oblique and = ω, draw PM and QN parallel to AY, and QS parallel to AX; also let PR be drawn perpendicular to QS; then the square upon QP = the square on QS + the square on PS + twice the rectangle QS, SR;



and, $Q\,S = a - a'$,

$$P\,S = b - b',$$

$$S\,R = P\,S \cos. P\,S\,R$$

$$= P\,S \cos. Y\,A\,X$$

$$= (b - b') \cos. \omega;$$

$$\therefore D^2 = (a - a')^2 + (b - b')^2 + 2(a - a')(b - b') \cos. \omega;$$

and when the point Q is at the origin, and therefore $a' = 0$, and $b' = 0$,

$$D^2 = a^2 + b^2 + 2\,a\,b \cos. \omega.$$

THE LOCUS OF AN EQUATION OF THE FIRST DEGREE.

31. To find the locus of an equation of the first degree between two unknown quantities.

The most general form of such an equation is,

$A\,y + B\,x + C = 0$, or $y = -\dfrac{B}{A}x - \dfrac{C}{A}$, or $y = \alpha x + b$ if $-\dfrac{B}{A} = \alpha$ and $-\dfrac{C}{A} = b$; we will in the first place consider the equation in its most simple form $y = \alpha\,x$.

Let A X, A Y be the rectangular axes, then a point in the locus will be determined by giving to x a particular value as 1, 2, 3, &c.; let A M, M P and A N, N Q be the respective co-ordinates of two points P and Q thus determined;

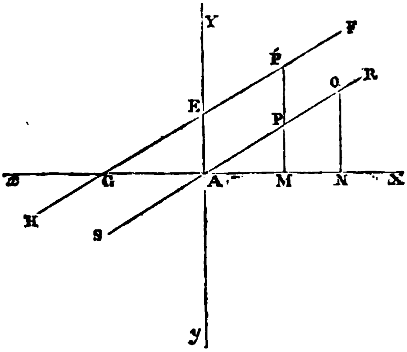
since $y = \alpha\,x$, we have

$$M\,P = \alpha . A\,M$$

$$\text{and } N\,Q = \alpha . A\,N$$

$$\therefore M\,P : A\,M :: N\,Q : A\,N;$$

therefore the triangles A M P, A N Q are similar, and the angles M A P, N A Q, equal to one another: hence the two lines A P, A Q coincide. If a third point R be taken in the locus, then, as before, A R will coincide with A P and A Q. Consequently all the lines drawn from A to the several points of the locus coincide; that is, all the points P, Q, R, &c., are in the same straight line A R, and by giving negative values to x we can determine the point S, &c., to be in the same straight line R A produced. Hence the straight line R A S produced both ways indefinitely, being the assemblage of all the points determined by the equation $y = \alpha\,x$, is the *locus* of that equation.



In considering the equation $y = \alpha x + b$, we observe that the new ordinate y always exceeds the former by the quantity b; hence taking A E in the axis A Y equal to b, and drawing the line H E F parallel to S R, the line H E F is the locus required.

Hence the equation of the first order belongs to the straight line.

32. To explain the nature of the equation more clearly, we will take the converse problem. To find the equation to a straight line H F, that is, to find the relation which exists between the co-ordinates, x and y, of each of its points?

Let A be the origin of co-ordinates, A X, A Y the axes; from A draw A R parallel to H F, and from any point P′ in the given line draw P′ P M perpendicular to A X and cutting A R in P.

$$\text{Let } AM = x,\ MP' = y, \text{ and } AE = b;$$

$$\begin{aligned} \text{then } MP' &= PM + PP' \\ &= AM \tan. PAM + AE \\ &= x \tan. FGX + b; \end{aligned}$$

$$\text{or } y = \alpha x + b, \quad \text{if } \tan. FGX = \alpha.$$

If A G $= a$, we have A E = A G. tan. E G A, or $b = a\,\alpha$, and therefore the equation to the straight line may be written under the form $y = \alpha x + \alpha a$.

33. In general, therefore, the equation to the straight line contains two constant quantities b and α; the former is the distance A E or is the ordinate of the point in which the line cuts the axis of y, the latter is the tangent of the angle which the line makes with the axis of x, for the angle F G A = the angle P A M: hence

$$\tan. FGA = \tan. PAM = \frac{y - b}{x} = \alpha.$$

It is to be particularly observed that, in calling α the tangent of the angle which the line makes with the axis of x, we understand the angle F G X and not F G x.

34. In the equation $y = \alpha x + b$, the quantities α and b may be either both positive, or both negative, or one positive and the other negative; let us then examine the course of the line to which the equation belongs in each case. Now it is clear that the knowledge of two points in a straight line is sufficient to determine the position of that line; hence we shall only find the points where it cuts the axes since they are the most easily obtained.

1. α and b positive; $\therefore y = \alpha x + b$;

$$\text{Let } x = 0; \quad \therefore y = b; \text{ in A Y take } AD = b;$$

$$y = 0; \quad \therefore x = -\frac{b}{\alpha}; \text{ in A}x \text{ take } AB = \frac{b}{\alpha};$$

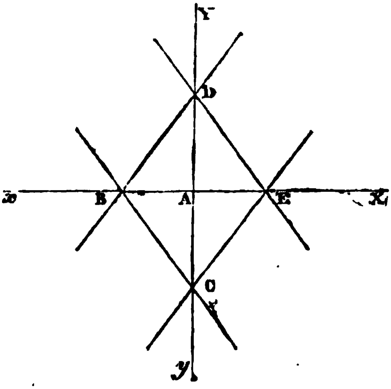
join B D; B D produced is the required locus.

2. α positive and b negative; $\therefore y = \alpha x - b$;

$$\text{Let } x = 0; \quad \therefore y = -b, \text{ in A}y \text{ take } AC = b;$$

$$y = 0; \quad \therefore x = \frac{b}{\alpha}; \text{ in A X take } AE = \frac{b}{\alpha};$$

join C E; C E produced is the required locus.



3. α negative and b positive; $\therefore y = -\alpha x + b$;
Let $x = 0$; $\therefore y = b$; in A Y take AD $= b$;
$y = 0$; $\therefore x = \frac{b}{\alpha}$; in A X take A E $= \frac{b}{\alpha}$;
join D E; D E produced is the required locus.

4. α negative and b negative; $\therefore y = -\alpha x - b$;
Let $x = 0$; $\therefore y = -b$; in A y take A C $= b$;
$y = 0$; $\therefore x = -\frac{b}{\alpha}$; in A x take AB $= \frac{b}{\alpha}$;
join B C; B C produced is the required locus.

35. The quantities α and b may also change in absolute value.

Let $b = 0$; $\therefore y = \pm \alpha x$; and the loci are two straight lines passing through the origin and drawn at angles with the axis of x whose respective tangents are $\pm \alpha$.

Let $\alpha = 0$; $\therefore y = 0x \pm b$; $\therefore y = \pm b$ and $x = \frac{0}{0}$; the former of these results shows that every point in the locus is equidistant from the axis of x, and the latter (or $0x = 0$) that every value of x satisfies the original equation; hence the loci are two straight lines drawn through D and C both parallel to the axis of x.

It has been stated (28), that the system of equations $y = b, x = 0$ refers to a point; we here see that the system $y = b$, $x = \frac{0}{0}$ refers to a straight line; hence, although the equation $x = \frac{0}{0}$ is generally omitted, yet it must be considered as essential to the locus.

Let $\alpha = \frac{1}{0}$; referring to article 32, the equation to the straight line may be written $y = \alpha x \pm \alpha a$ or $\frac{y}{\alpha} = x \pm a$, which when $\alpha = \frac{1}{0}$ becomes $0y = x \pm a$; hence, as before, the system $x = \pm a$, $y = \frac{0}{0}$, or more simply the equation $x = \pm a$ denotes two straight lines parallel to the axis of y and at a distance $\pm a$ from that axis.

Again let both $\alpha = 0$, and $b = 0$; and $\therefore$ the equation $y = \alpha x + b$ becomes $y = 0\,x + 0$; and hence, $y = 0$, $x = \frac{0}{0}$, and the locus is the axis of x.

If $\alpha = \frac{1}{0}$, and $b = 0$, the equation becomes $0\,y = x + 0$; $\therefore x = 0$ and $y = \frac{0}{0}$. Hence the equation $x = 0$ denotes the axis of y.

36. By the above methods the line to which any equation of the first order belongs may be drawn.

In the following examples reference is made to parts of the last figure.

Ex. 1. $3\,y - 5\,x - 1 = 0$; let $x = 0$, $\therefore y = \frac{1}{3}$; on the axis A Y take A D one-third of the linear unit, then the line passes through D: again let $y = 0$, $\therefore x = -\frac{1}{5}$; on the axis A x take A B $= \frac{1}{5}$ of the unit, then the line passes through B; hence the line joining the points B and D is the locus required.

Ex. 2. $10\,y - 21\,x + 6 = 0$; a line situated like C E.

Ex. 3. $y - x = 0$; let $x = 0$ $\therefore y = 0$, and the line passes through the origin; also α or the tangent of the angle which the line makes with the axis of $x = 1$, therefore that angle $= 45°$; hence the straight line drawn through the origin and bisecting the angle Y A X is the required locus.

Ex. 4. $5\,y - 2\,x = 0$. The line passes through the origin as in the last example, but to find another point through which the line passes, let $x = 5$; $\therefore y = 2$: hence take A E $= 5$, and from E draw E P $(= 2)$ perpendicular to A X; then the line joining the points A, P is the locus required.

Ex. 5. $ay + bx = 0$; a line drawn through A, and parallel to B C.

Ex. 6. $y^2 - 3\,x^2 = 0$; two straight lines making angles of 60° with the axis of x.

Ex. 7. $3y - 4 = 0$; take A D $= \frac{4}{3}$ of the unit, a line through D drawn parallel to A X is the locus.

Ex. 8. $x^2 + x - 2 = 0$; take A E $= 1$, and A B $= 2$, the lines drawn through E and B parallel to A Y are the required loci.

Ex. 9. $y + 2\,x = 4$. The equation to a straight line may be put under the convenient form $\frac{y}{b} + \frac{x}{a} = 1$, and since when $y = 0$, $x = a$, and when $x = 0$, $y = b$, the quantities b and a are respectively the distances of the origin from the intersection of the line with the axes of y and x. Thus Ex. 9. in this form is $\frac{y}{4} + \frac{x}{2} = 1$, take A D $= 4$, and A E $= 2$, join D E, this line produced is the required locus.

37. If the equation involve the second root of a negative quantity its locus will not be a straight line, but either a point or altogether imaginary:

thus the locus of the equation $y + 2x\sqrt{-1} - a = 0$ is a point whose co-ordinates are $x = 0$ and $y = a$, for no other real value of x can give a real value to y; but the locus of the equation $y + x + a\sqrt{-1} = 0$ is imaginary, for there are no corresponding real values of x and y. (24)

38. We have thus seen that the equation to a straight line is of the form $y = \alpha x + b$, and that its position depends entirely upon α and b.

By a given line we understand one whose position is given, that is, that α and b are given quantities; when we seek a line we require its position, so that assuming $y = \alpha x + b$ to be its equation, α and b are the two indeterminate quantities to be found by the conditions of the question: if only one can be found the conditions are insufficient to fix the position of the line.

By a given point we understand one whose co-ordinates are given; we shall generally use the letters x_1 and y_1 for the co-ordinates of a given point, and to avoid useless repetition, the point whose co-ordinates are x_1 and y_1 will be called " the point x_1, y_1." Similarly the line whose equation is $y = \alpha x + b$ will be called " the line $y = \alpha x + b$."

If in the same problem we use the equations to two straight lines as $y = \alpha x + b$ and $y = \alpha' x + b'$, it must be carefully remembered that x and y are not the same quantities in both equations; we might have used the equations $y = \alpha x + b$, and $Y = \alpha' X + b'$, X and Y being the variable coordinates of the second line, but the former notation is found to be the more convenient.

39. We regret much that in the following problems on straight lines we cannot employ an homogeneous equation as $\frac{y}{b} + \frac{x}{a} = 1$. In algebraical geometry the formulas most in use are very simple, much more so indeed than they would be if homogeneous: moreover the advantage of a uniform system of symbols and formulas is so great to mathematicians that it should not be violated without very strong reasons. To remedy in some degree this want of regularity, the student should repeatedly consider the meaning of the constants at his first introduction to the subject of straight lines.

PROBLEMS ON STRAIGHT LINES.

40. To find the equation to a straight line passing through a given point.

The point being given its co-ordinates are known; let them be $x_1 y_1$, and let the equation to the straight line be $y = \alpha x + b$; we signify that this line passes through the point $x_1 y_1$, by saying that when the variable abscissa x becomes x_1, then y will become y_1: hence the equation to the line becomes

$$y_1 = \alpha x_1 + b$$

$$\therefore b = y_1 - \alpha x_1$$

substituting this value for b in the first equation, we have

$$y = \alpha x + y_1 - \alpha x_1$$

$$\text{or } y - y_1 = \alpha(x - x_1)$$

The shortest method of eliminating b is by subtracting the second equation from the first, and this is the method generally adopted.

Since α, which fixes the direction of the line, is not determined, there may be an infinite number of straight lines drawn through a given point; this is also geometrically apparent.

If the given point be on the axis of x, $y_1 = 0$, and $\therefore\ y = \alpha\,(x - x_1)$; and if it be on the axis of y, $x_1 = 0\ \therefore\ y - y_1 = \alpha\,x$.

If either or both of the co-ordinates of the given point be negative, the proper substitutions must be made: thus if the point be on the axis of x and in the negative direction from A, its co-ordinates will be $-\,x_1$ and 0; therefore the equation to the line passing through that point will be

$$y = \alpha\,(x + x_1).$$

41. To find the equation to a straight line passing through two given points x_1, y_1 and x_2, y_2.

Let the required equation be $y = \alpha\,x + b \quad (1)$
then since the line passes through the given points, we have the equations

$$y_1 = \alpha\,x_1 + b \quad (2)$$

$$y_2 = \alpha\,x_2 + b \quad (3)$$

Subtracting (2) from (1)

$$y - y_1 = \alpha\,(x - x_1) \quad (4)$$

Subtracting (3) from (2)

$$y_1 - y_2 = \alpha\,(x_1 - x_2)$$

$$\therefore\ \alpha = \frac{y_1 - y_2}{x_1 - x_2}$$

Substituting this value of α in (4), we have finally

$$y - y_1 = \frac{y_1 - y_2}{x_1 - x_2}\,(x - x_1)$$

The two conditions have sufficed to determine α and b, and by their elimination the position of the line is fixed, as it ought to be, since only one straight line can be drawn through the same two points.

This equation will assume different forms according to the particular situation of the given points.

Thus if the point x_2, y_2 be on the axis of x, we have $y_2 = 0$;

$$\therefore\ y - y_1 = \frac{y_1}{x_1 - x_2}\,(x - x_1);$$

if it be on the axis of y, $x_2 = 0$; $\therefore\ y - y_1 = \dfrac{y_1 - y_2}{x_1}\,(x - x_1)$;
and if it be at the origin both y_2 and $x_2 = 0$;

$$\therefore\ y - y_1 = \frac{y_1}{x_1}(x - x_1) = \frac{y_1}{x_1}x - y_1$$

$$\therefore\ y = \frac{y_1}{x_1}\,x.$$

This last equation is also thus obtained; the line passing through the origin, its equation must be of the form $y = \alpha x$ (31) where α is the tangent of the angle which the line makes with the axis of x, and this line passing through the point x_1, y_1, α must be equal to $\frac{y_1}{x_1}$ $\therefore y = \frac{y_1}{x_1} x$.

If a straight line pass through three given points, the following relation must exist between the co-ordinates of those points:

$$(y_1 x_2 - x_1 y_2) - (y_1 x_3 - x_1 y_3) + (y_2 x_3 - x_2 y_3) = 0.$$

42. To find the equation to a straight line passing through a given point x_3, y_3, and bisecting a finite portion of a given straight line.

Let the portion of the straight line be limited by the points $x_1 y_1$ and $x_2 y_2$, and therefore the co-ordinates of the bisecting point are $\frac{x_1 + x_2}{2}$, $\frac{y_1 + y_2}{2}$; hence the required equation is

$$y - y_3 = \alpha (x - x_3) = \frac{y_1 + y_2 - 2 y_3}{x_1 + x_2 - 2 x_3}(x - x_3)$$

43. To find the equation to a straight line parallel to a given straight line.

Let $y = \alpha x + b$ (1) be the given line

$y = \alpha' x + b'$ (2) . . . . required line.

then since the lines are parallel they must make equal angles with the axis of x or $\alpha' = \alpha$ $\therefore$ the required equation is

$$y = \alpha x + b' \quad (3).$$

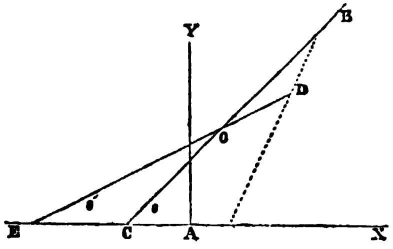
Of course b' could not be determined by the single condition of the parallelism of the lines, since an infinite number of lines may be drawn parallel to the given line; but if another condition is added, b' will be then determined: thus if the required line passes also through a given point $x_1 y_1$, equation (2) is

$$y - y_1 = \alpha' (x - x_1)$$

$$\therefore \text{(3) becomes } y - y_1 = \alpha (x - x_1)$$

44. To find the intersection of two given straight lines C B, E D.

This consists in finding the co-ordinates of the point O of intersection. Now it is evident that at this point they have the same abscissa and ordinate; hence if in the equations to two lines we regard x as representing the same abscissa and y the same ordinate, it is in fact saying that they are the co-ordinates X, Y of the point of intersection O.



Let $y = \alpha x + b$ be the equation to C B

and $y = \alpha' x + b'$ . . . . E D

then at O we have $Y = \alpha X + b = \alpha' X + b'$

$$\therefore X = \frac{b' - b}{\alpha - \alpha'}$$

$$\text{and } Y = \alpha X + b = \frac{\alpha b' - \alpha b}{\alpha - \alpha'} + b = \frac{\alpha b' - \alpha' b}{\alpha - \alpha'}.$$

Ex. 1. To find the intersection of the lines whose equations are

$y = 3x + 1$ and $y - 2x - 4 = 0$. $X = 3$ and $Y = 10$.

Ex. 2. To find the intersection of the lines whose equations are

$y - x = 0$ and $3y - 2x = 1$. $X = 1$ and $Y = 1$.

If a third line, whose equation is $y = \alpha'' x + b''$, passes through the point of intersection, the relation between the coefficients is

$$(\alpha b' - \alpha' b) - (\alpha b'' - \alpha'' b) + (\alpha' b'' - \alpha'' b') = 0.$$

45. To find the tangent, sine and cosine of the angle betwen two given straight lines.

Let $y = \alpha x + b$ be the equation to C B

$y = \alpha' x + b'$ . . . . E D

θ and θ' the angles which they make respectively with the axis of x; then

$$\tan. DOB = \tan. EOC = \tan. (\theta - \theta') = \frac{\tan. \theta - \tan. \theta'}{1 + \tan. \theta \tan. \theta'} = \frac{\alpha - \alpha'}{1 + \alpha\alpha'}$$

$$\text{also } \cos. DOB = \frac{1}{\sec. DOB} = \frac{1}{\sqrt{1 + (\tan. DOB)^2}} = \frac{1 + \alpha\alpha'}{\sqrt{(1+\alpha^2)(1+\alpha'^2)}}$$

$$\text{and sine } DOB = \tan. DOB \times \cos. DOB = \frac{\alpha - \alpha'}{\sqrt{1 + \alpha^2} \, . \, \sqrt{1 + \alpha'^2}}.$$

46. To find the equation to a straight line making a given angle with another straight line.

Let $y = \alpha x + b$ be the given line C B,

$y = \alpha' x + b'$ required line E D,

β = tangent of the given angle D O B.

Then $\alpha' = \tan. DEC = \tan. (BCX - BOD)$

$$= \frac{\tan. BCX - \tan. BOD}{1 + \tan. BCX \, . \, \tan. BOD} = \frac{\alpha - \beta}{1 + \alpha\beta}.$$

Substituting this value for α' in the second equation,

$$y = \frac{\alpha - \beta}{1 + \alpha\beta} x + b'.$$

If the required line passes also through a given point x_1, y_1, the equation is

$$y - y_1 = \frac{\alpha - \beta}{1 + \alpha\beta} (x - x_1).$$

If D be considered the given point x_1, y_1, then not only the line D O E but another (the dotted line in the figure) might be drawn, making a given angle with B C, and its equation is found, as above, to be

$$y - y_1 = \frac{\alpha + \beta}{1 - \alpha\beta}(x - x_1);$$

so that both lines are included in the equation

$$y - y_1 = \frac{\alpha \mp \beta}{1 \pm \alpha\beta}(x - x_1).$$

For example, the two straight lines which pass through the point D and cut B C at an angle of 45° are given by the equations

$$y - y_1 = \frac{\alpha - 1}{\alpha + 1}(x - x_1),$$

$$y - y_1 = \frac{1 + \alpha}{1 - \alpha}(x - x_1).$$

Also the equation to the straight line passing through D and cutting the axis of x at an angle of 135° is

$$y - y_1 = \beta(x - x_1) = \tan. 135^\circ (x - x_1) = -(x - x_1),$$

$$\text{or } y + x = y_1 + x_1.$$

47. If the required line is to be perpendicular to the given line, β is infinitely great; therefore the fraction $\dfrac{\alpha - \beta}{1 + \alpha\beta} = \dfrac{\frac{\alpha}{\beta} - 1}{\frac{1}{\beta} + \alpha} = -\dfrac{1}{\alpha}$, or $\alpha' = -\dfrac{1}{\alpha}$; hence the equation to a straight line perpendicular to a given line $y = \alpha x + b$, is $y = -\dfrac{1}{\alpha}x + b'$.

This may be also directly proved, for drawing O E perpendicular to B C, as in the next figure, we have $\alpha' = \tan.$ O E X $= -\tan.$ O E C $= -\cot.$ O C X $= -\dfrac{1}{\alpha}$: hence in the equations to two straight lines which are perpendicular to one another we have $\alpha\alpha' + 1 = 0$; and, conversely, if in the equations to two straight lines, we find $\alpha\alpha' + 1 = 0$, these lines are perpendicular to one another.

If the perpendicular line pass also through a given point $x_1 y_1$, its equation is

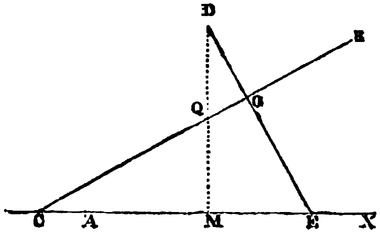
$$y - y_1 = -\frac{1}{\alpha}(x - x_1);$$

and, of course, this equation will assume various forms, agreeing with the position of the point $x_1 y_1$; thus, for example, the line drawn through the origin perpendicular to the line $y = \alpha x + b$, is one whose equation is $y = -\dfrac{1}{\alpha}x$, for here both x_1 and $y_1 = 0$.

48. To find the length of a perpendicular from a given point D $(x_1 y_1)$ on a given straight line C B.

Let $y = \alpha x + b$ (1) be the equation to C B,

then $y - y_1 = -\frac{1}{\alpha}(x - x_1)$ (2) is the equation to the perpendicular line D O E,



Let $p = $ D O; then if X and Y be the co-ordinates of O determined from (1) and (2) we have $p^2 = (X - x_1)^2 + (Y - y_1)^2$; (29)

$$\text{from (2) } Y = y_1 - \frac{1}{\alpha}(X - x_1) = \alpha X + b \text{ from (1)}$$

$$= \alpha(X - x_1) + \alpha x_1 + b,$$

$$\therefore (\alpha + \frac{1}{\alpha})(X - x_1) = y_1 - \alpha x_1 - b,$$

$$\therefore X - x_1 = \frac{\alpha}{1+\alpha^2}(y_1 - \alpha x_1 - b), \text{ also } Y - y_1 = -\frac{1}{\alpha}(X - x_1)$$

$$\therefore p^2 = (X - x_1)^2 + (Y - y_1)^2$$

$$= (X - x_1)^2 + \frac{1}{\alpha^2}(X - x_1)^2$$

$$= \frac{1 + \alpha^2}{\alpha^2}(X - x_1)^2$$

$$= \frac{1 + \alpha^2}{\alpha^2} \frac{\alpha^2}{(1 + \alpha^2)^2} \cdot (y_1 - \alpha x_1 - b)^2 = \frac{1}{1 + \alpha^2}(y_1 - \alpha x_1 - b)^2$$

$$\therefore p = \pm \frac{y_1 - \alpha x_1 - b}{\sqrt{1 + \alpha^2}}.$$

The superior sign is to be taken when the given point is above the given straight line, and the inferior in the contrary case.

If the given line pass through the origin $b = 0$; $\therefore p = \pm \frac{y_1 - \alpha x_1}{\sqrt{1 + \alpha^2}}$.

If the origin be the given point, $x_1 = 0$ and $y_1 = 0$; $\therefore p = \frac{\pm b}{\sqrt{1 + \alpha^2}}$.

There is another way of obtaining the expression for p.

Since the equation $y = \alpha x + b$ applies to all points in C O B, it must to Q, where M D or y_1 cuts C O B; $\therefore$ M Q $= \alpha x_1 + b$.

Now D O = D Q sin. D Q O,

but DQ = D M − M Q $= y_1 - \alpha x_1 - b$,

and sin. D Q O = sin. C Q M = cos. Q C M $= \frac{1}{\text{sec. Q C M}} =$

$$\frac{1}{\sqrt{1+(\text{tan. } QCM)^2}} = \frac{1}{\sqrt{1+\alpha^2}};$$

$\therefore$ D O or $p = \dfrac{y_1 - \alpha x_1 - b}{\sqrt{1+\alpha^2}}$, or $\dfrac{\alpha x_1 + b - y_1}{\sqrt{1+\alpha^2}}$, if D was below the line.

49. If the line D E had been drawn making a given angle [illegible]se tangent was β with the given line C O, the distance D O might be found; for instead of equation (2) we shall have

$$y - y_1 = \frac{\alpha - \beta}{1 + \alpha\beta}(x - x_1) \quad (46);$$

hence, following the same steps as above, we shall find

$$p = \pm \frac{y_1 - \alpha x_1 - b}{\sqrt{1+\alpha^2}} \; \frac{\sqrt{1+\beta^2}}{\beta}.$$

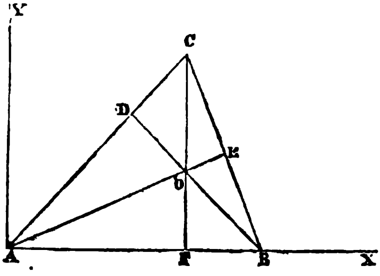
This expression is also very easily obtained trigonometrically.

Let γ = sine of the given angle, then

$$DO = DQ \, . \, \frac{\text{sin. } DQO}{\text{sin. } DOQ}$$

$$= \frac{y_1 - \alpha x_1 - b}{\sqrt{1+\alpha^2}} \, . \, \frac{1}{\gamma}.$$

50. The equation to the straight line may be used with advantage in the demonstration of the following theorem:—



If from the angles of a plane triangle perpendiculars be let fall on the opposite sides, these perpendiculars will meet in one point.

In the triangle A B C, let A E, B D, C F be perpendiculars from A, B and C on the opposite sides; let O be the point where A E and B D meet, then the theorem will be established by showing that the abscissa to the point O is A F.

Let A be the origin of co-ordinates,
A B the axis of x,
and A Y, perpendicular to A B, the axis of y.

Let the co-ordinates of C be x_1, y_1
.... B $x_2, 0$:

we have then the following equations,

$$\text{to AC} \quad y = \frac{y_1}{x_1}x \quad (41)$$

$$\therefore \text{ to B D, } y = \alpha(x - x_2) = -\frac{x_1}{y_1}(x - x_2) \quad (47)$$

$$\text{to B C, } y - y_2 = \frac{y_1 - y_2}{x_1 - x_2}(x - x_2) \quad (41)$$

$$\text{or } y = \frac{y_1}{x_1 - x_2}(x - x_2) \text{ since } y_2 = 0.$$

$$\therefore \text{ to A E, } y = \alpha x = -\frac{x_1 - x_2}{y_1} x;$$

for the intersection O of B D and A E we have, by equating the values of y,

$$-\frac{x_1}{y_1}(x - x_2) = -\frac{x_1 - x_2}{y_1} x,$$

$$\therefore x_1 x - x_1 x_2 = x_1 x - x_2 x; \quad \therefore x_2 x = x_1 x_2 \text{ and } x = x_1,$$

that is, the abscissa of the point O is found to be that of C.

In the same manner it may be proved that if perpendiculars be drawn from the bisections of the sides, they will meet in one point.

Similarly we may prove that the three straight lines F C, K B, and A L, in the 47th proposition of Euclid, meet in one point within the triangle A B C.

51. We have hitherto considered the axes as rectangular, but if they be oblique, the coefficient of x, in the equation to a straight line, is not the tangent of the angle which the line makes with the axis of x.

Let ω = the angle between the axes,
θ = the angle which the line makes with the axis of x;

$$\text{then } \alpha = \frac{y-b}{x} = \frac{\sin.\ \theta}{\sin.\ (\omega - \theta)} \quad (33);$$

b remains, as before, the distance of the origin from the intersection of the line with the axis of y: hence the equation to a straight line referred to oblique axes is

$$y = \frac{\sin.\ \theta}{\sin.\ (\omega - \theta)} x + b.$$

Since this equation is of the form $y = \alpha x + b$ all the results in the preceding articles which do not affect the ratio of $\frac{\sin.\ \theta}{\sin.\ (\omega - \theta)}$ will be equally true when the axes are oblique.

Thus, articles 40, 41, 42, 43, and 44, require no modification.

To find the tangent (β) of the angle between two given straight lines.

$$\text{Let } \left.\begin{array}{l} y = \alpha x + b \\ y = \alpha' x + b' \end{array}\right\} \text{be the equations;}$$

from $\alpha = \frac{\sin.\ \theta}{\sin.\ (\omega - \theta)}$ we have $\tan.\ \theta = \frac{\alpha \sin.\ \omega}{1 + \alpha \cos.\ \omega}$; and, similarly,

$$\tan.\ \theta' = \frac{\alpha' \sin.\ \omega}{1 + \alpha' \cos.\ \omega}; \text{ hence } \beta = \tan.\ (\theta - \theta') = \frac{(\alpha - \alpha') \sin.\ \omega}{1 + \alpha\alpha' + (\alpha + \alpha') \cos.\ \omega}$$

To find the equation to a straight line passing through a given point $x_1 y_1$, and making a given angle with a given straight line.

Let β be the tangent of the given angle,

$y = \alpha x + b$, the given line,
$y - y_1 = \alpha' (x - x_1)$, the required line.

From the last formula we have

$$\alpha' = \frac{\alpha \sin.\ \omega - \beta (1 + \alpha \cos.\ \omega)}{\sin.\ \omega + \beta (\alpha + \cos.\ \omega)}$$

and the required equation is

$$\therefore y - y_1 = \frac{\alpha \sin.\ \omega - \beta (1 + \alpha \cos.\ \omega)}{\sin.\ \omega + \beta (\alpha + \cos.\ \omega)} (x - x_1).$$

If the lines be perpendicular to each other $\beta = \frac{1}{0}$;

$$\therefore \alpha' = -\frac{1+\alpha \cos.\ \omega}{\alpha + \cos.\ \omega}.$$

and the required equation is

$$y - y_1 = -\frac{1+\alpha \cos.\ \omega}{\alpha + \cos.\ \omega}(x - x_1).$$

To find the length of the perpendicular from a given point upon a given straight line.

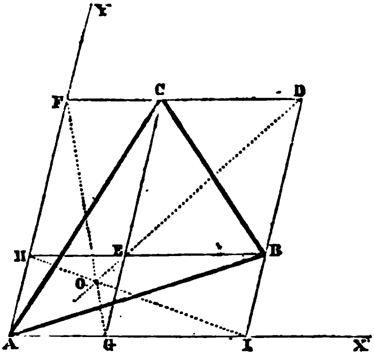
Instead of equation (2), in article 48, we must use the equation just found, and then proceeding as usual we shall find

$$p = \pm \frac{(y_1 - \alpha x_1 - b) \sin.\ \omega}{\sqrt{\{1 + 2\alpha \cos.\ \omega + \alpha^2\}}}.$$

It will be concluded from an observation of these formulas, that oblique axes are to be avoided as much as possible; they may be used with advantage where points and lines, but not angles, are the subjects of discussion. As an instance, we shall take the following theorem.

52. If, upon the sides of a triangle as diagonals, parallelograms be described, having their sides parallel to two given lines, the other diagonals of the parallelograms will intersect each other in the same point.

Let A B C be the triangle, A X, A Y the given lines, E B D C, C F A G,



H A I B the parallelograms, the opposite diagonals D E, F G, and H I will meet in one point O.

Let A be the origin, A X, A Y the oblique axes

$x_1 y_1$ the co-ordinates of B

$x_2 y_2$. . . . . C.

To find the equation to D E;

let it be $y = \alpha x + b$

$y_2 = \alpha x_1 + b$ at D

$$\therefore y - y_2 = \alpha (x - x_1)$$

$y_1 - y_2 = \alpha (x_2 - x_1)$ at E

$$\therefore y - y_2 = \frac{y_1 - y_2}{x_2 - x_1}(x - x_1) \quad (1).$$

To find the equation to F G;

$$y = \alpha x + b$$

$$y_2 = 0 + b \text{ at F}$$

$$\therefore y - y_2 = \alpha x$$

$$0 - y_2 = \alpha x_2 \text{ at G}$$

$$\therefore y - y_2 = -\frac{y_2}{x_2} x \quad (2).$$

To find the equation to H I;

$$y = \alpha x + b$$

$$y_1 = 0 + b \text{ at H}$$

$$\therefore y - y_1 = \alpha x$$

$$0 - y_1 = \alpha x_1 \text{ at I}$$

$$\therefore y - y_1 = -\frac{y_1}{x_1} x \quad (3).$$

Equating the values of y in (1) and (2) we find $X = \dfrac{x_1 x_2 (y_1 - y_2)}{y_1 x_2 - x_1 y_2}$; also equating the values of y in (2) and (3) we find the same value for X; hence the abscissa for intersection being the same for any two of the lines, they must all three intersect in the same point.

Similarly we may prove that if from the angles of a plane triangle straight lines be drawn to the bisections of the opposite sides, they will meet in one point.

CHAPTER IV.

THE TRANSFORMATION OF CO-ORDINATES.

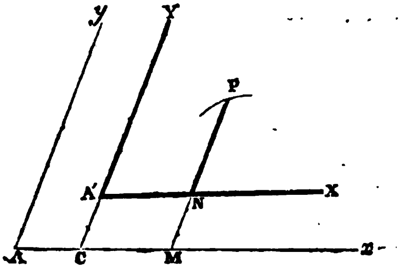
53. Before we proceed to the discussion of equations of higher orders, it is necessary to investigate a method for changing the position of the co-ordinate axes.

The object is to place the axes in such a manner that the equation to a given curve may appear in its most simple form, and conversely by the introduction of indeterminate constants into an equation to reduce the number of terms, so that the form and properties of the corresponding locus may be most easily detected.

An alteration of this nature cannot in the least change the form of the curve, but only the algebraical manner of representing it; thus the general equation to the straight line $y = \alpha x + b$ becomes $y = \alpha x$ when the origin is on the line itself. Also on examining articles 46 and 51 we see that the simplicity of an equation depends very much on the angle between the axes.

Hence in many cases not only the position of the origin but also the direction of the axes may be altered with advantage. The method of performing these operations is called the transformation of co-ordinates.

54. To transform an equation referred to an origin A, to an equation referred to another origin A′, the axes in the latter case being parallel to those in the former.



Let A x, A y be the original axes

A′ X, A′ Y the new axes

$$\left.\begin{array}{l} AM = x \\ MP = y \end{array}\right\}\text{ original co-ordinates of P}$$

$$\left.\begin{array}{l} A'N = X \\ NP = Y \end{array}\right\}\text{ new co-ordinates of P}$$

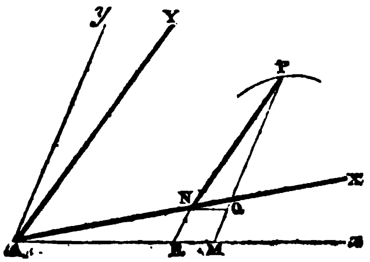
$$\left.\begin{array}{l} AC = a \\ CA' = b \end{array}\right\}\text{ the co-ordinates of the new origin A}',$$

then $MP = MN + NP$, that is, $y = b + Y$,

$AM = AC + CM$, ... $x = a + X$;

substituting these values for y and x in the equation to the curve, we have the transformed equation between Y and X referred to the origin A′.

55. To transform the equation referred to oblique axes, to an equation referred to other oblique axes having the same origin.



Let A x, A y be the original axes,

AX, AY be the new axes,

$$\left.\begin{array}{l} AM = x \\ MP = y \end{array}\right\}\text{ original co-ordinates of P},$$

$$\left.\begin{array}{l} AN = X \\ NP = Y \end{array}\right\}\text{ new co-ordinates of P}.$$

Let the angle $xAy = \omega$, $xAX = \theta$, $xAY = \theta'$;
Draw NR parallel to PM, and NQ parallel to AM,
then $y = MP = MQ + QP = NR + QP$

$$= AN\,\frac{\sin.\ NAR}{\sin.\ NRA} + PN\,\frac{\sin.\ PNQ}{\sin.\ PQN}$$

$$= X\,\frac{\sin.\ \theta}{\sin.\ \omega} + Y\,\frac{\sin.\ \theta'}{\sin.\ \omega};$$

and $x = AM = AR + RM = AR + NQ$

$$= AN\,\frac{\sin.\ ANR}{\sin.\ ARN} + PN\,\frac{\sin.\ NPQ}{\sin.\ NQP}$$

$$= X\,\frac{\sin.\ (\omega - \theta)}{\sin.\ \omega} + Y\,\frac{\sin.\ (\omega - \theta')}{\sin.\ \omega}$$

$$\therefore\ y = \frac{X \sin.\ \theta + Y \sin.\ \theta'}{\sin.\ \omega}$$

$$x = \frac{X \sin.\ (\omega - \theta) + Y \sin.\ (\omega - \theta')}{\sin.\ \omega}.$$

56. Let the original axes be oblique, and the new rectangular, or $\theta' - \theta = 90°$.

$$\therefore\ y = \frac{X \sin.\ \theta + Y \cos.\ \theta}{\sin.\ \omega}$$

$$x = \frac{X \sin.\ (\omega - \theta) - Y \cos.\ (\omega - \theta)}{\sin.\ \omega}.$$

57. Let the original axes be rectangular, or $\omega = 90°$.

$$\therefore\ y = X \sin.\ \theta + Y \sin.\ \theta',$$
$$x = X \cos.\ \theta + Y \cos.\ \theta'$$

58. Let both systems be rectangular, or $\omega = 90°$ and $\theta' - \theta = 90°$

$$\therefore\ y = X \sin.\ \theta + Y \cos.\ \theta,$$
$$x = X \cos.\ \theta - Y \sin.\ \theta$$

59. These forms have been deduced from the first, but each of them may be found by a separate process. The first and last pairs are the most useful. Perhaps they may be best remembered if expressed in the following manner.

Both systems oblique, the formulas (55) become

$$y = \{X \sin.\ XAx + Y \sin.\ YAx\}\,\frac{1}{\sin.\ xAy}$$

$$x = \{X \sin.\ XAy + Y \sin.\ YAy\}\,\frac{1}{\sin.\ xAy}.$$

Both systems rectangular, the formulas (58) become

$$y = X \cos.\ XAy + Y \cos.\ YAy$$
$$x = X \cos.\ XAx + Y \cos.\ YAx.$$

If the situation of the origin be changed as well as the direction of the axes, we have only to add the quantities a and b to the values of x and y respectively; however, in such a case, it is most convenient to perform each transformation separately.

If the new axis of X falls below the original axis of x, the angle θ must be considered as negative, therefore its sine will be negative and its cosine positive. Hence the formulas of transformation will require a slight alteration before applied to this particular case.

Since the values of x and y are in all cases expressed by equations of the first order, the degree of an equation is never changed by the transformation of co-ordinates.

60. Hitherto we have determined the situation of a point in a plane by its distance from two axes, but there is also another method of much use. Let S be a fixed point, and S B a fixed straight line; then the point P is also evidently determined if we know the length S P and the angle P S B.

If $SP = r$ and $PSB = \theta$, r and θ are called the polar co-ordinates of P. S is called the pole, and S P the radius vector, because a curve may be supposed to be described by the extremity of the line S P revolving round S, the length of S P being variable. The fixed straight line S B is also called the axis.

To transform an equation between co-ordinates x and y into another between polar co-ordinates r and θ,

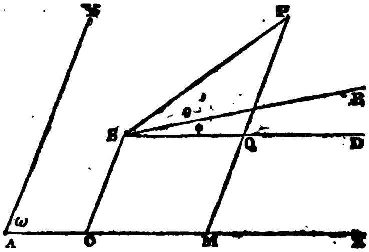
Draw S D parallel to A X, and let the angle $BSD = \phi$, and the angle $YAX = \omega$.

Let $AM = x$, $MP = y$, $AC = a$, $CS = b$,

$$\left.\begin{aligned} \text{then } y &= MP = MQ + QP = b + r\frac{\sin.(\theta+\phi)}{\sin.\omega} \\ x &= AM = AC + SQ = a + r\frac{\sin.\{\omega-(\theta+\phi)\}}{\sin.\omega.} \end{aligned}\right\} (1)$$

Let S B coincide with S D, or $\phi = 0$;

$$\left.\begin{aligned} \therefore y &= b + r\frac{\sin.\theta}{\sin.\omega} \\ x &= a + r\frac{\sin.(\omega-\theta)}{\sin.\omega} \end{aligned}\right\} (2)$$



61. Let the original axes be also rectangular, or $\omega = \frac{\pi}{2}$;

$$\left.\begin{aligned} \therefore y &= b + r\sin.\theta \\ x &= a + r\cos.\theta \end{aligned}\right\} (3)$$

and if the origin A be the pole, we have $a = 0$ and $b = 0$.

$$\left.\begin{aligned} \therefore y &= r\sin.\theta \\ x &= r\cos.\theta \end{aligned}\right\} (4).$$

Of these formulas (3) and (4) are the most useful.

62. Conversely, to find r and θ in terms of x and y: from (I) we have

$$\frac{x-a}{y-b} = \frac{\sin.\{\omega-(\theta+\phi)\}}{\sin.(\theta+\phi)} = \sin.\omega\cot.(\theta+\phi) - \cos.\omega;$$

$$\therefore \tan.(\theta+\phi) = \frac{(y-b)\sin.\omega}{x-a+(y-b)\cos.\omega}$$

$$\therefore \theta+\phi = \tan.^{-1}\left\{\frac{(y-b)\sin.\omega}{x-a+(y-b)\cos.\omega}\right\},$$

where the symbol $\tan.^{-1}a$ is equivalent to the words "a circular arc whose radius is unity, and tangent a."

$$\text{also } r^2 = (x-a)^2 + (y-b)^2 + 2(x-a)(y-b)\cos.\omega \ldots (30)$$

63. If the axes are rectangular, or $\omega = \frac{\pi}{2}$, the pole at the origin, and therefore $a = 0$ and $b = 0$, and also $\phi = 0$, we have $\frac{y}{x} = \tan.\ \theta$, and therefore

$$\theta = \tan.^{-1} \frac{y}{x}$$

$$\text{and } r^2 = x^2 + y^2 \ldots. (29)$$

and these are the formulas generally used.

From $\tan.\ \theta = \frac{y}{x}$ we have $\cos.\ \theta = \frac{1}{\sqrt{1 + (\tan.\ \theta)^2}} = \frac{1}{\sqrt{1 + \frac{y^2}{x^2}}} = \frac{x}{\sqrt{y^2 + x^2}}$ and $\sin.\ \theta = \cos.\ \theta \times \tan.\ \theta = \frac{y}{\sqrt{y^2 + x^2}}$; hence the value of θ may also be expressed by the equations

$$\theta = \sin.^{-1} \frac{y}{\sqrt{y^2 + x^2}} \text{ or } \theta = \cos.^{-1} \frac{x}{\sqrt{y^2 + x^2}}.$$

CHAPTER V.

ON THE CIRCLE.

64. FOLLOWING the order of this treatise, our next subject of discussion would be the loci of the general equation of the second degree; but there is one curve among these loci, remarkable for the facility of its description and the simplicity of its equation: this curve, we need scarcely say, is the circle; and as the discussion of the circle is admirably fitted to prepare the reader for other investigations, we proceed to examine its analytical character.

The common definition of the circle states, that the distance of any point on the circumference of the figure from the centre is equal to a given line called the radius.

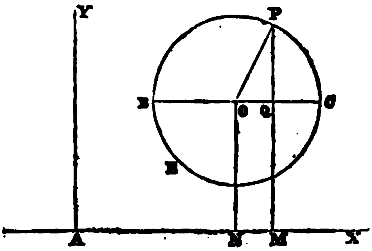
If a and b be the co-ordinates of the centre, x and y those of any point on the circumference, and r the radius, the distance between those points is $\sqrt{\{(y - b)^2 + (x - a)^2\}}$ (29): hence the equation to the circle is

$$(y - b)^2 + (x - a)^2 = r^2$$

65. To obtain this equation directly from the figure,
let A be the origin,
A X, A Y the rectangular axes,
$\left.\begin{array}{l} AN = a \\ NO = b \end{array}\right\}$ the co-ordinates of the centre,

$\left.\begin{array}{l} AM = x \\ MP = y \end{array}\right\}$ those of any point P on the circumference;

B O Q C a line parallel to the axis of x.



Then the square upon O P = the square upon P Q + the square upon O Q,

and $PQ = PM - QM = y - b$

also $OQ = AM - AN = x - a$;

$\therefore r^2 = (y-b)^2 + (x-a)^2$

or $(y-b)^2 + (x-a)^2 = r^2$ (1)

If the axis of x or that of y passes through the centre, the equation (1) becomes respectively

$$\left.\begin{array}{l} (y-b)^2 + x^2 = r^2 \\ \text{or } y^2 + (x-a)^2 = r^2 \end{array}\right\} (2)$$

If the origin be at any point of the circumference as E, we have then the equation of condition $a^2 + b^2 = r^2$; expanding (1) and reducing it by means of this condition, we have

$$y^2 - 2by + x^2 - 2ax = 0 \ (3).$$

If the origin is at B, B O being the axis of x, we have $b = 0$ and $a = r$;

$$\therefore y^2 + x^2 - 2rx = 0$$

$$\text{or } y^2 = 2rx - x^2 \ (4).$$

Again, placing the origin at the centre O, we have $b = 0$ and $a = 0$;

$$\therefore y^2 + x^2 = r^2 \ (5).$$

The above equations are all useful, but those most required are (1), (4), and (5).

66. Equation (1), if expanded, is

$$y^2 + x^2 - 2by - 2ax + a^2 + b^2 - r^2 = 0.$$

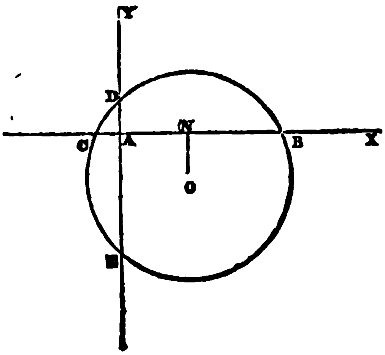
This differs from the complete equation of the second order (23) in having the coefficients of x^2 and y^2 unity, and by having no term containing the product xy.

Any equation of this form being given, we can, by comparing it with the above equation, determine the situation of its locus, that is, find the position of the centre, and the magnitude of the corresponding circle.

Ex. 1. $y^2 + x^2 + 4y - 8x - 5 = 0$.

here $b = -2$, $a = 4$, and $a^2 + b^2 - r^2 = -5$;

$$\therefore r^2 = a^2 + b^2 + 5 = 25.$$



Let A be the origin of co-ordinates, A X, A Y the axes.

In AX take AN = 4 times the linear unit, from N draw NO perpendicular to A X, but downwards, and equal to 2; then O is the centre of the circle. With centre O and radius 5 describe a circle; this is the locus required. The points where it cuts the axis of x are determined by putting $y = 0$;

$$\therefore x^2 - 8x - 5 = 0;$$

$$\therefore x = 4 \pm \sqrt{21};$$

$$\text{hence } AB = 4 + \sqrt{21} \text{ and } AC = 4 - \sqrt{21}.$$

Similarly putting $x = 0$, we find $AD = 1$ and $AE = 5$.

67. The shortest way of describing the locus is to put the equation into the form $(y - b)^2 + (x - a)^2 = r^2$.

For example, the equation

$$y^2 + x^2 + cy + dx + e = 0,$$

becomes, by the addition and subtraction of $\frac{c^2}{4}$ and $\frac{d^2}{4}$,

$$y^2 + cy + \frac{c^2}{4} + x^2 + dx + \frac{d^2}{4} + e - \frac{c^2}{4} - \frac{d^2}{4} = 0,$$

$$\text{or } (y + \frac{c}{2})^2 + (x + \frac{d}{2})^2 = \frac{c^2 + d^2}{4} - e,$$

where we observe directly that $-\frac{d}{2}$ and $-\frac{c}{2}$ are the co-ordinates of the centre, and that $\sqrt{\{\frac{c^2 + d^2}{4} - e\}}$ is the radius of the required locus.

Ex. 2. $y^2 + x^2 + 4y - 4x - 8 = 0$

add and subtract 8, and the equation becomes

$$y^2 + 4y + 4 + x^2 - 4x + 4 - 16 = 0$$

$$\text{or } (y + 2)^2 + (x - 2)^2 = 16$$

hence the co-ordinates of the centre are $a = 2$ and $b = -2$, and the radius is 4.

Ex. 3. $2y^2 + 2x^2 - 4y - 4x + 1 = 0$; $a = 1$, $b = 1$, $r = \sqrt{\frac{3}{2}}$.

4. $y^2 + x^2 - 6y + 4x - 3 = 0$; $a = -2$, $b = 3$, $r = 4$.

5. $6y^2 + 6x^2 - 21y - 8x + 14 = 0$; $a = -\frac{2}{3}$, $b = \frac{7}{4}$, $r = \frac{13}{12}$

6. $y^2 + x^2 + 4y - 3x = 0$; $a = \frac{3}{2}$, $b = -2$, $r = \frac{5}{2}$.

7. $y^2 + x^2 - 4y + 2x = 0$; $a = -1$, $b = 2$, $r = \sqrt{5}$.

In these last two examples there is no occasion to calculate the length of the radius, for the circumference of the circle passes through the origin of co-ordinates, as do the loci of all equations which want the last or constant term.

8. $y^2 + x^2 - 4y = 0$; $a = 0$, $b = 2$, $r = 2$.

9. $y^2 + x^2 + 6x = 0$; $a = -3$, $b = 0$, $r = 3$.

10. $y^2 + x^2 - 6x + 8 = 0$; $a = 3$, $b = 0$, $r = 1$.

In the last three examples the centre of the circle is on the axes.

68. We have seen that the equation to the circle referred to rectangular axes does not contain the product xy, and also that the coefficients of y^2 and x^2 are each unity; we have, moreover, seen that generally an equation of the second degree of this form has a circle for its locus, but there are some exceptions to this last rule.

For example, the equation $y^2 + x^2 - 8y - 12x + 52 = 0$ is apparently of the circular form; its locus, however, is not a circle, but a point whose co-ordinates are $x = 6$ and $y = 4$, for it may be put under the form $(y - 4)^2 + (x - 6)^2 = 0$, the only real solution of which is $x = 6$ and $y = 4$; and this will always be the case when $r^2 = 0$, hence a point may be considered as a circle whose radius is indefinitely small.

Again, the equation $y^2 + x^2 - 4y + 2x + 9 = 0$, may be put under the form $(y - 2)^2 + (x + 1)^2 = -4$; but there are no possible values of x and y that can satisfy this equation, therefore the locus is imaginary (24).

69. To find the equation to the tangent to a circle.

Let the origin of co-ordinates be at the centre, and x', y' any point on the circumference.

Then the equation to the straight line through x', y' is

$$y - y' = \alpha(x - x');$$

the equation to the radius through x', y' is $y = \frac{y'}{x'}x$;

but the tangent being perpendicular to the radius, we have $\alpha = -\frac{x'}{y'}$ (47.)

$$\therefore y - y' = -\frac{x'}{y'}(x - x'),$$

$$\text{or } yy' - y'^2 = -xx' + x'^2;$$

$$\therefore yy' + xx' = y'^2 + x'^2 = r^2.$$

The equation $yy' + xx' = r^2$, thus found, may be easily remembered, from the similarity of its form to that of the equation to the circle, it being obtained at once from $y^2 + x^2 = r^2$ by changing y^2 or yy into yy', and x^2 or xx into xx'.

If we take the general equation to the circle, $(y - b)^2 + (x - a)^2 = r^2$, the equation to the radius is

$$y - b = \frac{y' - b}{x' - a}(x - a) \ldots . (41)$$

$$\therefore \alpha = -\frac{x' - a}{y' - b}, \text{ and the equation to the tangent is}$$

$$y - y' = \alpha(x - x') = -\frac{x' - a}{y' - b}(x - x')$$

The equation $(y' - b)^2 + (x' - a)^2 = r^2$ enables us finally to reduce the equation to the tangent to the form

$$(y - b)(y' - b) + (x - a)(x' - a) = r^2$$

70. To find the equation to the tangent of a circle parallel to a given straight line.

Let $y = \alpha x + b$ be the given line,
and $yy' + xx' = r^2$ the required tangent, in which x', y' are unknown.

∴ Since these lines are parallel, $-\frac{x'}{y'} = \alpha$, (43) or $-\frac{\sqrt{r^2 - y'^2}}{y'} = \alpha$,

$$\therefore y' = \pm \frac{}{\sqrt{1 + \alpha^2}}.$$

Hence by substitution in the equation $y = -\frac{x'}{y'} x + \frac{r^2}{y'}$, we have

$$y = \alpha x \pm r \sqrt{1 + \alpha^2};$$

consequently two tangents can be drawn parallel to the given line.

71. To find the intersection of a straight line and circle:

Let the centre of the circle be the common origin, and let the equations be $y = \alpha x + b$, and $y^2 + x^2 = r^2$; at the point of intersection, y and x must be the same for both. $\therefore r^2 - x^2 = (\alpha x + b)^2$,

$$\text{whence } x = \frac{-\alpha b \pm \sqrt{\{r^2(1+\alpha^2) - b^2\}}}{1 + \alpha^2};$$

there being two values of x, we have two intersections; these values may be constructed, and the points of intersection found.

If $r^2(1+\alpha^2) = b^2$ the two values of x are equal, and the line will touch the circle. If $r^2(1+\alpha^2)$ is less than b^2 the line will not meet the circle.

Ex. 1. $y^2 + x^2 = 25$, $y + x = 1$; $x = 4$ and -3, $y = -3$ and 4

Ex. 2. $y^2 + x^2 = 25$, $y + x = 5$; $x = 5$ and 0, $y = 0$ and 5

Ex. 3. $y^2 + x^2 = 25$, $4y + 3x = 24$; The line touches the circle.

We may observe that the combination of an equation of the first order with any equation of two dimensions will, as above, give an equation of the second order for solution; and hence there can be only two intersections of their loci.

72. If the axes be oblique and inclined to each other at an angle ω, the equation to the circle is

$$(y - b)^2 + (x - a)^2 + 2(y - b)(x - a) \cos. \omega = r^2, \qquad (30)$$

and $y^2 + x^2 + 2xy \cos. \omega = r^2$, if the origin be at the centre; hence the equation $y^2 + cxy + x^2 + dy + ex + f = 0$, belongs to the circle in the particular case where the co-ordinate angle is one whose cosine $= \frac{c}{2}$.

Comparing it with the general equation to the circle, we find

$$2 \cos. \omega = c, \qquad -2b - 2a \cos. \omega = d,$$

$$-2a - 2b \cos. \omega = e, \qquad a^2 + b^2 + 2ab \cos. \omega - r^2 = f;$$

whence, by elimination, we obtain $a = \frac{2e - cd}{c^2 - 4}$, $b = \frac{2d - ce}{c^2 - 4}$,

$$\text{and } r^2 = \frac{ced - e^2 - d^2}{c^2 - 4} - f;$$

hence the co-ordinates of the centre and the radius being known, the locus can be drawn.

Ex. 1. $$y^2 + xy + x^2 + y + x - 1 = 0;$$

here $2 \cos. \omega = 1$; $\therefore \omega = 60°$; hence this equation will give a circle if the axes be inclined at an angle of 60°; the co-ordinates of the centre are $a = -\frac{1}{3}$, $b = -\frac{1}{3}$; and the radius $= \frac{2}{\sqrt{3}}$

The equation to this circle, when referred to the centre as origin, and to rectangular axes, is obviously $y^2 + x^2 = r^2 = \frac{4}{3}$.

Ex. 2. $y^2 + \sqrt{2} \,.\, xy + x^2 - 9 = 0$.

This will give a circle if the axes be inclined at an angle of 45°, the centre is at the origin of co-ordinates, and the radius $= 3$.

Of course c must never be equal to, or greater than, ± 2, for cos. ω must be less than unity.

If the circle be referred to oblique co-ordinates, the equation to the radius is $y - b = \frac{y' - b}{x' - a} (x - a) \ldots . (41)$

and the equation to the tangent is

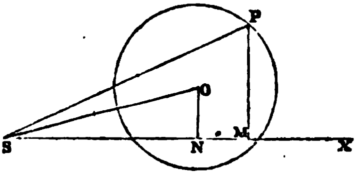
$$y - y' = -\frac{(x' - a) + (y' - b)\cos.\omega}{(y' - b) + (x' - a)\cos.\omega}(x - x') \ldots . (51)$$

and reducing as in article 69 we have the equation to the tangent

$$(y - b)(y' - b) + (x - a)(x' - a) + (x - a)(y' - b)\cos.\omega + (x' - a)(y - b)\cos.\omega = r^2.$$

73. To find the polar equation to the circle.

Let the pole be at the origin S, and the angle P S M ($= \theta$) be measured from the axis of x.



$$\left.\begin{array}{l}\text{Let } SM = x \\ MP = y\end{array}\right\} \text{be rectangular co-ordinates of P}$$

$$\left.\begin{array}{l}\text{and } SN = a \\ NO = b\end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots O$$

Let $SP = u$, $SO = c$, and angle $OSX = \alpha$; then by the formulas (61) or by the figure $y = u \sin.\theta$, $x = u \cos.\theta$, $a = c \cos.\alpha$, and $b = c \sin.\alpha$.

Substituting these values of x and y in the equation to the circle,

$$y^2 + x^2 - 2by - 2ax + a^2 + b^2 - r^2 = 0,$$

we have

$$u^2 (\sin.\theta)^2 + u^2 (\cos.\theta)^2 - 2cu \sin.\alpha \sin.\theta - 2cu \cos.\alpha \cos.\theta + c^2 (\cos.\alpha)^2 + c^2 (\sin.\alpha)^2 - r^2 = 0,$$

$$\text{or} \quad u^2 - 2cu \{ \sin.\theta \sin.\alpha + \cos\theta \cos.\alpha \} + c^2_1 - r^2 = 0,$$

$$\text{or} \quad u^2 - 2cu \cos.(\theta - \alpha) + c^2 - r^2 = 0.$$

74. If a and b are not expressed in terms of the polar co-ordinates c and α, the polar equation is then of the form

$$u^2 - 2\{ b \sin.\theta + a \cos.\theta \} u + a^2 + b^2 - r^2 = 0.$$

If the origin be on the circumference we have $a^2 + b^2 = r^2$, and therefore the equation to the circle becomes

$$u = 2\,(b \sin.\,\theta + a \cos.\,\theta)$$

If the axis of x passes through the centre, $b = 0$, and the equation is

$$u^2 - 2\,a\,u \cos.\,\theta + a^2 - r^2 = 0.$$

$$\text{Whence } u = a \cos.\,\theta \pm \sqrt{r^2 - a^2\,(\sin.\theta)^2};$$

which equation may also be directly obtained from the triangle S P O.

CHAPTER VI.

DISCUSSION OF THE GENERAL EQUATION OF THE SECOND ORDER.

75. The most general form in which this equation appears is

$$a y^2 + b x y + c x^2 + d y + e x + f = 0;$$

where a, b, c, &c., are constant coefficients.

Let the equation be solved with respect to y and x separately, then

$$y = -\frac{bx+d}{2a} \pm \frac{1}{2a}\sqrt{\{(b^2-4\,ac)\,x^2+2\,(bd-2\,ae)\,x+d^2-4\,af\}}; \quad (1)$$

$$x = -\frac{by+e}{2c} \pm \frac{1}{2c}\sqrt{\{(b^2-4ac)\,y^2+2\,(be-2\,cd)\,y+e^2-4\,cf\}} \quad (2).$$

On account of the double sign of the root in (1), there are, in general, two values of y; hence there are two ordinates corresponding to the same abscissa: these ordinates may be constructed whenever the values of x render the radical quantity real; but if these values render it nothing, there is only one ordinate, and if they make it imaginary, no corresponding ordinate can be drawn, and therefore there is no point of the curve corresponding to such a value of x. Hence, to know the extent and limits of the curve, we must examine when the quantity under the root is real, nothing, or imaginary.

This will depend on the algebraical sign of the quantity

$$(b^2 - 4ac)\,x^2 + 2\,(b\,d - 2\,a\,e)\;x + d^2 - 4\,a\,f.$$

In an expression of this form, a value may be given to x, so large that the sign of the whole quantity depends only upon that of its first term, or upon that of its coefficient $b^2 - 4\,a\,c$, since x^2 is always positive for any real value of x.

For, writing the expression in the form $m\,(x^2 + \frac{n}{m}\,x + \frac{p}{m})$ let q be the absolute value of the greater of the two quantities $\frac{n}{m}$ and $\frac{p}{m}$; then substituting $r = \pm\,(q + 1)$ for x, the expression becomes

$$m\left\{q^2+2q+1\pm\frac{n}{m}q\pm\frac{n}{m}+\frac{p}{m}\right\},$$

which, whatever be the values of $\frac{n}{m}$ and $\frac{p}{m}$, is positive, and the same is true for any magnitude greater than $\pm r$; hence the sign of the expression depends upon that of m.

When b^2-4ac is negative, real values may be given to x, either positive or negative, greater than $\pm r$, which will render y imaginary. The curve will then be limited in both the positive and negative directions of x.

When b^2-4ac is positive, all values of x not less than $\pm r$ will render y real, and therefore the curve is of infinite extent in both directions of x.

Lastly, when b^2-4ac is nothing, the quantity under the root becomes

$$2(bd-2ae)x+d^2-4af.$$

If $bd-2ae$ be positive, real positive values may be given to x, which shall render y real; but if a negative value be given to x greater than $\frac{d^2-4af}{2(bd-2ae)}$, y is imaginary; therefore the curve will be of indefinite extent in the direction of x positive and limited in the opposite direction.

But if $bd-2ae$ be negative, exactly opposite results will follow, that is, the curve will be of indefinite extent in the direction of x negative and limited in the opposite direction.

Taking equation (2) we should find similar results.

The curves corresponding to the equation of the second degree, may therefore be divided into three distinct classes.

1. b^2-4ac negative, curves limited in every direction.
2. b^2-4ac positive, curves unlimited in every direction.
3. b^2-4ac nothing, curves limited in one direction, but unlimited in the opposite direction.

76. *First class b^2-4ac negative.*

Let $-\frac{b}{2a}=\alpha,\ -\frac{d}{2a}=l,\ \frac{b^2-4ac}{4a^2}=-\mu,$

and let x_1 and x_2 be the roots of the equation

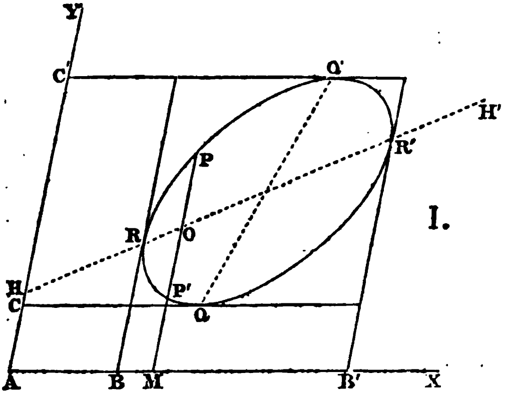
$$(b^2-4ac)x^2+2(bd-2ae)x+d^2-4af=0.$$

Then equation (1) or

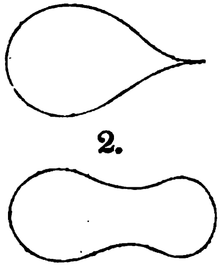
$$y=-\frac{bx+d}{2a}\pm\sqrt{\left\{\frac{b^2-4ac}{4a^2}\left(x^2+2\frac{bd-2ae}{b^2-4ac}x+\frac{d^2-4af}{b^2-4ac}\right)\right\}}$$

becomes by substitution

$$y=\alpha x+l\pm\sqrt{\{-\mu(x-x_1)(x-x_2)\}}$$



1.



2.

Let A be the origin of co-ordinates, A X, A Y the oblique axes.

Let H H′ be the line represented by the equation $y = \alpha x + l$, M O one of its ordinates corresponding to any value of x between x_1 and x_2.

Along the line M O take O P and O P′ each equal to $\sqrt{\{-\mu(x-x_1)(x-x_2)\}}$, then P and P′ are two points in the curve, for

$$\text{M P} = \text{M O} + \text{O P} = \alpha x + l + \sqrt{\{-\mu(x-x_1)(x-x_2)\}}$$
$$\text{M P}' = \text{M O} - \text{O P}' = \alpha x + l - \sqrt{\{-\mu(x-x_1)(x-x_2)\}}$$

If we repeat this construction for all the real values of x which render the root real we obtain the different points of the curve.

The line H H′ is called a diameter of the curve, for it bisects all the chords P P′ which are parallel to the axis of y.

The reality of y depends on the reality of the radical quantity, which last depends on the form of the factors $(x - x_1)$ and $(x - x_2)$, that is, on the roots x_1 and x_2. Now these may enter the equation in three forms—real and unequal—real and equal—or both imaginary.

Case 1. Let x_1 and x_2 be real and unequal, take A B $= x_1$, A B′ $= x_2$, then if $x = x_1$ or x_2 the quantity $-\mu(x-x_1)(x-x_2)$ vanishes, and the ordinate to the curve coincides with the ordinate to the diameter, therefore drawing through B and B′ two lines B R and B′ R′ parallel to A Y the curve cuts the diameter in R and R′.

For all values of x between x_1 and x_2 there are two real values of y, for $x - x_1$ is positive and $x - x_2$ is negative, and therefore $-\mu(x-x_1)(x-x_2)$ is positive.

For all values of $x > x_2$ or $< x_1$, $-\mu(x-x_1)(x-x_2)$ is negative, the root being impossible cannot be constructed, hence there is no real value of y corresponding to such values of x, and therefore the curve is entirely confined between the two lines B R and B′ R′.

Similarly by taking equation (2) in (75), we shall find that a straight line Q Q′ is a diameter; that the curve cuts it in two points Q, Q′: that drawing lines parallel to A X through Q and Q′ the curve is confined between those parallels.

We have thus determined that the curve exists and only exists between certain parallel lines: its form is not yet ascertained. We might by giving a variety of values to x between x_1 and x_2 determine a variety of points P, Q, &c., and thus arrive at a tolerably exact idea of its course, but independently of this method, its form cannot much differ from that in the figure, for supposing it to be such as in fig. (2) a straight line could be drawn cutting it in more points than two which is impossible (71).

This oval curve is called the Ellipse.

If we require the points where the curve cuts A X, put $y = 0$, then the roots of the equation $cx^2 + ex + f = 0$ are the abscissas of the points of intersection, and the curve will cut the axis in two points, touch it in one, or never meet it, according as these roots are real and unequal, real and equal, or imaginary. Similarly putting $x = 0$ we find the points, if any, where the curve meets the axis of y.

Case 2. Let the roots x_1 and x_2 be real and equal,

$$\therefore y = ax + l \pm (x - x_1)\sqrt{-\mu}$$

which is imaginary except when $x = x_1$, therefore the locus is the point whose co-ordinates are x_1 and $ax_1 + l$, or $\dfrac{2ae - bd}{b^2 - 4ac}$ and $\dfrac{2cd - be}{b^2 - 4ac}$.

Case 3. Let x_1 and x_2 be impossible, then no real value can be given to x to make $(x - x_1)(x - x_2)$ negative, for the roots are of the form $\pm p + q\sqrt{-1}$ and $\pm p - q\sqrt{-1}$ $\therefore (x - x_1)(x - x_2) = x^2 \pm 2px + p^2 + q^2 = (x \pm p)^2 + q^2$ which quantity is always positive for a real value of x. Hence in this case the radical quantity being impossible there is no locus.

We have not examined the equation of x in terms of y at length, for the results of the latter are dependent on those of the former. By comparing equations (1) and (2) in (75), we see that c stands in one equation where a stands in the other, and therefore that the radical quantities are contemporaneously possible, equal, or impossible, provided that a and c have the same sign, which is the case when $b^2 - 4ac$ is negative.

In discussing a particular example reduce it to the forms

$$y = ax + l \pm \sqrt{\{-\mu(x - x_1)(x - x_2)\}}$$
$$x = a'y + l' \pm \sqrt{\{-\mu'(y - y_1)(y - y_2)\}}$$

there are then three cases.

Case 1. x_1 and x_2 real and unequal. The locus is called an ellipse, its boundaries are determined from x_1, x_2, y_1 and y_2, its diameters are drawn from $y = ax + l$ and $x = a'y + l'$, and its intersections with the axes found by putting x and y successively $= 0$ in the original equation.

Case 2. x_1 and x_2 real and equal: the locus is a point.

Case 3. x_1 and x_2 impossible: the locus is imaginary.

Ex. 1. $y^2 - 2xy + 2x^2 - 2y - 4x + 9 = 0$. Case 1. Fig. 1.

$AB = 2$, $AB' = 4$, $AC = 4 - \sqrt{2}$, $AC' = 4 + \sqrt{2}$, $AH = 1$

Ex. 2. $y^2 + xy + x^2 + y + x - 5 = 0$. Case 1.

The curve cuts the diameters when $AB = 2\frac{1}{3}$, $AB' = -3$, $AC = 2\frac{1}{3}$, $AC' = -3$, and it cuts the axes at distances $1 \cdot 7$ and $-2 \cdot 7$ nearly;

These six points are sufficient to determine its course.

Ex. 3. $y^2 + 2xy + 3x^2 - 4x = 0$. Case 1.

Ex. 4. $y^2 - 2xy + 2x^2 - 2y + 2x = 0$. Case 1.

Ex. 5. $y^2 + 2x^2 - 10x + 12 = 0$. Case 1.

Ex. 6. $y^2 - 2xy + 3x^2 - 2y - 10x + 19 = 0$. Case 2. The intersection of the diameters in Ex. 1.

Ex. 7. $y^2 - 4xy + 5x^2 + 2y - 4x + 2 = 0$. Case 3.

It is to be observed that no accurate form of the curve is here found, that will be hereafter ascertained, all that we can at present do, is to obtain an idea of the situation of the locus.

77. *Second class, $b^2 - 4ac$ positive.*

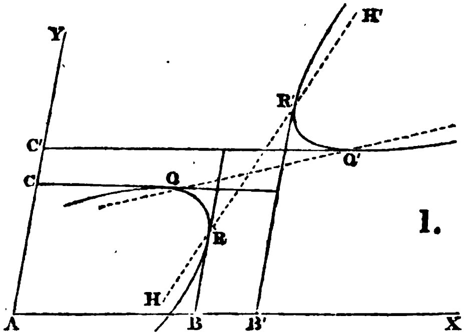
Arranging and substituting as in (76) the equation becomes

$$y = \alpha x + l \pm \sqrt{\{\mu (x - x_1)(x - x_2)\}}$$

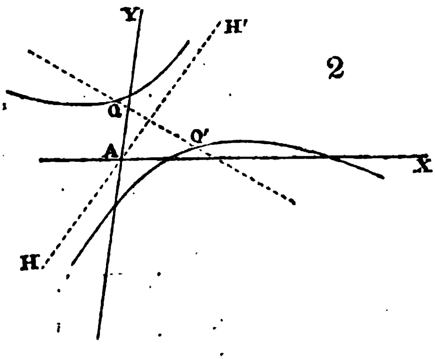
Let H H′ be the diameter whose equation is $y = \alpha x + l$.

Then as before there are three forms of the roots x_1 and x_2.

Case 1. Let x_1 and x_2 be real and unequal, let A B $= x_1$ and A B′ $= x_2$ draw B R, B′ R′ parallel to A Y, the curve meets the diameter in R and R′. The radical quantity is imaginary for all values of x between x_1 and x_2 but real beyond these limits, hence no part of the curve is between the parallels B R, B′ R′, but it extends to infinity beyond them.



1.



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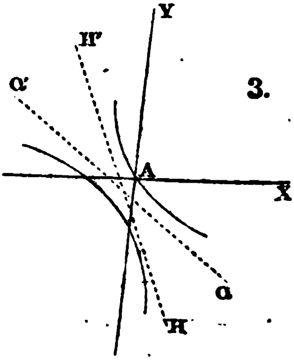


3.

Taking the equation for x in terms of y, we may draw the diameter QQ′ and determine the lines CQ, C′Q′ parallel to AX between which no part of the curve is found, and beyond which x is always possible.

From this examination it results that the form of the locus must be something like that in fig. 1, consisting of two opposite arcs with branches proceeding to infinity.

This curve is called the Hyperbola.

We must observe that the second diameter does not necessarily meet the curve, for the contemporaneous possibility or impossibility of the radical quantities depends on the signs of a and c, and these may be different in the hyperbola; so that one radical quantity may have possible and the other impossible roots.

Case 2. x_1 and x_2 real and equal.

$$y = \alpha x + l \pm (x - x_1)\sqrt{\mu}$$

this is the equation to two straight lines.

Case 3. x_1 and x_2 imaginary; whatever real values be given to x the radical quantity is real, and therefore there must be four infinite branches. Also since $\mu(x-x_1)(x-x_2)$ can never vanish, (76, Case 3.) the diameter HH′ never meets the curve, but we may draw the other diameter as in the first case.

If neither diameter meets the curve, yet they will at least determine where the curve does not pass, we must then find the intersections with the axes. If these will not give a number of points sufficient to determine the locality of the curve we must have recourse to other methods to be explained hereafter.

In discussing a particular example reduce it to the forms

$$y = \alpha x + l \pm \sqrt{\{\mu(x-x_1)(x-x_2)\}}$$
$$x = \alpha' y + l' \pm \sqrt{\{\mu(y-y_1)(y-y_2)\}}$$

there are then three cases.

Case 1. x_1 and x_2 real and unequal. The locus is an hyperbola, its boundaries determined from x_1, x_2, y_1 and y_2, the diameters are drawn from $y = \alpha x + l$ and $x = \alpha' y + l'$, and its intersections with the axes found by putting x and y separately equal nothing.

Case 2. x_1 and x_2, real and equal. The locus consists of two straight lines which intersect each other.

Case 3. x_1 and x_2 impossible. The locus is an hyperbola, draw the diameters, and find the intersection, if any, of the curve with either diameter and with the axes.

Ex. 1. $y^2 - 3xy + x^2 + 1 = 0$. Case 1. Fig. 1. The origin being at the intersection of the dotted lines.

The equations to the diameters are $y = \frac{3x}{2}$ and $y = \frac{2x}{3}$, $AB' = \frac{2}{\sqrt{5}}$, $AB = -\frac{2}{\sqrt{5}}$, $AC' = \frac{2}{\sqrt{5}}$, $AC = -\frac{2}{\sqrt{5}}$.

Ex. 2. $y^2 - 2xy - x^2 + 2 = 0$. Case 1. The two diameters pass through the origin and make an angle of 45° with the axes, the second QQ′ never meets the curve, $AB' = 1$ and $AB = -1$; the curve intersects the axis of x at distances $\pm\sqrt{2}$.

Ex. 3. $4y^2 - 4xy - 3x^2 + 8y + 4x + 16 = 0$. Case 1.

Ex. 4. $y^2 - 4xy - 5x^2 - 2y + 40x - 26 = 0$. Case 1.

Ex. 5. $y^2 - 6xy + 8x^2 + 2x - 1 = 0$. Case 2. The equations to the two straight lines are $y - 4x + 1 = 0$, and $y - 2x - 1 = 0$.

Ex. 6. $y^2 + 3xy + 2x^2 + 2y + 3x + 1 = 0$. Case 2.

Ex. 7. $y^2 - 4xy - x^2 + 10x - 10 = 0$. Case 3. Fig. 2.

Ex. 8. $y^2 + 3xy + x^2 + y + x = 0$. Case 3. Fig. 3.

Here neither diameter meets the curve; but the curve passes through the origin and cuts the axis of x at a distance -1, and that of y also at a distance -1.

Ex. 9. $y^2 - x^2 - 2y + 5x - 3 = 0$. Case 3.

The diameters are parallel to the axes, but the curve never meets that diameter whose equation is $x = \frac{5}{2}$.

Ex. 10. $y^2 - x^2 - y = 0$. Case 3.

78. *Third Class.* $b^2 - 4ac = 0$.

In this case the general equation becomes

$$y = -\frac{bx+d}{2a} \pm \frac{1}{2a}\sqrt{\{2(bd-2ae)x + d^2 - 4af\}} \quad (1)$$

$$\text{Let } -\frac{b}{2a} = \alpha,\ -\frac{d}{2a} = l,\ \frac{bd-2ae}{2a^2} = \nu$$

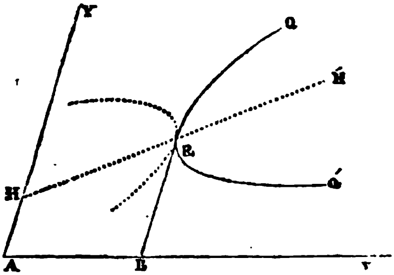
And let x_1 be the root of the equation

$$2(bd - 2ae)x + d^2 - 4af = 0.$$

Substituting equation (1) becomes

$$y = \alpha x + l \pm \sqrt{\{\nu(x - x_1)\}}$$

The locus of $y = \alpha x + l$ is a diameter H H′ as before.



Let ν be positive, then if $x = x_1$ the root vanishes; or if $AB = x_1$ and BR be drawn parallel to AY, the curve cuts the diameter in R. As x increases from x_1 to ∞, y increases to ∞, hence there are two arcs RQ, RQ′ extending to infinity. If x be less than x_1, y is impossible, or no part of the curve extends to the negative side of B.

Let ν be negative, then the results are contrary, and the curve only extends on the negative side of B; this case is represented by the dotted curve.

This curve is called the Parabola.

If $bd - 2ae = 0$, $y = \alpha x + l \pm \sqrt{\left\{\frac{d^2 - 4af}{4a^2}\right\}}$, and the locus consists of two parallel straight lines; and, according as $d^2 - 4af$ is positive, nothing, or negative, these lines are both real, or unite into one, or are both imaginary.

In discussing a particular example, reduce it to the form

$$y = \alpha x + l \pm \sqrt{\{\nu(x - x_1)\}}$$

Case 1. ν positive or negative. The locus is called a parabola; draw the diameter and find the points where the curve cuts the axes and diameter.

Case 2. $\nu = 0$. The locus consists of two parallel straight lines, or one straight line, or is imaginary.

Ex. 1. $y^2 - 2xy + x^2 - 2y - 1 = 0$. Case 1.

Ex. 2. $y^2 - 2xy + x^2 - 2y - 2x = 0$. Case 1.

Ex. 3. $y^2 + 2xy + x^2 + 2y + x + 3 = 0$. Case 1.

Ex. 4. $y^2 - 2xy + x^2 - 1 = 0$. Case 2. Two parallel straight lines.

Ex. 5. $y^2 - 2xy + x^2 + 2y - 2x + 1 = 0$. Case 2. One straight line.

Ex. 6. $y^2 + 2xy + x^2 + 1 = 0$. Case 2. Imaginary locus.

79. Before we leave this subject, it may be useful to recapitulate the results obtained from the investigation of the general equation

$$ay^2 + bxy + cx^2 + dy + ex + f = 0.$$

If $b^2 - 4ac$ be negative, the locus is an ellipse admitting of the following varieties:—

1. $c = a$, and $\frac{b}{2a}$ = cosine of the angle between the axes, locus a circle. (72.)

2. $(bd - 2ae)^2 = (b^2 - 4ac)(d^2 - 4af)$. Locus a point.

3. $(bd - 2ae)^2$ less than $(b^2 - 4ac)(d^2 - 4af)$. Locus imaginary.

If $b^2 - 4ac$ be positive, the locus is an hyperbola admitting of one variety.

1. $(bd - 2ae)^2 = (b^2 - 4ac)(d^2 - 4af)$. Locus two straight lines.

Lastly, if $b^2 - 4ac = 0$, the locus is a parabola admitting of the following varieties,—

1. $bd - 2ae = 0$. Locus two parallel straight lines.

2. $bd - 2ae = 0$, and $d^2 - 4af = 0$. Locus one straight line.

3. $bd - 2ae = 0$, and d^2 less than $4af$. Locus imaginary.

Apparently another relation between the coefficients would be obtained in each variety, by taking the equation of x in terms of y; but on examination, it will be found that in each case the last relation is involved in the former.

CHAPTER VII.

REDUCTION OF THE GENERAL EQUATION OF THE SECOND ORDER.

80. In order to investigate the properties of lines of the second order more conveniently, we proceed to reduce the general equation to a more simple form, which will be effected by the transformation of co-ordinates. Taking the formulas in (54.)

$$y = y' + n, \text{ and } x = x' + m$$

and substituting in the general equation, we have

$$a(y' + n)^2 + b(x' + m)(y' + n) + c(x' + m)^2 + d(y' + n) + e(x' + m) + f = 0;$$

or arranging

$$ay'^2 + bx'y' + cx'^2 + (2an + bm + d)y' + (2cm + bn + e)x' + an^2 + bmn + cm^2 + dn + emf = 0.$$

As we have introduced two indeterminate quantities, m and n, we are at liberty to make two hypotheses respecting the new co-efficients in the last equation; let, therefore, the co-efficients of x' and y' each $= 0$.

$$\therefore 2an + bm + d = 0, \text{ and } 2cm + bn + e = 0;$$

whence we find by elimination, $m = \dfrac{2ae - bd}{b^2 - 4ac}$, and $n = \dfrac{2cd - be}{b^2 - 4ac}$.

The value of the constant term, or f', may be obtained from the equation,

$$f' = an^2 + bmn + cm^2 + dn + em + f = 0;$$

or, since $2an + bm + d = 0$, and $2cm + bn + e = 0$,

Multiply the first of these two equations by n, and the second by m; and, adding the results, we have

$$2an^2 + 2bmn + 2cm^2 + dn + em = 0;$$

$$\therefore an^2 + bmn + cm^2 = -\frac{dn + em}{2};$$

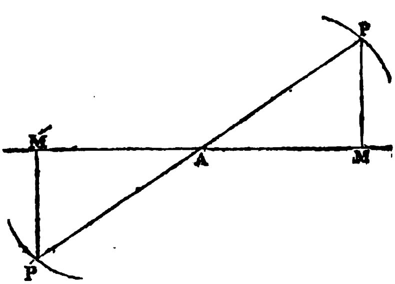
hence $f' = -\dfrac{dn + em}{2} + dn + em + f = \dfrac{dn + em}{2} + f$, which, by the substitution of the values of m and n, becomes

$$f' = \frac{ae^2 + cd^2 - bde}{b^2 - 4ac} + f.$$

The reduced equation is now of the form

$$ay'^2 + bx'y' + cx'^2 + f' = 0.$$

81. The point A, which is the new origin of co-ordinates, is called the centre of the curve, because every chord passing through it is bisected in that point. For the last equation remains the same when $-x$ and $-y$ are substituted for $+x$ and $+y$: hence, for every point P in the curve, whose co-ordinates are x and y, there is another point P′, whose co-ordinates are $-x$ and $-y$, or A M′ and M′ P′; hence, by comparing the right angled triangles, A M P, A M′ P′, we see that the vertical angles at A are equal, and therefore, the line P A P′ is a straight line bisected in A.



Whenever, therefore, the equation remains the same on the substitution of $-x$ and $-y$ for $+x$ and $+y$ respectively, it belongs to a locus referred to its centre.

If the equation be of an even order, this condition will be satisfied if the sum of the exponents of the variables in every term be even; thus, in the general equation of the second order, $a y^2 + b x y + c x^2 + d y + e x + f = 0$, the sum of the exponents in each of the three first terms is 2, and in the two next terms is 1; changing the signs of x and y, the equation does not remain the same; or for one point P, there is not another point P′ opposite and similarly situated with respect to the origin; hence that origin is not the centre of the curve. But the equation $a y^2 + b x y + c x^2 + f' = 0$, refers just as much to the point P′ as to P, and thus the origin is here the centre of the curve.

If the equation be of an odd order, the sum of the exponents in each term must be odd, and the constant term also must vanish; for if both these conditions are not fulfilled, the equation would be totally altered by putting $-x$ and $-y$ for $+x$ and $+y$ respectively. Therefore a locus may be referred to a centre if it be expressed by an equation which, by transformation, can be brought under either of the two following conditions:—

(1) Where the sum of the indices of every term is even, whether there be a constant or not, as $a y^2 + b x y + c x^2 + f = 0$.

(2) Where the sum of the indices in every term is odd, and there is no constant term, as $a y^3 + b x y^2 + c x^2 y + d x^3 + e y + f x = 0$.

Now it has been stated (59) that no equation can be so transformed that the new equation shall be of a lower or higher degree than the original one. Hence, if the original equation be of an even degree, the transformed equation will be so too, and the locus can be transferred to a centre only where the equation can be brought under the first condition; but if the original equation be of an odd degree, the transformed equation also will be of an odd degree, and the locus can only be transferred to a centre when the equation can be brought under the second condition. Hence we have a test, whether a locus with a given equation can be referred to a centre or not. If the axes can be transferred so that (1) The original equation being of an even degree, the co-efficients of all the terms, the sum of whose exponents is odd, vanish. (2) The original equation being of an odd degree, the co-efficients of all the terms, the sum of whose exponents is even, and also the constant term, vanish, then the locus may be referred to a centre, and not otherwise.

Now in the transformation which we effect by making $y = y' + n$, and $x = x' + m$, we can destroy only two terms; we cannot therefore bring, by any substitution, an equation of higher dimensions than the second under the necessary conditions, unless from some accidental relation of the original co-efficients of that equation. But in the case of equations of the second degree, we can always bring them under the first condition, unless the values of the indeterminate quantities, m and n, are found to be impossible or infinite.

In curves of the second order, we see that the values of m and n are real and finite, unless $b^2 - 4\,a\,c = 0$; consequently the ellipse and hyperbola have a centre and the parabola has not; hence arises the division of these curves into two classes, central and non-central.

In the case where $b^2 - 4\,a\,c = 0$, and at the same time $2\,a\,e - b\,d$ or $2\,c\,d - b\,e$ vanish, the equation becomes that to a straight line, as appears on inspecting the equations (1) and (2) in (75).

If by the transformation the term f' should vanish, the equation becomes of the form $a\,y^2 + b\,x\,y + c\,x^2 = 0$; whence

$y = \{-b \pm \sqrt{b^2 - 4\,a\,c}\}\dfrac{x}{2\,a}$; and the curve is reduced to two straight lines which pass through the centre; or if $b^2 - 4\,a\,c$ is negative, the locus is the centre itself (25).

82. The central class may have their general equation still further reduced by causing the term containing the product of the variables to vanish, which is done by another transformation of co-ordinates. Taking the formulas in (58) let

$$\text{Let } y' = x'' \sin.\theta + y'' \cos.\theta,$$

$$x' = x'' \cos.\theta - y'' \sin.\theta,$$

substituting in the equation $a\,y'^2 + b\,x'\,y' + c\,x'^2 + f' = 0$, we have

$$a(x''\sin.\theta + y''\cos.\theta)^2 + b(x''\sin.\theta + y''\cos.\theta)(x''\cos.\theta - y''\sin.\theta) + c(x''\cos.\theta - y''\sin.\theta)^2 + f' = 0,$$

$$\therefore y''^2\{a(\cos.\theta)^2 - b\sin.\theta\cos.\theta + c(\sin.\theta)^2\} + x''^2\{a(\sin.\theta)^2 + b\sin.\theta\cos.\theta + c(\cos.\theta)^2\}$$

$$+ x''\,y''\{2\,a\sin.\theta\cos.\theta + b(\cos.\theta)^2 - b(\sin.\theta)^2 - 2\,c\sin.\theta\cos.\theta\} + f' = 0.$$

Let the co-efficient of $x''\,y'' = 0$,

$$\therefore 2\,a\sin.\theta\cos.\theta + b(\cos.\theta)^2 - b(\sin.\theta)^2 - 2\,c\sin.\theta\cos.\theta = 0,$$

$$\text{or } (a - c)\,2\sin.\theta\cos.\theta + b\{(\cos.\theta)^2 - (\sin.\theta)^2\} = 0,$$

$$\therefore (a - c)\sin.2\,\theta + b\cos.2\,\theta = 0;$$

and dividing by cos. $2\,\theta$, we have

$$\tan.2\,\theta = \frac{-b}{a - c}.$$

Here θ is the angle which the new axis of x makes with the original one (58); hence, if the original rectangular axes be transferred through an angle θ, such that $\tan.2\,\theta = \dfrac{-b}{a - c}$, the transformed equation will have no term containing the product $x'' y''$, that is, the equation, when referred to its new rectangular axes, will be reduced to the simple form

$$a' y''^2 + c' x''^2 + f' = 0.$$

83. As a tangent is capable of expressing all values from 0 to ∞, positive or negative, it follows that the angle θ has always a real value, whatever be the values of a, b, and c, and thus it is always possible to destroy the term containing $x y$.

The values of sin. 2θ and cos. 2θ are thus obtained from that of tan. 2θ;

$$\cos. 2\theta = \frac{1}{\pm\sqrt{1 + (\tan. 2\theta)^2}} = \frac{1}{\pm\sqrt{1 + \left(\frac{b}{a - c}\right)^2}} = \frac{a - c}{\pm\sqrt{(a - c)^2 + b^2}},$$

$$\text{And } \sin. 2\theta = \cos. 2\theta . \tan. 2\theta = \frac{-b}{\pm\sqrt{(a - c)^2 + b^2}}.$$

Since θ must be less than 90°, and therefore sin. 2θ positive, the sign of the radical quantity must be taken positive or negative, according as b is itself negative or positive.

84. To express the co-efficients a' and c' of the transformed equation in terms of the co-efficients in the original equation.

Taking the expressions for the co-efficients in article (82) we have

$$a' = a (\cos. \theta)^2 - b \sin. \theta \cos. \theta + c (\sin. \theta)^2$$

$$c' = a (\sin. \theta)^2 + b \sin. \theta \cos. \theta + c (\cos. \theta)^2,$$

$$\therefore a' - c' = a\{(\cos. \theta)^2 - (\sin. \theta)^2\} - 2b \sin. \theta \cos. \theta + c\{(\sin. \theta)^2 - (\cos. \theta)^2\}$$
$$= a \cos. 2\theta - b \sin. 2\theta - c \cos. 2\theta$$
$$= (a - c) \cos. 2\theta - b \sin. 2\theta;$$

$$\text{but } \cos. 2\theta = \frac{a - c}{\pm\sqrt{(a - c)^2 + b^2}}, \text{ and } \sin. 2\theta = \frac{-b}{\pm\sqrt{(a - c)^2 + b^2}};$$

hence substituting, we have

$$a' - c' = \frac{(a - c)^2}{\pm\sqrt{(a - c)^2 + b^2}} + \frac{b^2}{\pm\sqrt{(a - c)^2 + b^2}}$$
$$= \frac{(a - c)^2 + b^2}{\pm\sqrt{(a - c)^2 + b^2}},$$

$$\text{or } a' - c' = \pm\sqrt{(a - c)^2 + b^2}$$

$$\text{Also } a' + c' = a + c,$$

$$\therefore a' = \tfrac{1}{2}\{a + c \pm \sqrt{(a - c)^2 + b^2}.\}$$

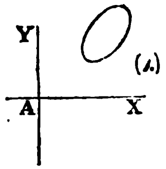
$$c' = \tfrac{1}{2}\{a + c \mp \sqrt{(a - c)^2 + b^2}.\}$$

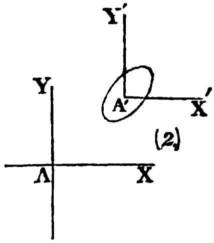
Hence the final equation is

$$\tfrac{1}{2}\{a + c \pm \sqrt{(a - c)^2 + b^2}\} y''^2 + \tfrac{1}{2}\{a + c \mp \sqrt{(a - c)^2 + b^2}\} x''^2 + \frac{ae^2 + cd^2 - bde}{b^2 - 4ac} + f = 0.$$

The upper or lower sign to be taken all through this article, according as the sign of b in the original equation is negative or positive.

85. Hitherto in this chapter we have been making a number of alterations in the *form* of the original equation: the following figures will show the corresponding alterations which have been made in the *position* of the curve. The ellipse is used in the figure, in preference to the hyperbola solely on account of its easier description.





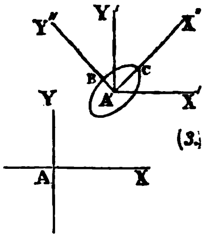


Fig. 1. We have here the original position of the curve referred to rectangular axes A X and A Y, and the corresponding equation is

$$a y^2 + b x y + c x^2 + d y + e x + f = 0.$$

Fig. 2. The origin is here transferred from A to the centre of the curve A′, the co-ordinates of which are $m = \frac{2\,a e - b d}{b^2 - 4\,a c}$, and $n = \frac{2\,c d - b e}{b^2 - 4\,a c}$. The new axes A′ X′ and A′ Y′ are parallel to the former axes, and the equation to the curve is

$$a y'^2 + b x' y' + c x'^2 + f' = 0.$$

Fig. 3. The origin remains at A′, but the curve is referred to the new rectangular axes A′ X″ and A′ Y″, instead of the former ones A′ X′ and A′ Y′. The axis A′ X′ has been transferred through an angle X′ A′ X″ into the position A′ X″, the angle X′ A′ X″, or θ, being determined by the equation $\tan.\ 2\,\theta = \frac{-\,b}{a - c}$, and the equation to the curve is now

$$a' y''^2 + c' x''^2 + f' = 0.$$

86. In the ellipse and hyperbola the word "axis" is used in a limited sense to signify that portion of the central rectangular axis which is bounded by the curve.

To find the lengths of the axes, put x'' and y'' successively $= 0$, we then obtain the points where the curve cuts the axes, or, in other words, we have the lengths of the semi-axes.

In the equation

$$a' y''^2 + c' x''^2 + f' = 0,$$

$$\text{Let } y'' = 0,\ \therefore\ c' x''^2 + f' = 0, \text{ and } x'' = \sqrt{\frac{-f'}{c'}}.$$

$$\text{Let } x'' = 0,\ \therefore\ a' y''^2 + f' = 0, \text{ and } y'' = \sqrt{\frac{-f'}{a'}}.$$

In fig. 3, the semi-axes are A′ C and A′ B, so that $\text{A}'\,\text{C} = \sqrt{\frac{-f'}{c'}}$, and $\text{A}'\,\text{B} = \sqrt{\frac{-f'}{a'}}$; putting for a', c', and f', their values in terms of the original co-efficients (80, 84) we have the squares upon the semi-axes both comprehended in the formula,

$$-\frac{2}{a+c\pm\sqrt{(a-c)^2+b^2}}\left(\frac{a\,e^2+c\,d^2-b\,d\,e}{b^2-4\,a\,c}+f\right).$$

Let the equation $a'\,y''^2+c'\,x''^2+f'=0$, be written in the form $\left(-\frac{a'}{f'}\right)y''^2+\left(-\frac{c'}{f'}\right)x''^2=1.$

Then, if the curve is an ellipse, we must have $b^2-4\,a\,c$ negative, or, since $b=0$ in the present case, we must have $-4\left(-\frac{a'}{f'}\right)\left(-\frac{c'}{f'}\right)$ negative, and therefore $-\frac{a'}{f'}$ and $-\frac{c'}{f'}$ both positive; thus both axes meet the curve, (the case where both $-\frac{a'}{f'}$ and $-\frac{c'}{f'}$ are negative, would give an imaginary locus). If the curve is an hyperbola, $b^2-4\,a\,c$ is positive, and therefore $-4\left(-\frac{a'}{f'}\right)\left(-\frac{c'}{f'}\right)$ must be positive, or one of the values, $\left(-\frac{a'}{f'}\right)$ positive, and the other $\left(-\frac{c'}{f'}\right)$ negative; hence, one of the axes in the hyperbola has an impossible value, and therefore does not meet the curve.

The relative lengths of the axes will depend entirely on the magnitude of $\frac{a'}{f'}$ and $\frac{c'}{f'}$.

87. Hitherto the original co-ordinates have been rectangular, but if they were oblique, considerable alterations must be made in some of the formulas.

Articles 80 and 81 are applicable in all cases, but 82, 83, and 84, must be entirely changed; the method pursued will be nearly the same as in the more simple case; but on account of the great length of some of the operations, we cannot do more than indicate a few steps, and give the results *.

To destroy the co-efficient of the term containing the product of the variables, take the formulas in (55)

$$y'=\frac{x''\,\sin.\,\theta+y''\,\sin.\,\theta'}{\sin.\,\omega}$$

$$x'=\frac{x''\,\sin.\,(\omega-\theta)+y''\,\sin.\,(\omega-\theta')}{\sin.\,\omega.}$$

Substituting in the equation $a\,y'^2+b\,x'\,y'+c\,x'^2+f'=0$, we have

$$y''^2\left\{a\,(\sin.\,\theta')^2+b\,\sin.\,\theta'\,\sin.\,\overline{\omega-\theta'}+c\,(\sin.\,\overline{\omega-\theta'})^2\right\}\frac{1}{(\sin.\,\omega)^2}$$

$$+\,x''^2\left\{a\,(\sin.\,\theta)^2+b\,\sin.\,\theta\,\sin.\,\overline{\omega-\theta}+c\,(\sin.\,\overline{\omega-\theta})^2\right\}\frac{1}{(\sin.\,\omega)^2}$$

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* This article, and the following ones marked with an asterisk, had better be omitted at the first reading of the subject.

$$+ x'' y'' \Big\{ 2\, a \sin.\theta.\sin.\theta' + b \sin.\theta' \sin.\overline{\omega-\theta} + b \sin.\theta \sin.\overline{\omega-\theta'} +$$

$$2\, c \sin.\overline{\omega-\theta}.\sin.\overline{\omega-\theta'} \Big\} \frac{1}{(\sin.\omega)^2} + f' = 0.$$

Let the co-efficient of $x'' y'' = 0$; expanding $\sin.\overline{\omega-\theta}$, $\sin.\overline{\omega-\theta'}$ and dividing by $\cos.\theta.\cos\theta'$ we shall obtain the equation

$$\{a - b\cos.\omega + c(\cos.\omega)^2\}\, 2\tan.\theta.\tan.\theta' + \{b - 2c\cos.\omega\}\sin.\omega.$$
$$\{\tan.\theta + \tan.\theta'\} + 2c(\sin.\omega)^2 = 0.$$

Whence for any given value of θ a value of θ' and consequently of $(\theta' - \theta)$ may be found, so that there are an infinite number of pairs of axes to which if the curve be referred, its equation may assume the form

$$a' y^2 + c' x^2 + f' = 0.$$

Let us now examine these pairs of axes, to find what systems can be rectangular:

For this purpose we must have $\theta' - \theta = \frac{\pi}{2}$ and therefore $\tan.\theta' = -\frac{1}{\tan.\theta}$.

By substituting this value of $\tan.\theta'$ in the equation containing $\tan.\theta$ and $\tan.\theta'$, we have $-2\{a - b\cos.\omega + c(\cos.\omega)^2\} + \{2c\cos.\omega - b\}$

$$\sin.\omega.\frac{2}{\tan.2\theta} + 2c(\sin.\omega)^2 = 0.$$

$$\therefore \tan.2\theta = \frac{c\sin.2\omega - b\sin.\omega}{a - b\cos.\omega + c\cos.2\omega}.$$

There are two angles which have got the same tan. 2θ separated from each other by 180°, therefore there are two angles θ, which would satisfy the above equation; however, as they are separated by an angle of 90°, the second value only applies to the new axis of y''.

Hence there is only one *system* of rectangular axes, and their position is fully determined by the last formula.

*88. To find the co-efficients a' and c' in terms of the co-efficients of the original equation, the new axes being supposed rectangular. Taking the co-efficients in the general transformed equation given above, putting $\theta' = \frac{\pi}{2} + \theta$, and multiplying by $(\sin.\omega)^2$, we have,

$$a'(\sin.\omega)^2 = a(\cos.\theta)^2 - b\cos.\theta\cos.\overline{\omega-\theta} + c(\cos.\overline{\omega-\theta})^2$$
$$c'(\sin.\omega)^2 = a(\sin.\theta)^2 + b\sin.\theta\sin.\overline{\omega-\theta} + c(\sin.\overline{\omega-\theta})^2$$

$$\therefore (a' - c')(\sin.\omega)^2 = a\{(\cos.\theta)^2 - (\sin.\theta)^2\} - b\{\cos.\theta\cos.\overline{\omega-\theta} +$$
$$\sin.\theta\sin.\overline{\omega-\theta'}\} + c\{(\cos.\overline{\omega-\theta})^2 - (\sin.\overline{\omega-\theta})^2\}$$
$$= a\cos.2\theta - b\cos.(\omega - 2\theta) + c\cos.(2\omega - 2\theta)$$
$$= \{a - b\cos.\omega + c\cos.2\omega\}\cos.2\theta + (c\sin.2\omega - b\sin.\omega)\sin.2\theta.$$

Also, following the method in (83) we find from tan. 2θ that

$$\cos.2\theta = \frac{a - b\cos.\omega + c\cos.2\omega}{\pm M}, \text{ and } \sin.2\theta = \frac{c\sin.2\omega - b\sin.\omega}{\pm M}$$

$$\text{Where } \mathrm{M} = \pm \sqrt{\{a^2 + b^2 + c^2 - 2b(a+c)\cos.\omega + 2ac\cos.2\omega\}}$$

$$\text{or } = \pm \sqrt{\{(a + c - b\cos.\omega)^2 + (b^2 - 4ac)(\sin.\omega)^2\}}.$$

$$\text{Hence } (a' - c')(\sin.\omega)^2 = \pm \mathrm{M}$$

$$\text{and } (a' + c')(\sin.\omega)^2 = a - b\cos.\omega + c$$

$$\therefore a' = \{a - b\cos.\omega + c \pm \mathrm{M}\}\frac{1}{2(\sin.\omega)^2}$$

$$\text{and } c' = \{a - b\cos.\omega + c \mp \mathrm{M}\}\frac{1}{2(\sin.\omega)^2}$$

Hence the final equation is

$$\{a - b\cos.\omega + c \pm \mathrm{M}\}\frac{y''^2}{2(\sin.\omega)^2} + \{a - b\cos.\omega + c \mp \mathrm{M}\}\frac{x''^2}{2(\sin.\omega)^2} + \frac{ae^2 + cd^2 - bde}{b^2 - 4ac} + f = 0 .$$

And the ± sign is to be used according as $c \sin. 2\omega - b \sin.\omega$ is positive or negative, since 2θ is assumed to be positive.

These analytical transformations may be geometrically represented as in (85). In figures (1), (2), and (3) we must suppose the axes A X, A Y, and also the axes A′X′ and A′ Y′, to contain the angle ω.

The article (86) will equally apply when the original axes are oblique; the value of the square on the semi-axis is,

$$\frac{-2(\sin.\omega)^2}{a - b\cos.\omega + c \pm \mathrm{M}} \cdot \left(\frac{ae^2 + cd^2 - bde}{b^2 - 4ac} + f\right)$$

89. We shall conclude the discussion of the central class by the application of the results already obtained to a few examples.

The original axes rectangular.

$$ay^2 + bxy + cx^2 + dy + ex + f = 0 \quad (1)$$

$$\left.\begin{array}{l} y = y' + n \\ x = x' + m \end{array}\right\} \text{formulas to be used.}$$

$$m = \frac{2ae - bd}{b^2 - 4ac}\ (2),\quad n = \frac{2cd - be}{b^2 - 4ac}\ (3)$$

$$f' = \frac{dn + em}{2} + f = \frac{ae^2 + cd^2 - bde}{b^2 - 4ac} + f\ (4)$$

$$ay'^2 + bx'y' + cx' + f' = 0\ (5)$$

$$\left.\begin{array}{l} y' = x''\sin.\theta + y''\cos.\theta \\ x = x''\cos.\theta - y''\sin.\theta \end{array}\right\} \text{formulas to be used.}$$

$$\tan.2\theta = \frac{-b}{a-c}\quad (6)$$

$$a' = \frac{1}{2}\left\{a + c \pm \mathrm{M}\right\} \quad (7) \qquad c' = \frac{1}{2}\left\{a + c \mp \mathrm{M}\right\} \quad (8)$$

$$\mathrm{M} = \pm \sqrt{(a-c)^2 + b^2}, \ \pm \text{ according as } b \text{ is } \mp$$

$$\left(-\frac{a'}{f'}\right)y''^2 + \left(-\frac{c'}{f'}\right)x''^2 = 1 \qquad (9);$$

(2) and (3) determine the situation of the centre, and together with (4) reduce the equation to the form (5); (6) determines the position of the rectangular axes passing through the centre, (7) and (8) enable us to reduce the equation to its most simple form (9): and the co-efficients of y''^2 and x''^2 inverted are respectively the squares upon the semi-axes measured along the axes of y'' and x''.

Ex. 1. $y^2 - xy + x^2 + y + x - 1 = 0$; locus an ellipse.

$$m = -1;\ n = -1;\ f' = -2;$$

$$y'^2 - x'y' + x'^2 - 2 = 0$$

tan. $2\theta = \frac{1}{0}$ $\therefore 2\theta = 90°$ and $\theta = 45°$; b is negative, and $\therefore$ $\mathrm{M} = +1$

$$a' = \frac{3}{2} \text{ and } c' = \frac{1}{2}$$

$$\therefore \frac{3}{2}y''^2 + \frac{1}{2}x''^2 - 2 = 0$$

$$\text{or } \frac{3}{4}y''^2 + \frac{1}{4}x''^2 = 1$$

The squares on the semi-axes are $\frac{4}{3}$ and 4; hence the semi-axes themselves are $\frac{2}{\sqrt{3}}$ and 2, and therefore the lengths of the axes are 4 and $\frac{4}{\sqrt{3}}$.

Ex. 2. $3y^2 - 4xy + 3x^2 + y - x - \frac{9}{10} = 0$; locus an ellipse.

The reduced equation is $5y''^2 + x''^2 = 1$. The axes are 2 and $\frac{2}{\sqrt{5}}$

Ex. 3. $2y^2 + xy + x^2 - 2y - 4x + 3 = 0$; locus an ellipse.

The reduced equation is $\frac{3 - \sqrt{2}}{2}y''^2 + \frac{3 + \sqrt{2}}{2}x''^2 = 1.$

Ex. 4. $5y^2 + 6xy + 5x^2 - 22y - 26x + 29 = 0$; locus an ellipse.

$$m = 2,\ n = 1,\ f' = \frac{dn + em}{2} + f = -8$$

$$\therefore 5y'^2 + 6x'y' + 5x'^2 - 8 = 0$$

tan. $2\theta = \infty$ $\therefore \theta = 45°$; hence the formulas of transformation are

$$y' = \frac{x'' + y''}{\sqrt{2}} \text{ and } x' = \frac{x'' - y''}{\sqrt{2}};$$

$$\therefore \frac{5}{2}(x'' + y'')^2 + 6\,\frac{x''^2 - y''^2}{2} + \frac{5}{2}(x'' - y'')^2 - 8 = 0$$

$$\text{or } 4\,y''^2 + x''^2 = 4$$

Ex. 5. $5y^2 + 2xy + 5x^2 - 12y - 12x = 0$; locus an ellipse.

$$2\,y''^2 + 3\,x''^2 = 6$$

Ex. 6. $2y^2 + x^2 + 4y - 2x - 6 = 0$; locus an ellipse.

Let $y = y' + n$ and $x = x' + m$, hence the transformed equation is

$$2\,(y' + n)^2 + (x' + m)^2 + 4\,(y' + n) - 2\,(x' + m) - 6 = 0$$

or $2y'^2 + x'^2 + 4\,(n+1)\,y' + 2\,(m-1)\,x' + 2n^2 + m^2 + 4n - 2m - 6 = 0.$

Let $n + 1 = 0$ and $m - 1 = 0$ $\therefore m = 1$ and $n = -1$ and $f' = -9$; hence the transformed equation is

$$2\,y'^2 + x'^2 = 9$$

and no further transformation is requisite. The axes are 6 and $3\sqrt{2}$.

Ex. 7. $y^2 - 10\,xy + x^2 + y + x + 1 = 0$; locus an hyperbola.

$$6\,y''^2 - 4\,x''^2 + \frac{9}{8} = 0.$$

Ex. 8. $4y^2 - 8\,xy - 4x^2 - 4y + 28x - 15 = 0$; locus an hyperbola.

$$y''^2 - x''^2 = \frac{-1}{2\sqrt{2}}$$

Here the axes are each $= \sqrt[4]{2}$, that which is measured along the new axis of x'' alone meeting the curve.

Ex. 9. $y^2 - 2\,xy - x^2 - 2 = 0$; locus an hyperbola.

The origin is already at the centre, and thus only one transformation is necessary.

$\tan. 2\,\theta = 1 \therefore 2\,\theta = 45°$; $M = \sqrt{8}$, $a' = \sqrt{2}$, $c' = -\sqrt{2}$, $y'^2 - x'^2 = \sqrt{2}$.

*90. The axes oblique.

The values of m, n, and f' remain as for rectangular axes.

$$\tan. 2\,\theta = \frac{c \sin. 2\,\omega - b \sin. \omega}{a - b \cos. \omega + c \cos. 2\,\omega}.$$

$$a' = \{a - b \cos. \omega + c \pm M\}\,\frac{1}{2\,(\sin \omega)^2}.$$

$$c' = \{a - b \cos. \omega + c \mp M\}\,\frac{1}{2\,(\sin. \omega)^2}.$$

$$M = \pm \sqrt{\{a^2 + b^2 + c^2 - 2\,b\,(a + c) \cos. \omega + 2\,a\,c \cos. 2\,\omega\}}.$$

$$\pm \text{ as } c \sin. 2\,\omega - b \sin \omega \text{ is } \pm.$$

Ex. 1. $y^2 + xy + x^2 + y + x - \frac{1}{6} = 0$; the angle between the axes being 45°.

$m = -\frac{1}{3}$, $n = -\frac{1}{3}$; $\tan. 2\,\theta = 1$ $\therefore 2\,\theta = 45°$; $M = +(1 - \sqrt{2})$.

$a' = 3 - \frac{3}{\sqrt{2}}$, $c' = 1 + \frac{1}{\sqrt{2}}$ and $f' = -\frac{1}{2}$.

The reduced equation is

$$3\,(2 - \sqrt{2})\,y''^2 + (2 + \sqrt{2})\,x''^2 = 1.$$

The curve is an ellipse, and the squares upon the semi-axes are

$$\frac{1}{3\,(2 - \sqrt{2})} \text{ and } \frac{1}{2 + \sqrt{2}}.$$

Ex. 2. $7\,y^2 + 16\,x\,y + 16\,x^2 + 32\,y + 64\,x + 28 = 0$. The angle $\omega = 60°$.

$$m = -2,\ n = 0,\ f' = \frac{d\,n + e\,m}{2} + f = -36.$$

The form of the equation is now

$$7\,y'^2 + 16\,x'\,y' + 16\,x'^2 - 36 = 0;$$

since tan. $2\,\theta' = 0$, the reduction to rectangular axes is effected by merely transferring the axis of y' through 30°; hence, putting $\theta = 0$, and $\omega = 60$, the formulas (56) of transformation become

$$y' = \frac{2\,y''}{\sqrt{3}}, \text{ and } x' = x'' - \frac{y''}{\sqrt{3}}.$$

Substituting these values in the last equation, it becomes

$$4\,y''^2 + 16\,x''^2 - 36 = 0;$$

Hence the axes of the ellipse are 3 and 6.

Ex. 3. $y^2 - 3\,x\,y + x^2 + 1 = 0$; the angle $\omega = 60°$.

$$m = 0,\ n = 0,\ \text{tan. } 2\,\theta = \sqrt{3},\ \therefore\ \theta = 30°;\ M = +4.$$

$a' = 5$, $c' = \dfrac{-1}{3}$, $f' = 1$, and the reduced equation is

$$5\,y''^2 - \frac{1}{3}\,x''^2 = -1.$$

The curve is an hyperbola, of which the axes are $2\sqrt{3}$ and $\dfrac{2}{\sqrt{5}}$; the first of these, which is the greatest, is measured along the new axis of x''. The second axis never meets the curve.

91. It was observed, at the end of art. 81, that the curves corresponding to the general equation of the second order were divided into two classes, one class having a centre or point such that every chord passing through it is bisected in that point, and another class having no such peculiar point. This fact was ascertained from the inspection of the values of the two indeterminate quantities m and n introduced into the equation by means of the transformation of co-ordinates, and for the purpose of destroying certain terms in the general equation. The values of m and n were found to be infinite, that is, there was no centre when the relation among the three first terms of the co-efficients of the general equation was such that $b^2 - 4\,a\,c = 0$.

This relation $b^2 - 4\,a\,c = 0$ being characteristic of the parabola, it follows that the general equation of the second order belonging to a parabola is not capable of the reduction performed in art. (80); that is, we cannot destroy the co-efficients of both x and y, or reduce the equation to the form $a\,y^2 + b\,x\,y + c\,x^2 + f = 0$, or, finally, to the form $a\,y^2 + c\,x^2 + f = 0$.

Although, however, we cannot thus reduce the parabolic equation, we are yet able to reduce it to a very simple form, in fact to a much more simple form than that of either of the above equations. This will be effected by a process similar to that already used for the general equation, only in a different order. We shall commence by transferring the axes through an angle θ, and thus destroy two terms in the equation, so that it will be reduced to the form $a\,y^2 + d\,y + e\,x + f = 0$; we shall then transfer the axes parallel to themselves, and by that means destroy two other terms, so that the final equation will be of the form

$$a\,y^2 + e\,x = 0.$$

92. Taking the formulas in (58), let

$$y = x' \sin.\,\theta + y' \cos.\,\theta$$
$$x = x' \cos.\,\theta - y' \sin.\,\theta$$

substituting these values in the general equation

$$a\,y^2 + b\,x\,y + c\,x^2 + d\,y + e\,x + f = 0,$$

and arranging, we obtain the equation

$$\begin{array}{r|r|r|r|r|l}
a\,(\cos.\,\theta)^2 & y'^2 + 2\,a \sin.\,\theta \cos.\,\theta & x'\,y' + a\,(\sin.\,\theta)^2 & x'^2 + d \cos.\,\theta & y' + d \sin.\,\theta & x' + f = 0 \\
-b \sin.\,\theta \cos.\,\theta & + b\,(\cos.\,\theta)^2 & + b \sin.\,\theta \cos.\,\theta & - e \sin.\,\theta & + e \cos.\,\theta & \\
+ c\,(\sin.\,\theta)^2 & - b\,(\sin.\,\theta)^2 & + c\,(\cos.\,\theta)^2 & & & \\
 & - 2\,c \sin.\,\theta \cos.\,\theta & & & &
\end{array}$$

Let the co-efficient of $x'\,y' = 0$

$$\therefore\ 2\,(a - c) \sin\,\theta \cos.\,\theta + b\,\{(\cos.\,\theta)^2 - (\sin.\,\theta)^2\} = 0,$$
$$\text{or } (a - c) \sin.\,2\,\theta + b \cos.\,2\,\theta = 0,$$
$$\text{and } \tan.\,2\,\theta = \frac{-b}{a - c}, \text{ as in (82.)}$$

Hence, if the axes be transfered through an angle θ such that $\tan.\,2\,\theta = \frac{-b}{a - c}$ the transformed equation will have no term containing the product of the variables; that is, it will be of the form

$$a'\,y'^2 + c'\,x'^2 + d'\,y' + e'\,x' + f = 0.$$

But, since this last equation belongs to a parabola, the relation among the co-efficients of the three first terms must be such that the general condition $b^2 - 4\,a\,c = 0$ holds good. In this case, since $b' = 0$, we must have $-\,4\,a'\,c' = 0$; hence either a' or c' must $= 0$; that is, the transformation which has enabled us to destroy the co-efficient of the term containing $x'\,y'$ will of necessity destroy the co-efficient of either x'^2 or y'^2. And this will soon be observed upon examining the values of the co-efficients of x'^2 and y'^2.

93. Let the co-efficient b in the original equation be negative, that is, let $b = -\,2\,\sqrt{a\,c}$.

From tan. 2 θ we have $\cos.\,2\,\theta = \dfrac{1}{\sqrt{1 + (\tan.\,2\,\theta)^2}} = \dfrac{1}{\sqrt{1 + \left(\dfrac{b}{a - c}\right)^2}}$

$$= \frac{a-c}{\pm\sqrt{b^2+(a-c)^2}} = \frac{a-c}{\pm(a+c)} \text{ and } \sin 2\theta = \frac{-b}{\pm(a+c)} = \frac{-b}{(a+c)},$$

since sin. 2θ must be positive, and b is itself negative;

$$\text{hence } \cos.\theta = \sqrt{\frac{1+\cos.2\theta}{2}} = \sqrt{\tfrac{1}{2}\left(1+\frac{a-c}{a+c}\right)} = \sqrt{\frac{a}{a+c}},$$

$$\text{and } \sin.\theta = \sqrt{\frac{1-\cos.2\theta}{2}} = \sqrt{\frac{c}{a+c}}.$$

Substituting these values of sin. θ and cos. θ in the general transformed equation, we have

$$a' = \frac{a\,a}{a+c} - \frac{b\sqrt{a\,c}}{a+c} + \frac{c\,c}{a+c} = \frac{a^2+2\,a\,c+c^2}{a+c} = a+c.$$

$$c' = \frac{a\,c}{a+c} + \frac{b\sqrt{a\,c}}{a+c} + \frac{c\,a}{a+c} = \frac{a\,c-2\,a\,c+a\,c}{a+c} = 0.$$

$$d' = \frac{d\sqrt{a}-e\sqrt{c}}{\sqrt{a+c}}$$

$$e' = \frac{d\sqrt{c}+e\sqrt{a}}{\sqrt{a+c}}$$

And the transformed equation is now

$$(a+c)\,y'^2 + \frac{d\sqrt{a}-e\sqrt{c}}{\sqrt{a+c}}\,y' + \frac{d\sqrt{c}+e\sqrt{a}}{\sqrt{a+c}}\,x' + f = 0.$$

And it is manifest that if b had been positive all the way through this article, the reduced equation would have been

$$(a+c)\,x'^2 + \frac{d\sqrt{c}-e\sqrt{a}}{\sqrt{a+c}}\,y' + \frac{d\sqrt{a}+e\sqrt{c}}{\sqrt{a+c}}\,x' + f = 0.$$

94. In order to reduce the equation still lower, let us transfer the axes parallel to themselves by means of the formulas $y' = y'' + n$ and $x' = x'' + m$ (54.)

then the equation $a'\,y'^2 + d'\,y' + e'\,x' + f = 0$ becomes

$$a'\,(y''+n)^2 + d'\,(y''+n) + e'\,(x''+m) + f = 0,$$

$$\text{or } a'\,y''^2 + (2\,a'\,n + d')\,y'' + e'\,x'' + a'\,n^2 + d'\,n + e'\,m + f = 0.$$

And since we have two independent quantities, m and n, we can make two hypotheses respecting them; let, therefore, their values be such that the co-efficient of y' and the constant term in the equation each $= 0$; that is, let

$$2\,a'\,n + d' = 0, \text{ and } a'\,n^2 + d'\,n + e'\,m + f = 0;$$

$$\text{whence } n = \frac{-d'}{2\,a'} \text{ and } m = \frac{d'^2-4\,a'\,f}{4\,a'\,e'},$$

and the reduced equation is now of the form

$$a'\,y''^2 + e'\,x'' = 0;$$

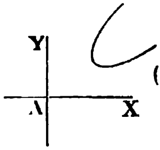
and it is manifest that if b had been positive, the equation $c' x'^2 + d' y' + e' x' + f = 0$ would have been reduced to the form

$$c' x''^2 + d' y'' = 0,$$

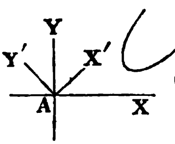
where the values of m and n would be found from the equations

$$m = -\frac{e'}{2\,c'}, \text{ and } n = \frac{e'^2 - 4\,c' f}{4\,c'\,d'}.$$

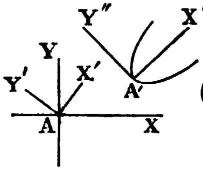
95. The following figures will exhibit the changes which have taken place in regard to the *position* of the locus corresponding to each analytical change in the *form* of the equation :—



(1.)



(2.)



(3.

In fig. 1, the curve is referred to rectangular axes A X and A Y, and the equation is

$$a\,y^2 + b\,x\,y + c\,x^2 + d\,y + e\,x + f = 0.$$

In fig. 2, the axes are transferred into the position A X', A Y', the angle X A X' or θ being determined by the equation $\tan.\ 2\,\theta = \frac{-b}{a - c}$, the corresponding equation is, for b negative,

$$a' y'^2 + d' y' + e' x' + f = 0.$$

If b is positive, the curve would originally have been situated at right angles to its present position, and the reduced equation would be

$$c' x'^2 + d' y' + e' x' + f = 0.$$

In fig. 3, the position of the origin is changed from A to A', the coordinates of A' being measured along A X' and A Y', and their values determined by the equations

$$\text{for } b \text{ negative, } n = \frac{-d'}{2\,a'} \text{ and } m = \frac{d'^2 - 4\,a' f}{4\,a'\,e'}$$

$$\text{for } b \text{ positive, } m = \frac{-e'}{2\,c'} \text{ and } n = \frac{e'^2 - 4\,c' f}{4\,c'\,d'}.$$

The reduced equation is

$$\text{for } b \text{ negative, } a' y''^2 + e' x'' = 0.$$
$$\text{for } b \text{ positive, } c' x''^2 + d' y'' = 0.$$

96[*]. If the original axes are oblique, the transformation of the general equation must be effected by means of the formulas in (55). The values of a', b', and c' will be exactly the same as in (87).

We may then let $b' = 0$, and also find $\tan.\ 2\,\theta$ when the axes are rectangular, whence, as in (87), we shall find that there is but one such system of axes.

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[*] See Note, Art. 87.

The same value of θ which destroys the term in $x' y'$ will, as in (93), also destroy the term in x'^2 or y'^2; hence the reduced equation will be

for $c \sin. 2\,\omega - b \sin. \omega$ positive, $a' y'^2 + d' y' + e' x' + f = 0$.
for $c \sin. 2\,\omega - b \sin. \omega$ negative, $c' x'^2 + d' y' + e' x' + f = 0$.

97. To find the values of a', c', d', and e'.

The values of a' and c' are best deduced from those in art. (88), Since $b^2 - 4\,a\,c = 0$, we have for $c \sin. 2\,\omega - b \sin. \omega$ positive

$$M = a - b \cos. \omega + c$$

$$a' = \{a - b \cos. \omega + c\} \frac{1}{(\sin. \omega)^2}$$

$$c' = 0$$

$$\cos. 2\,\theta = \frac{a - b \cos. \omega + c \cos. 2\,\omega}{a - b \cos. \omega + c};$$

$$\sin. \theta = \frac{\sin. \omega \sqrt{c}}{\sqrt{a - b \cos. \omega + c}}, \text{ and } \cos. \theta = \sqrt{\frac{a - b \cos. \omega + c\,(\cos. \omega)^2}{a - b \cos. \omega + c}}:$$

Also $d' = \dfrac{d \cos. \theta - e \cos. (\omega - \theta)}{\sin \omega}$ from the transformed equation

$$= \frac{(d - e \cos. \omega) \sqrt{\{a - b \cos. \omega + c\,(\cos. \omega)^2\}} - e \sqrt{c}\,(\sin. \omega)^2}{\sin. \omega \sqrt{\{a - b \cos. \omega + c\}}}$$

and $e' = \dfrac{d \sin. \theta + e \sin. (\omega - \theta)}{\sin. \omega}$

$$= \frac{(d - e \cos. \omega) \sqrt{c} + e \sqrt{\{a - b \cos. \omega + c\,(\cos. \omega)^2\}}}{\sqrt{\{a - b \cos. \omega + c\}}}$$

and the reduced equation is now of the form

$$a' y'^2 + d' y' + e' x' + f = 0;$$

For $c \sin. 2\,\omega - b \sin. \omega$ negative, the corresponding values of a', c', M, d', and e' are

$$M = -(a - b \cos. \omega + c)$$

$$a' = 0$$

$$c' = (a - b \cos. \omega + c) \frac{1}{(\sin. \omega)^2}$$

$\sin. \theta$ and $\cos. \theta$ merely change values,

hence $d' = \dfrac{(d - e \cos. \omega) \sqrt{c} - e \sqrt{\{a - b \cos. \omega + c\,(\cos. \omega)^2\}}}{\sqrt{\{a - b \cos. \omega + c\}}}$

and $e' = \dfrac{(d - e \cos. \omega) \sqrt{\{a - b \cos. \omega + c\,(\cos. \omega)^2\}} + e \sqrt{c}\,(\sin. \omega)^2}{\sin. \omega \sqrt{\{a - b \cos. \omega + c\}}}$

and the reduced equation is now of the form

$$c' x'^2 + d' y' + e' x' + f = 0.$$

The transformation required to reduce the equations still lower is performed exactly as in (94); and, by making the angle between the original axes oblique, the figures in (95) will exhibit the changes in the *position* of the curve.

98. We shall conclude the discussion of this class of curves by the application of the results already obtained to a few examples.

Ex. 1. $y^2 - 6xy + 9x^2 + 10y + 1 = 0$; locus a parabola.

$$\tan. 2\theta = \frac{-b}{a-c} = -\frac{3}{4}; \text{ hence } \theta \text{ may be found by the tables.}$$

b is negative;

$$\therefore \text{by (93) } a' = a + c = 10, c' = 0, d' = \sqrt{10} \text{ and } e' = 3\sqrt{10},$$

$$\therefore 10y'^2 + \sqrt{10}\,y' + 3\sqrt{10}\,x' + 1 = 0.$$

Also by (94) $n = \dfrac{-d'}{2a'} = \dfrac{-1}{2\sqrt{10}}$, and $m = \dfrac{-1}{4\sqrt{10}}$;

and the final equation is

$$y''^2 + \frac{3}{\sqrt{10}}x = 0.$$

Ex. 2. $y^2 + 2xy + x^2 + y - 3x + 1 = 0$; locus a parabola.

$$x''^2 + \sqrt{2}\,y = 0.$$

Ex. 3. $\sqrt{y} + \sqrt{x} = \sqrt{d}$. This equation may be put under the form $y + x - d = 2\sqrt{xy}$; or

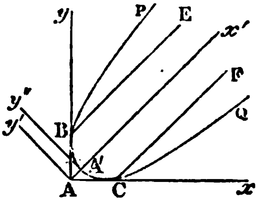
$$y^2 - 2xy + x^2 - 2dy - 2dx + d^2 = 0;$$

and the locus is a parabola because it satisfies the condition

$$b^2 - 4ac = 0.$$

By tracing the curve as in (78) we shall find its position to be that of PBCQ in the figure; and $y = x \pm d$ are the equations to the diameters BE and CF.

A x', A y', are the new axes, θ or x A x' being 45°.



$$a' = 2, d' = 0, e' = -2d\sqrt{2}, n = o, m = \frac{d}{2\sqrt{2}};$$

the last two quantities are to be measured along the new axes, therefore take A A$' = \dfrac{d}{2\sqrt{2}}$, and A$'$ is the new origin.

The final equation is

$$y^2 = dx\sqrt{2}.$$

Ex. 4. $y = d + ex + fx^2$. The locus is a parabola, since $b^2 - 4ac$ or $0 - 4.0.f = 0$.

Let $y = y' + n$, and $x = x' + m$;

$$\therefore y' + n = d + e\,(x' + m) + f\,(x' + m)^2;$$

$$\therefore f x'^2 + (2\,m\,f + e)\,x' - y' + f\,m^2 + e\,m + d - n = 0.$$

Let $2\,m\,f + e = 0$, and $f\,m^2 + e\,m + d - n = 0$

$$\therefore m = \frac{-e}{2\,f} \text{ and } n = \frac{4\,d\,f - e^2}{4\,f},$$

and the equation is reduced at once to the form

$$f\,x'^2 - y' = 0.$$

99. The axes oblique.

$y^2 - 2\,x\,y + x^2 - 6\,x = 0$; the angle between the axes being 60°.

Here, $c \sin. 2\,\omega - b \sin. \omega$ is positive.

$$\sin. \theta = \frac{\sin. 60}{\sqrt{3}} = \frac{1}{2} \quad \therefore \theta = 30^\circ$$

$$M = 3,\ a' = 4,\ c' = 0,\ d' = 6,\ e' = -2\sqrt{3},\ m = -\frac{3\sqrt{3}}{8},$$

$$n = -\frac{3}{4};$$

$$\therefore 4\,y^2 - 2\sqrt{3}\ x = 0.$$

CHAPTER VIII.

THE ELLIPSE.

100. In the discussion of the general equation of the second order, we have seen that, supposing the origin of co-ordinates in the centre, there is but one system of rectangular axes to which, if the corresponding ellipse be referred, its equation is of the simple form

$$\left(\frac{-a'}{f'}\right) y^2 + \left(\frac{-c'}{f'}\right) x^2 = 1$$

$$\text{or, } P\,y^2 + Q\,x^2 = 1$$

where the coefficients P and Q are both positive. (86, 87.)

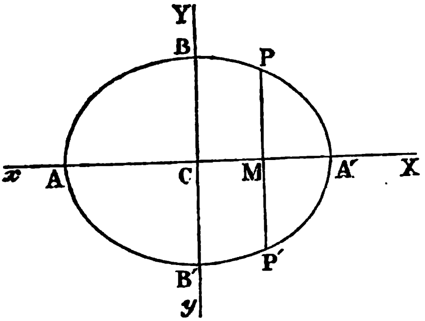
We now proceed to deduce from this equation the various properties of the ellipse.

To exhibit the coefficients in a better form; let C be the centre of the curve; X x, Y y, the rectangular axes meeting in C; C M $= x$, M P $= y$.

Then at the points where the curve cuts the axes, we have

$$y = 0, \quad Q\,x^2 = 1, \quad \therefore x = \pm \frac{1}{\sqrt{Q}}.$$

$$x = 0, \quad P\,y^2 = 1, \quad \therefore y = \pm \frac{1}{\sqrt{P}}.$$



In the axis of x take $CA' = \frac{1}{\sqrt{Q}}$ and $CA = -\frac{1}{\sqrt{Q}}$,

also in the axis of y take $CB = \frac{1}{\sqrt{P}}$ and $CB' = -\frac{1}{\sqrt{P}}$,

then the curve cuts the axes at the points A, A′, B, and B′.

Also if $CA = a$ and $CB = b$, and a be greater than b, we have $Q = \frac{1}{a^2}$ and $P = \frac{1}{b^2}$, therefore the equation to the curve becomes

$$\frac{y^2}{b^2} + \frac{x^2}{a^2} = 1$$

$$\text{or} \quad a^2 y^2 + b^2 x^2 = a^2 b^2$$

$$\text{or} \quad y^2 = \frac{b^2}{a^2}(a^2 - x^2).$$

101. We have already seen (76) that the curve is limited in every direction.

The points A, A′, B, and B′ determine those limits. From the last equation we have

$$y = \pm \frac{b}{a}\sqrt{a^2 - x^2} \quad (1), \quad \text{and } x = \pm \frac{a}{b}\sqrt{b^2 - y^2} \quad (2),$$

from (1) if x is greater than $\pm a$, y is impossible, and from (2) if y is greater than $\pm b$, x is also impossible ;. hence straight lines drawn through the points A, A′, B and B′ parallel to the axes, completely enclose the curve.

Again from (1) for every value of x less than a we have two real and equal values of y, that is, for any abscissa C M less than C A′ we have two equal ordinates M P, M P′, the ± sign determining their opposite directions.

Also as x increases from 0 to $+ a$ these values of y decrease from $\pm b$ to 0, hence we have two equal arcs B P A′, B′ P′ A′ exactly similar and opposite to one another.

If x be negative, and decrease from 0 to $- a$, x^2 is positive, and the same values of y must recur, hence there are two equal and opposite arcs B A, B′A. Therefore the whole curve is divided into two equal parts by the axis of x.

From (2) the curve appears in the same way to be divided into two equal parts by the axis of y: hence it is said to be symmetrical with respect to those axes.

Its concavity must also be turned towards the centre, otherwise it might be cut by a straight line in more points than two, which is impossible (71).

102. From the equation $y^2 = \frac{b^2}{a^2}(a^2 - x^2)$ we have

$$CP = \sqrt{x^2 + y^2} = \sqrt{x^2 + \frac{b^2}{a^2}(a^2 - x^2)} = \sqrt{b^2 + \frac{a^2 - b^2}{a^2}x^2}$$

hence C P is greatest when x is greatest, that is, when $x = a$, in which case C P becomes also equal to a, hence C A or C A′ is the greatest line that can be drawn from the centre to the curve. Again C P is least when $x = 0$, in which case C P becomes equal to b, hence C B is the least line that can be drawn from C to the curve. The axes A A′ and B B′ are thus shown to be the greatest and least lines that can be drawn through the centre. The greater A A′ is called the axis major, or greater axis, or transverse axis, and B B′ the axis minor or lesser axis.

103. The points A, B, A′ and B′ are called the vertices or summits of the curve. Any of these points may be taken for the origin, thus let A be the origin, A C the axis of x, and let the axis of y be parallel to C B, and A M $= x'$.

Then $x = CM = AM - AC = x' - a$

$$\therefore\ y^2 = \frac{b^2}{a^2}(a^2 - x^2) = \frac{b^2}{a^2}\{a^2 - (x' - a)^2\} = \frac{b^2}{a^2}(2ax' - x'^2)$$

or suppressing the accents, $y^2 = \frac{b^2}{a^2}(2ax - x^2) = \frac{b^2}{a^2}x(2a - x)$.

This last equation is geometrically expressed by the following proportion. The square upon M P : the rectangle A M, M A′ :: the square upon B C : the square upon A C.

Hence the square upon the ordinate varies as the rectangle contained by the segments of the axis major.

If the origin be at C, C A′ the axis of y and C B the axis of x, we have, putting x for y and y for x, the equation $y^2 = \frac{a^2}{b^2}(b^2 - x^2)$, and if the origin be at B, $y^2 = \frac{a^2}{b^2}(2bx - x^2)$.

104. If the axes major and minor were equal to one another, the equation to the ellipse would become $y^2 = a^2 - x^2$, which is that to a circle whose diameter is $2a$, hence we see as in (79) that the circle is a species of ellipse. As we advance we shall have frequent occasion to remark the analogy existing between these two curves.

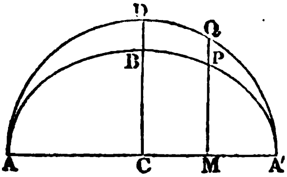
Let A D Q A′ be the circle described upon A A′ as diameter, and M Q or Y be an ordinate corresponding to the abscissa C M or x, let M P $(= y)$ be the corresponding ordinate to the ellipse, then we have

$$Y^2 = a^2 - x^2$$

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2)$$

$$\therefore y^2 = \frac{b^2}{a^2}Y^2 \text{ and } y = \frac{b}{a}Y$$

$$\therefore y : Y :: b : a$$



thus the ordinate to the ellipse has to the corresponding ordinate of the circle the constant ratio of the axis minor to the axis major.

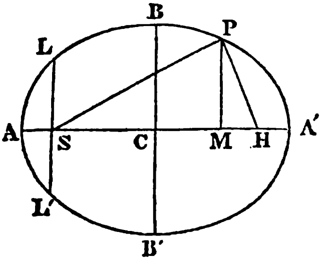
Since b is less than a the circle is wholly without the ellipse, except at A and A′ where they meet. Similarly if a circle be described on the axis minor, it is wholly within the ellipse except at B and B′. Thus the elliptic curve lies between the two circumferences.

THE FOCUS.

105. The equation $y^2 = \frac{b^2}{a^2}(2ax - x^2)$ may be put under the form $y^2 = lx - \frac{l}{2a}x^2$, in which case the quantity $l = \frac{2b^2}{a}$ is called the principal Parameter or Latus Rectum.

Since $l = \frac{2b^2}{a} = \frac{4b^2}{2a}$ the Latus Rectum is a third proportional to the axis major and minor.

106. To find from what point in the axis major a double ordinate can be drawn equal to the Latus Rectum.



Here $4y^2 = l^2$ or $\frac{4b^2}{a^2}(a^2 - x^2) = \frac{4b^4}{a^2}$

$$\therefore a^2 - x^2 = b^2$$

$$\text{or } x^2 = a^2 - b^2$$

$$\text{and } x = \pm\sqrt{a^2 - b^2}.$$

With centre B and radius a describe a circle cutting the axis major in the points S and H, then we have $CH = +\sqrt{a^2 - b^2}$ and $CS = -\sqrt{a^2 - b^2}$, thus S and H are the points through either of which if an ordinate a L S L′ be drawn, it is equal to the Latus Rectum; henceforward then we shall consider this line as the Latus Rectum or principal parameter of the ellipse.

The two points S and H thus determined are called the Foci, for a reason to be hereafter explained.

107. The fraction $\frac{\sqrt{a^2 - b^2}}{a}$ which represents the ratio of C S to C A

is called the excentricity, because the deviation of this curve from the circular form, that is, its ex-centric course, depends upon the magnitude of this ratio.

If the excentricity, which is evidently less than unity, be represented by the letter e, we have $\frac{\sqrt{a^2 - b^2}}{a} = e$ whence $e^2 = \frac{a^2 - b^2}{a^2} = 1 - \frac{b^2}{a^2} \therefore \frac{b^2}{a^2} = 1 - e^2$ and the equation to the ellipse may be put under the form

$$y^2 = (1 - e^2)(a^2 - x^2).$$

108. The line S C is sometimes called the ellipticity; its value, as above, is $a e$; but it is also expressed by the letter c. Also since $a^2 - b^2 = a^2 e^2$ we have $b^2 = a^2 - a^2 e^2 = (a - ae)(a + ae)$; hence

The rectangle A S, S A′ = The square upon B C.

109. To find the distance from the focus to any point P in the curve.

Let $SP = r$, $HP = r'$,

$$\therefore r^2 = (y - y')^2 + (x - x')^2 \ldots (29)$$

also y', x' being the co-ordinates of S, we have $y' = 0$ and $x' = -ae$,

$$\begin{aligned} \therefore r^2 &= y^2 + (x + ae)^2 \\ &= (1 - e^2)(a^2 - x^2) + (x + ae)^2 \\ &= a^2 - x^2 - e^2 a^2 + e^2 x^2 + x^2 + 2aex + a^2 e^2 \\ &= a^2 + 2aex + e^2 x^2 \\ &= (a + ex)^2 \end{aligned}$$

$$\therefore SP = a + ex; \text{ similarly } HP = a - ex.$$

In all questions referring to the absolute magnitude of S P or H P we must give to x its proper sign; thus if P is between B and A, the absolute magnitude of S P is $a - ex$, because x is itself negative.

By the addition of S P and H P, we have $SP + HP = 2a = AA'$; that is, the sum of the distances of any point on the curve from the foci is equal to the axis major.

This property is analogous to that of the circle, where the distance of any point from the centre is constant.

110. This property of the ellipse is so useful, that we shall prove the converse. To find the locus of a point P, the sum of whose distances from two fixed points S and H is constant or equal $2a$.

Let $SH = 2c$, bisect S H in C, which point assume to be the origin of rectangular axes C A′, C B; let $CM = x$, and $MP = y$,

$$\text{then } SP = \sqrt{(c + x)^2 + y^2}$$

$$HP = \sqrt{(c - x)^2 + y^2}$$

$$\text{but } SP + HP = 2a, \quad \text{or } SP = 2a - HP$$

$$\therefore \sqrt{(c + x)^2 + y^2} = 2a - \sqrt{(c - x)^2 + y^2}$$

$$\therefore (c + x)^2 + y^2 = 4a^2 - 4a\sqrt{(c - x)^2 + y^2} + (c - x)^2 + y^2;$$

hence, transposing and dividing by 4, we have

$$a\sqrt{(c - x)^2 + y^2} = a^2 - cx$$

$$\therefore a^2y^2 = a^4 - a^2c^2 + c^2x^2 \quad a^2x^2$$
$$= (a^2 - c^2)(a^2 - x^2)$$
$$\text{and } y^2 = \frac{a^2 - c^2}{a^2}(a^2 - x^2)$$

Hence the locus is an ellipse whose axes are $2a$ and $2\sqrt{a^2 - c^2}$, and whose foci are S and H.

THE TANGENT.

111. To find the equation to the tangent to the ellipse at any point.

Let x' y' be the point P

. . . x'' y'' be any other point Q;

the equation to the line P Q through these two points is

$$y - y' = \frac{y' - y''}{x' - x''}(x - x'), \quad (41).$$

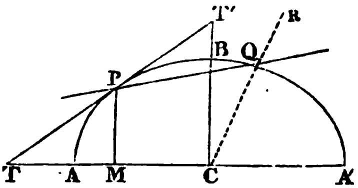
Now this cutting line or secant PQ will come to the position T P T' or just touch the curve when Q comes to P, and the equation P Q will become the equation to the tangent P T when $x'' = x'$ and $y'' = y'$.

In this case the term $\frac{y' - y''}{x' - x''}$ becomes $\frac{0}{0}$, but its value may yet be found, for since the points x' y', x'' y'' are on the curve, we have

$$a^2y'^2 + b^2x'^2 = a^2b^2$$
$$a^2y''^2 + b^2x''^2 = a^2b^2$$
$$\therefore a^2(y'^2 - y''^2) + b^2(x'^2 - x''^2) = 0;$$
$$\text{or } a^2(y' - y'')(y' + y'') + b^2(x' - x'')(x' + x'') = 0,$$
$$\therefore \frac{y' - y''}{x' - x''} = -\frac{b^2}{a^2}\frac{x' + x''}{y' + x''}$$
$$= -\frac{b^2}{a^2}\frac{x'}{y'} \text{ when } x'' = x' \text{ and } y'' = y.'$$

∴ The equation to the tangent is

$$y - y' = -\frac{b^2}{a^2}\frac{x'}{y'}(x - x')$$



By multiplication $a^2yy' - a^2y'^2 = -b^2xx' + b^2x'^2$

$$\therefore a^2yy' + b^2xx' = a^2y'^2 + b^2x'^2$$
$$= a^2b^2.$$

In the figure C M is x' and M P is y', and x and y are the co-ordinates of any point in T P T′.

The equation to the tangent is easily recollected, since it may be obtained from that to the curve $a^2y^2 + b^2x^2 = a^2b^2$ by putting yy' for y^2 and xx' for x^2.

112. That PT is a tangent is evident, since a straight line cannot cut the curve in more points than two, and here those two have gradually coalesced; it may, however, be satisfactory to show that every point in P T except P is without the curve.

Let x_1 and y_1 be the co-ordinates of any point R; then if $a^2y_1^2 + b^2x_1^2$ is greater than a^2b^2, the point R is without the curve. For, join the point R with the centre of the ellipse by a line cutting the curve in Q, and let x and y be the co-ordinates of Q, then if $a^2y_1^2 + b^2x_1^2$ is greater than a^2b^2, or than $a^2y^2 + b^2x^2$, we have $b^2(x_1^2 - x^2)$ greater than $a^2(y^2 - y_1^2)$; but b is less than a, therefore $x_1^2 - x^2$ must be greater than $y^2 - y_1^2$, or $x_1^2 + y_1^2$ greater than $x^2 + y^2$, and therefore C R greater than C Q (29), or R is without the curve.

In the present case we have the two equations.

$$a^2yy' + b^2xx' = a^2b^2$$

$$a^2y'^2 + b^2x'^2 = a^2b^2$$

$$\therefore\ a^2y'^2 - 2a^2yy' + b^2x'^2 - 2b^2xx' = -a^2b^2$$

$$\text{or } a^2(y' - y)^2 + b^2(x' - x)^2 = a^2y^2 + b^2x^2 - a^2b^2$$

$$\therefore\ a^2y^2 + b^2x^2 = a^2b^2 + a^2(y' - y)^2 + b^2(x' - x)^2$$

which is greater than a^2b^2.

But y and x are the co-ordinates of any point in the tangent; therefore generally any point on the tangent is without the curve; in the particular case where $y = y'$, and $x = x'$, that is at P, we have the equation $a^2y^2 + b^2x^2 = a^2b^2$, therefore at that point the tangent coincides with the curve.

113. If the vertex A be the origin, the equation to the curve is

$$y^2 = \frac{b^2}{a^2}(2ax - x^2) \text{ or } a^2y^2 + b^2x^2 - 2ab^2x = 0,$$

and the equation to the tangent, found exactly as above, is

$$a^2yy' + b^2xx' - ab^2(x + x') = 0;$$

If the equation to the ellipse be $y^2 = mx - nx^2$, the equation to the tangent is $yy' = \frac{m}{2}(x + x') - nxx'$.

Generally, if the equation to the curve be

$$ay^2 + bxy + cx^2 + dy + ex + f = 0,$$

the equation to the tangent is

$$y - y' = -\frac{2cx' + by' + e}{2ay' + bx' + d}(x - x'),$$

$$\text{or } (2ay' + bx' + d)y + (2cx' + by' + e)x + dy + ex + 2f = 0.$$

Again let $y = \alpha x + d$ be the equation to a tangent to the ellipse; then, comparing this with the equation $a^2 y y' + b^2 x x' = a^2 b^2$, and eliminating x' and y' by means of the equation $a^2 y'^2 + b^2 x'^2 = a^2 b^2$, we have

$$a^2 \alpha^2 + b^2 = d^2,$$

and this is the necessary relation among the co-efficients of the equation $y = \alpha x + d$ when it is a tangent to the curve.

114. To find the point where the tangent cuts the axes.

In the equation $a^2 y y' + b^2 x x' = a^2 b^2$ put $y = 0$ $\therefore b^2 x x' = a^2 b^2$, and $x = \frac{a^2}{x'} = \mathrm{CT}$; similarly $y = \mathrm{CT}' = \frac{b^2}{y'}$; hence we have

The rectangle C T, C M = The square upon A C,
and The rectangle CT′, M P = The square upon B C.

Since $\mathrm{CT}\left(= \frac{a^2}{x'}\right)$ does not involve y', it is the same for all ellipses which have the same axis major, and same abscissa for the point of contact; and, as the circle on the axis major may be considered as one of these ellipses, the distance C T is the same for an ellipse and its circumscribing circle.

Again, since $\mathrm{CT} = \frac{a^2}{x'}$ is independent of the sign of y', the tangents, at the two extremities of an ordinate, meet in the same point on the axis. The equation to the lower tangent is found by putting $-y'$ for y' in the general equation to the tangent (111).

115. The distance MT from the foot of the ordinate to the point where the tangent meets the axis of x, is called the subtangent.

In the ellipse, $\mathrm{MT} = \mathrm{C\,T} - \mathrm{C\,M} = \frac{a^2}{x'} - x' = \frac{a^2 - x'^2}{x'}$;

Hence, The rectangle C M, M T = The rectangle A M, M A′.

116. The equation to the tangent being $a^2 y y' + b^2 x x' = a^2 b^2$, let $x' = a$; and $\therefore y' = 0$, $\therefore b^2 a x = a^2 b^2$ and $x = a$; hence the tangent, at the extremity of the axis major, is perpendicular to that axis. At B, the equation to the tangent is $y = b$; hence the tangent at B is perpendicular to the axis minor.

The equation to the tangent being $a^2 y y' + b^2 x x' = a^2 b^2$, or

$$y = -\frac{b^2 x'}{a^2 y'} x + \frac{b^2}{y'}.$$

If P C be produced to meet the curve again in P′, the signs of the co-ordinates of P′ are both contrary to those of P; hence the co-efficient $-\frac{b^2 x'}{a^2 y'}$ remains the same for the tangent at P′, or the tangents at P and P′ are parallel (43).

117. To find the equation to the tangent at the extremity of the Latus Rectum.

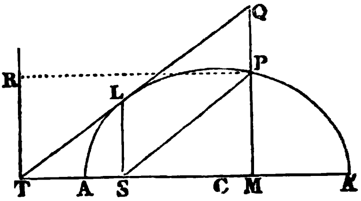
The equation to the tangent is generally

$$a^2 y y' + b^2 x x' = a^2 b^2$$

At L, $x' = -ae$ and $y' = \frac{b^2}{a}$,

$$\therefore a^2 y \frac{b^2}{a} - b^2 x\, a e = a^2 b^2$$

$$y = a + e x.$$



If the ordinate y, or M Q, cut the ellipse in P, we have S P $= a + e x$ (109)

$\therefore$ M Q = S P.

118. To find the point where this particular tangent cuts the axis, let $y = 0$; $\therefore x = \text{CT} = -\frac{a}{e}$.

From T draw T R perpendicular to A C, and from P draw P R parallel to A C; then, taking the absolute values of C M and C T, we have

$$\text{PR} = \text{MT} = \text{CT} + \text{CM} = \frac{a}{e} + x = \frac{a + e x}{e} = \frac{1}{e} . \text{SP};$$

Consequently, the distances of any point P from S, and from the line T R, are in the constant ratio of $e : 1$.

This line T R is called the directrix; for, knowing the position of this line and of the focus, an ellipse of any excentricity may be described, as will hereafter be shown.

If $x = 0$, we have $y = a$. Thus the tangent, at the extremity of the Latus Rectum, cuts the axis of y where that axis meets the circumscribing circle.

By producing Q M to meet the ellipse again in P′, it may be proved that

The rectangle Q P, Q P′ = The square on S M.

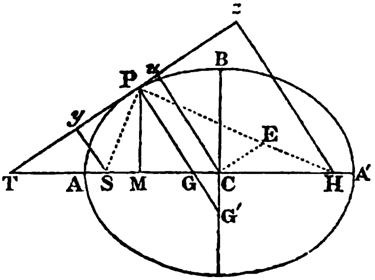
119. To find the length of the perpendicular from the focus on the tangent.

Let S y, H z, be the perpendiculars on the tangent P T.

Taking the expression in (48.) we have

$$p = -\frac{y_1 - \alpha x_1 - d}{\sqrt{1 + \alpha^2}} .$$

Where $y_1 = 0$ and $x_1 = -ae$ are co-ordinates of the point S, and $y = \alpha x + d$ is the equation to the line P T. But the equation to P T is also



$$y = -\frac{b^2 x'}{a^2 y'} x + \frac{b^2}{y'};$$

$$\therefore \alpha = -\frac{b^2 x'}{a^2 y'} \text{ and } d = \frac{b^2}{y'};$$

$$\therefore p = -\frac{-\frac{b^2 x'}{a^2 y'} a e - \frac{b^2}{y'}}{\sqrt{\left\{1 + \frac{b^4 x'^2}{a^4 y'^2}\right\}}}$$

$$= \frac{a b^2 . (a + e x')}{\sqrt{a^4 y'^2 + b^4 x'^2}}$$

And $a^4 y'^2 + b^4 x'^2 = a^2 (a^2 b^2 - b^2 x'^2) + b^4 x'^2 = a^2 b^2 \left\{a^2 - \frac{a^2 - b^2}{a^2} x'^2\right\}$

$$= a^2 b^2 (a^2 - e^2 x'^2);$$

$$\therefore p = \frac{a b^2 (a + e x')}{a b \sqrt{(a^2 - e^2 x'^2)}} = b \sqrt{\frac{a + e x'}{a - e x'}}.$$

Let S P or $a + e x' = r$, and H P or $a - e x' = 2a - r = r'$,

$$\therefore p^2 = \frac{b^2 r}{2a - r} = \frac{b^2 r}{r'}.$$

Similarly, if H $z = p'$, we have $p'^2 = b^2 \frac{r'}{r}$.

By multiplication we have $p p' = b^2$; Hence,

The rectangle S y, H z = The square upon B C.

120. To find the locus of y or z in the last article.

The equation to the curve at P is $a^2 y'^2 + b^2 x'^2 = a^2 b^2$ (1)

The equation to the tangent at P is $a^2 y y' + b^2 x x' = a^2 b^2$ (2).

The equation to the perpendicular S y (the co-ordinates of S being $-c, 0$) is $y = \alpha (x + c)$ and this line being perpendicular to the tangent (2), we have $\alpha = \frac{a^2 y'}{b^2 x'}$; and therefore the equation to S y is

$$y = \frac{a^2 y'}{b^2 x'} (x + c) \quad (3).$$

If we eliminate y' and x' from (1) (2) and (3), we shall have an equation involving x and y; but this elimination supposes x and y to be the same for both (2) and (3), and therefore can only refer to their intersection. Hence, the resulting equation is the locus of their intersection.

From (3) $\frac{y'}{x'} = \frac{b^2}{a^2} \frac{y}{x + c} = \frac{b^2}{x' y} - \frac{b^2 x}{a^2 y}$ from (2);

$$\therefore \frac{1}{x'} = \frac{y^2 + x (x + c)}{a^2 (x + c)}, \therefore x' = \frac{a^2 (x + c)}{y^2 + x (x + c)},$$

$$\text{and } y' = \frac{b^2 y x'}{a^2 (x + c)} = \frac{b^2 y}{y^2 + x (x + c)}.$$

Substituting these values of x' and y', in (1), we have

$$a^2 b^4 y^2 + b^2 a^4 (x + c)^2 = a^2 b^2 \{y^2 + x(x + c)\}^2$$

$$\therefore\ b^2 y^2 + a^2 (x + c)^2 = \{y^2 + x(x + c)\}^2;$$

$$\text{Or, } a^2 y^2 - c^2 y^2 + a^2 (x^2 + c)^2 = y^4 + 2x(x + c) y^2 + x^2 (x + c)^2;$$

$$\begin{aligned} \therefore\ a^2 \{y^2 + (x + c)^2\} &= y^4 + y^2 \{2x(x + c) + c^2\} + x^2 (x + c)^2 \\ &= y^4 + y^2 x^2 + y^2 (x + c)^2 + x^2 (x + c)^2 \\ &= y^2 (y^2 + x^2) + (y^2 + x^2)(x + c)^2 \\ &= (y^2 + x^2)\{y^2 + (x + c)^2\}; \end{aligned}$$

$$\therefore\ a^2 = y^2 + x^2.$$

This is the equation to a circle whose radius is a. Hence, the locus of y is the circle described on the axis major as diameter.

From the equation to S y, combined with that to C P $\left(y = \dfrac{y'}{x'} x\right)$, we may prove that C P and S y meet in the directrix.

121. To find the angle which the focal distance S P makes with the tangent P T.

The equation to the tangent is $y = -\dfrac{b^2 x'}{a^2 y'} x + \dfrac{b^2}{y'}$.

The equation to S P passing through S $(-c, 0)$ and P (x', y') is

$$y - y' = \frac{y' - y''}{x' - x''}(x - x') = \frac{y'}{x' + c}(x - x');$$

$$\text{And tan. SPT} = \text{tan. (PSC} - \text{PTC)} = \frac{\dfrac{y'}{x' + c} + \dfrac{b^2 x}{a^2 y'}}{1 - \dfrac{y'}{x' + c}\,\dfrac{b^2 x'}{a^2 y'}}$$

$$= \frac{a^2 y'^2 + b^2 x'^2 + b^2 c x'}{y'\{(x' + c) a^2 - b^2 x'\}} = \frac{a^2 b^2 + b^2 c x'}{y'\{(a^2 - b^2) x' + a^2 c\}}$$

$$= \frac{b^2 (a^2 + c x')}{c y' (a^2 + c x')} = \frac{b^2}{c y'}.$$

To pass from tan. S P T to tan. H P T we must put $-c$ for c in the preceding investigation; this would evidently lead us to the equation tan. H P T $= -\dfrac{b^2}{c y'}$; hence, tan. H P z = tan. (180 − H P T) = − tan. H P T $= \dfrac{b^2}{c y'}$, or the two angles S P T, H P z are equal; thus the tangent makes equal angles with the focal distances.

It is a property of light that, if a ray proceeding from H in the direction H P be reflected by the line z P y, the angle S P y of the reflected ray will equal the angle H P z. Now, in the ellipse, these angles are equal;

hence, if a light be placed at H, all rays which are reflected by the ellipse will proceed to S. Hence, these points, S and H, are called foci.

This very important property is also thus proved from article 119.

$$S y = p = b \sqrt{\frac{r}{r'}}; \text{ and } H z = p' = b \sqrt{\frac{r'}{r}};$$

$$\therefore S y : H z :: r : r' :: S P : H P;$$

hence the triangles S P y and H P z are similar, and the angle S P y equal to the angle H P z.*

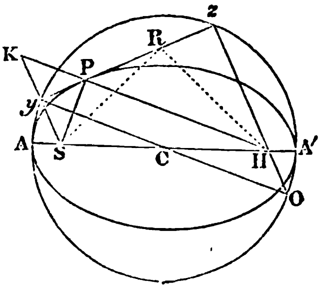
122. To find the length of the perpendicular C u, from the centre, on the tangent:

$$p = -\frac{y_1 - \alpha x_1 - d}{\sqrt{1 + \alpha^2}};$$

here $y_1 = 0$, $x_1 = 0$, $\alpha = -\frac{b^2 x'}{a^2 y'}$, and $d = \frac{b^2}{y'}$;

$$\therefore C u = \frac{\frac{b^2}{y'}}{\sqrt{\left\{1 + \frac{b^4 x'^2}{a^4 y'^2}\right\}}} = \frac{a^2 b^2}{\sqrt{\{a^4 y'^2 + b^4 x'^2\}}} = \frac{a^2 b^2}{a b \sqrt{a^2 - e^2 x'^2}} \quad (119)$$

$$= \frac{a b}{\sqrt{(a + e x')(a - e x')}} = \frac{a b}{\sqrt{r r'}}.$$



[*] The following geometrical method of drawing a tangent to the ellipse, and proving that the locus of the perpendicular from the focus on the tangent is the circumscribing circle, will be found useful.

Let A P A' be the ellipse, P any point on it. Join S P and H P, and produce H P to K, making P K = P S; bisect the angle K P S by the line y P z, and join S K, cutting P y in y.

1. P y is a tangent to the ellipse; for if R be any other point in the line P y, we have S R + R H = K R + R H, greater than K H, greater than 2 a; hence, R and every other point in z P y except P is without the ellipse.

2. The locus of y is the circumscribing circle. Draw H z parallel to S y, and join C y; then, because the triangles S P y, K P y are equal, we have the angle S y P a right angle, or S y and H z are perpendicular to the tangent. Also, since S y = K y, and S C = C H, we have C y parallel to K H, and C y = $\frac{1}{2}$ K H = $\frac{1}{2}$ (S P + P H) = C A.

3. The rectangle S y, H z = the square on B C. Let Z H meet the circle again in O and join C O; then, because the angle y z O is a right angle, and that the points y and O are in the circumference of the circle, the line y C O must be a straight line, and a diameter. Hence, the triangles C S y, C H O are equal; and the rectangle S y, H z = the rectangle Z H, H O = the rectangle A H, H A' = the square on B C (108).

4. Let S P = r, H P = 2a − S P = 2a − r, S y = p, and H z = p', then $p^2 = \frac{b^2 r}{2a - r}$;

For by similar triangles, S y : S P :: H z : H P $\therefore p = \frac{r}{2a - r} p'$; and, as above,

$$p p' = b^2 \therefore p^2 = \frac{b^2 r}{2a - r}.$$

123. To find the locus of u:

$$a^2 y'^2 + b^2 x'^2 = a^2 b^2 \quad (1).$$

$$a^2 y y' + b^2 x x' = a^2 b^2 \quad (2).$$

$$y = \frac{a^2 y'}{b^2 x'} x \quad . \; . \; . \; (3), \text{ the equation to C } u.$$

Proceeding, as in (120.), to eliminate x' and y', we arrive at the final equation $b^2 y^2 + a^2 x^2 = (y^2 + x^2)^2$; the locus is an oval meeting the ellipse at the extremities of the axes, and bulging out beyond the curve, something like the lowest of figures 2 in page 44. We shall have occasion to trace this curve hereafter.

124. To find the angle which the distance C P makes with the tangent, we have the equation

$$\text{to C P, } y = \frac{y'}{x'} x; \text{ and to P T, } y = -\frac{b^2 x'}{a^2 y'} x + \frac{b^2}{y'};$$

hence tan. C P T is found $= \dfrac{a^2 b^2}{c^2 x' y'}$.

125. From C u = C y sin. C y u, we have

$$\frac{a b}{\sqrt{r r'}} = a \text{ sin. C } y \, u \; \therefore \text{ sin. C } y \, u = \frac{b}{\sqrt{r r'}};$$

Also from H z = H P sin. H P z, we have

$$b \sqrt{\frac{r'}{r}} = r' \text{ sin. H P } z \; \therefore \text{ sin. H P } z, = \frac{b}{\sqrt{r r'}};$$

∴ angle C y u = angle H P z, and C y is parallel to H P.

Hence, if C E be drawn parallel to the tangent P T, and meeting H P in E, we have P E = C y = A C.

THE NORMAL.

126. The normal to any point of a curve is a straight line drawn through that point, and perpendicular to the tangent at that point.

To find the equation to the normal P G.

The equation to a straight line through the point P $(x' \, y')$ is

$$y - y' = \alpha (x - x')$$

This line must be perpendicular to the tangent whose equation is

$$y = -\frac{b^2 x'}{a^2 y'} x + \frac{b^2}{y'} \quad \therefore \alpha = \frac{a^2 y'}{b^2 x'};$$

and the equation to the normal is

$$y - y' = \frac{a^2 y'}{b^2 x'} (x - x').$$

127. To find the points where the normal cuts the axes:

Let $y = 0 \therefore - y' = \frac{a^2 y'}{b^2 x'}(x - x') \;\therefore\; x = x' - \frac{b^2 x'}{a^2} = \frac{a^2 - b^2}{a^2} x'$ $= e^2 x' = \mathrm{C\,G}$.

Let $x = 0 \;\therefore\; y = y' - \frac{a^2 y'}{b^2} = - \frac{a^2 - b^2}{b^2} y' = - \frac{a^2 e^2}{b^2} y' = \mathrm{C\,G'}$.

Hence $\mathrm{S\,G} = \mathrm{S\,C} - \mathrm{C\,G} = ae - e^2 x' = e\,(a - e x') = e.\ \mathrm{S\,P}$.

The distance M G, from the foot of the ordinate to the foot of the normal, is called the subnormal:

Its value is $x - x' = - \frac{b^2}{a^2} x'$.

128. From the above values of M G, C G and C G′ we

$$\text{have } \mathrm{P\,G} = \sqrt{\left\{ y'^2 + \frac{b^4}{a^4} x'^2 \right\}} = \sqrt{\left\{ \frac{b^2}{a^2}(a^2 - x'^2) + \frac{b^4 x'^2}{a^4} \right\}}$$

$$= \frac{b}{a} \sqrt{\left\{ a^2 - x'^2 + \frac{b^2 x'^2}{a^2} \right\}} = \frac{b}{a} \sqrt{\left\{ a^2 - \frac{a^2 - b^2}{a^2} x'^2 \right\}}$$

$$= \frac{b}{a} \sqrt{\left\{ a^2 - e^2 x'^2 \right\}} = \frac{b}{a} \sqrt{r r'}\,;$$

and similarly $\mathrm{P\,G'} = \frac{a}{b} \sqrt{r r'}$, consequently,

The rectangle, P G, P G′ $= r\,r'$ = the rectangle S P, H P.

The greatest value of the normal is when $x' = 0$; hence, at the extremity of the axis minor, we have the greatest value of the normal $= b$. Similarly, the least value of the normal is at the extremity of the axis major, the value being then $= \frac{b^2}{a}$, or half the Latus Rectum (105.).

Also, $\mathrm{S\,G'} = \frac{a e}{b} \sqrt{r r'}$, and $\mathrm{G\,G'} = \frac{a e^2}{b} \sqrt{r r'} \;\therefore\; \mathrm{G\,G'} = e.\mathrm{S\,G'}$.

If a perpendicular G L be drawn from G upon S P or H P, the triangles P G L, S P y, and H P z, are similar; hence

$$\mathrm{P\,L} = \mathrm{P\,G}.\frac{p}{r}, \text{ or } = \mathrm{P\,G}\,\frac{p'}{r'} = \frac{b^2}{a} = \tfrac{1}{2} \text{ the Latus Rectum.}$$

129. Since the tangent makes equal angles with the focal distances, the normal, which is perpendicular to the tangent, also makes equal angles with the focal distances. This theorem may be directly proved from the above value of C G; for $\mathrm{S\,G} : \mathrm{H\,G} :: \mathrm{S\,C} - \mathrm{C\,G} : \mathrm{H\,C} + \mathrm{C\,G}$ $:: a\,e - e^2 x' : a\,e + e^2 x' :: a - e x' : a + e x' :: \mathrm{S\,P} : \mathrm{H\,P}$; hence, the angle S P H is bisected by the line P G.—Euclid, VI. 3, or Geometry, ii. 50 *.

THE DIAMETERS.

130. A diameter was defined in (76.) to be a line bisecting a system of parallel chords. We shall now prove that all the diameters of the ellipse

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* The absolute values of S P and H P are here taken.—See 109.

are straight lines, and that they pass through the centre, which last circumstance is evident, since no line could bisect every one of a system of parallel chords without itself passing through the centre.

Let $y = \alpha x + c$ be the equation to any chord;

$$a^2 y^2 + b^2 x^2 = a^2 b^2, \text{ the equation to the curve.}$$

Transfer the origin to the bisecting point $x'\ y'$ of the chord, by putting $y + y'$ for y and $x + x'$ for x, then the equation to the chord becomes $y + y' = \alpha (x + x') + c$ or $y = \alpha x$, since $y' = \alpha x' + c$; also the equation to the curve becomes $a^2 (y + y')^2 + b^2 (x + x')^2 = a^2 b^2$.

To find where the chord intersects the curve, put αx for y in the second equation:

$$\therefore a^2 (\alpha x + y')^2 + b^2 (x + x')^2 = a^2 b^2;$$

$$\text{or, } (a^2 \alpha^2 + b^2) x^2 + 2 (a^2 \alpha y' + b^2 x') x + a^2 y'^2 + b^2 x'^2 = a^2 b^2.$$

But since the origin is at the bisection of the chord, the two values of x must be equal to one another, and have opposite signs, or the second term of the last equation must $= 0$.

$$\therefore a^2 \alpha y' + b^2 x' = 0.$$

This equation gives the relation between x' and y'; and, since it is independent of c, it will be the same for any chord parallel to $y = \alpha x + c$; hence, considering x' and y' as variable, it is the equation to the assemblage of all the middle points, or to their locus.

This equation is evidently that to a straight line passing through the centre. Conversely, any straight line passing through the centre is a diameter.

131. A pair of diameters are called conjugate when each bisects all the chords parallel to the other.

Hence, the axes major and minor are conjugate diameters, and the equation $a^2 y^2 + b^2 x^2 = a^2 b^2$, which we have generally employed, is that to the ellipse referred to its centre and rectangular conjugate diameters.

If the curve be referred to oblique co-ordinates, and its equation remains of the same form, that is, containing only x^2, y^2, and constant quantities, the new axes will also be conjugate diameters; for each value of one co-ordinate will give two equal and opposite values to the other. We shall, therefore, pass from the above equation to another referred to oblique conjugate diameters, by determining, through the transformation of co-ordinates, all the systems of axes, for which the equation to the ellipse preserves this same form.

Let the equation be $a^2 y^2 + b^2 x^2 = a^2 b^2$; the formulas for transformation are (57),

$$y = x' \sin.\theta + y' \sin.\theta',$$

$$x = x' \cos.\theta + y' \cos.\theta',$$

$$\therefore a^2 (x' \sin.\theta + y' \sin.\theta')^2 + b^2 (x' \cos.\theta + y' \cos.\theta')^2 = a^2 b^2,$$

$$\text{or } \{a^2 (\sin.\theta')^2 + b^2 (\cos.\theta')^2\} y'^2 + \{a^2 (\sin.\theta)^2 + b^2 (\cos.\theta)^2\} x'^2$$
$$+ 2 \{a^2 \sin.\theta \sin.\theta' + b^2 \cos.\theta \cos.\theta'\} x' y' = a^2 b^2.$$

In order that this equation may be of the conjugate form, it must not contain the term $x' y'$; but since we have introduced two indeterminate quantities, θ and θ', we are enabled to put the co-efficient of $x' y' = 0$; hence we have the condition

$$a^2 \sin.\theta \sin.\theta' + b^2 \cos.\theta \cos.\theta' = 0;$$

or dividing by $a^2 \cos.\ \theta \cos.\ \theta'$,

$$\tan.\ \theta.\ \tan.\ \theta' = -\frac{b^2}{a^2}.$$

Now this condition will not determine both the angles θ and θ', but for any value of the one angle it gives a real value for the other; and hence there is an infinite number of pairs of axes to which, if the curve be referred, its equation is of the required conjugate form.

If, in the next figure, we draw C P making any angle θ with C A$'$, and C D making an angle θ' (whose tangent is $-\frac{b^2}{a^2}\cot.\ \theta$) with C A$'$, then C P and C D are conjugate diameters. Also since the product of the tangents is negative, if C P be drawn in the angle A$'$ C B, C D must be drawn in the angle B C A.

132. There is no occasion to examine the above equation of condition in the case where θ or $\theta' = 0$, for then we have the original axes; but let us examine whether there are any other systems of rectangular axes.

Let $\theta' = 90° + \theta$, $\therefore \sin.\ \theta' = \cos.\ \theta$, and $\cos.\ \theta' = -\sin.\ \theta$,

hence the equation of condition becomes

$$(a^2 - b^2) \sin.\ \theta \cos.\ \theta = 0,$$

and since, by the nature of the ellipse, a^2 cannot $= b^2$, we must have $\theta = 0$, or $\theta = 90°$, both which values give the original axes again; hence the only system of rectangular diameters is that of the axes. This remark agrees with article 87.

We may observe in the above transformation that, although we have introduced *two* indeterminate quantities θ and θ', it does not follow that we can destroy *two* terms in the transformed equation, unless the values of these quantities are real: for example, if we attempt to destroy any other term as the second, we find $\tan.\ \theta = \frac{b}{a}\sqrt{-1}$, a value to which there is no corresponding angle θ; hence, in putting the co-efficient of $x'y' = 0$, we adopted the only possible hypothesis.

133. The equation to the curve is now

$$\{a^2 (\sin.\ \theta')^2 + b^2 (\cos.\ \theta')^2\} y'^2 + \{a^2 (\sin.\ \theta)^2 + b^2 (\cos.\ \theta)^2\} x'^2 = a^2 b^2.$$

If we successively make $y' = 0$, and $x' = 0$, we have the distances from the origin to the points in which the curve cuts the new axes; let these distances be represented by a_1 and b_1, the former being measured along the axis of x', and the latter along the axis of y'; then we have

$$y' = 0,\ \therefore \{a^2 (\sin.\ \theta)^2 + b^2 (\cos.\ \theta)^2\} a_1^2 = a^2 b^2,$$
$$x' = 0,\ \therefore \{a^2 (\sin.\ \theta')^2 + b^2 (\cos.\ \theta')^2\} b_1^2 = a^2 b^2,$$

And the transformed equation becomes

$$\frac{a^2 b^2}{b_1^2} y'^2 + \frac{a^2 b^2}{a_1^2} x'^2 = a^2 b^2,$$

$$\text{or, } \frac{y'^2}{b_1^2} + \frac{x'^2}{a_1^2} = 1,$$

$$\text{or, } a_1^2 y'^2 + b_1^2 x'^2 = a_1^2 b_1^2,$$

Where the lengths of the new conjugate diameters are $2\,a_1$ and $2\,b_1$.

134. From the transformation we obtain the three following equations:

$$a_1^2\{a^2(\sin.\theta)^2+b^2(\cos.\theta)^2\}=a^2b^2\quad(1),$$

$$b_1^2\{a^2(\sin.\theta')^2+b^2(\cos.\theta')^2\}=a^2b^2\quad(2),$$

$$\left.\begin{array}{r}a^2\sin.\theta\sin.\theta'+b^2\cos.\theta\cos.\theta'=0,\\ \text{or,}\quad\tan.\theta\tan.\theta'=-\dfrac{b^2}{a^2},\end{array}\right\}\quad(3).$$

Putting $1-(\sin.\theta)^2$ for $(\cos.\theta)^2$ in (1), we have

$$a_1^2(a^2-b^2)(\sin.\theta)^2=a^2b^2-a_1^2b^2,$$

$$\text{and }a_1^2(a^2-b^2)(\cos.\theta)^2=a_1^2a^2-a^2b^2,$$

$$\therefore(\tan.\theta)^2=\frac{b^2}{a^2}\,\frac{a^2-a_1^2}{a_1^2-b^2}.$$

Putting b_1 for a_1 in this expression, we have the value of $(\tan.\theta')^2$, as found from (2)

$$(\tan.\theta')^2=\frac{b^2}{a^2}\,\frac{a^2-b_1^2}{b_1^2-b^2};$$

hence by multiplication,

$$(\tan.\theta)^2(\tan.\theta')^2=\frac{b^4}{a^4}\,\frac{a^2-a_1^2}{a_1^2-b^2}\,\frac{a^2-b_1^2}{b_1^2-b^2}=\frac{b^4}{a^4}\text{ from (3)},$$

$$\therefore(a^2-a_1^2)(a^2-b_1^2)=(a_1^2-b^2)(b_1^2-b^2),$$

$$\text{or, }a^4-a^2b_1^2-a_1^2a^2+a_1^2b_1^2=a_1^2b_1^2-a_1^2b^2-b^2b_1^2+b^4;$$

$$\begin{aligned}\therefore\quad a^4-b^4&=a^2b_1^2+a_1^2a^2-a_1^2b^2-b^2b_1^2,\\&=a^2(a_1^2+b_1^2)-b^2(a_1^2+b_1^2),\\&=(a^2-b^2)(a_1^2+b_1^2),\end{aligned}$$

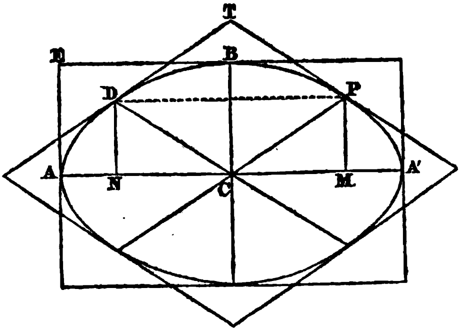
$$\therefore\quad a^2+b^2=a_1^2+b_1^2,$$

that is, the sum of the squares upon the conjugate diameters is equal to the sum of the squares upon the axes.

135. Again, multiplying (1) and (2) together, and (3) by itself, and then subtracting the results, we have

$$a_1^2b_1^2\{a^4(\sin.\theta)^2(\sin.\theta')^2+b^4(\cos.\theta)^2(\cos.\theta')^2+a^2b^2(\sin.\theta')^2(\cos.\theta)^2$$
$$+a^2b^2(\sin.\theta)^2(\cos.\theta')^2\}=a^4b^4,$$

$$a^4(\sin.\theta)^2(\sin.\theta')^2+b^4(\cos.\theta)^2(\cos.\theta')^2+2\,a^2b^2\sin.\theta\sin.\theta'\cos.\theta$$
$$\cos.\theta'=0;$$



$$\therefore\ a_1^2 b_1^2 a^2 b^2 \{ (\sin.\ \theta')^2 (\cos.\ \theta)^2 - 2 \sin.\ \theta \sin.\ \theta' \cos.\ \theta \cos.\ \theta' + (\sin.\ \theta)^2 (\cos.\ \theta')^2 \} = a^4 b^4,$$

$$\text{or, } a_1^2 b_1^2 \{ \sin.\ \theta' \cos.\ \theta - \sin.\ \theta \cos.\ \theta' \}^2 = a^2 b^2,$$

$$\text{or, } a_1^2 b_1^2 \{ \sin.\ (\theta' - \theta) \}^2 = a^2 b^2;$$

$$\therefore\ a_1 b_1 \sin.\ (\theta' - \theta) = a\,b.$$

Now $\theta' - \theta$ is the angle P C D, between the conjugate diameters C P and C D; hence drawing straight lines at the extremities of the conjugate diameters, parallel to those diameters, we have, from the above equation, the parallelogram P C D T = the rectangle A C B E, and therefore the whole parallelogram thus circumscribing the ellipse is equal to the rectangle contained by the axes *.

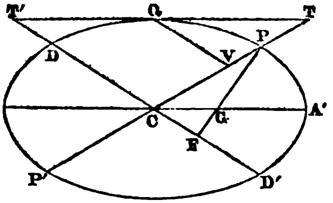
If the extremities of the conjugate diameters be joined, it is readily seen that the inscribed figure is a parallelogram, and that its area is equal to half that of the above circumscribed parallelogram.

We may remark, in passing, that the circumscribed parallelogram, having its sides parallel to a pair of conjugate diameters, is the least of all parallelograms circumscribing the ellipse; and that the inscribed parallelogram, having conjugate lines for its diameters, is the greatest of all inscribed parallelograms.

136. Returning to article (133.), the equation to the curve, suppressing the accents on x' and y', as no longer necessary, is

$$a_1^2 y^2 + b_1^2 x^2 = a_1^2 b_1^2.$$

In the figure, C P = a_1, C D = b_1 C V = x, and V Q = y.



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* The theorems in articles 134 and 135 may be proved also in the following manner: Referring the curve to its rectangular axes, as in article (138.), let the co-ordinates of P be x' and y'; then the equation to C D is $a^2 y y' + b^2 x x' = 0$, and eliminating x and y between this equation and that to the curve ($a^2 y^2 + b^2 x^2 = a^2 b^2$), we have the co-ordinates C N and D N, fig. 135, of the intersection of C D with the curve, C N $= x = \frac{a y'}{b}$ and D N $= y = \frac{b x'}{a}$; hence we have

$$a_1^2 + b_1^2 = x'^2 + y'^2 + x^2 + y^2 = x'^2 + y'^2 + \frac{a^2 y'^2}{b^2} + \frac{b^2 x'^2}{a^2} = \frac{b^2 x'^2 + a^2 y'^2}{b^2} + \frac{a^2 y'^2 + b^2 x'^2}{a^2} = \frac{a^2 b^2}{b^2} + \frac{a^2 b^2}{a^2} = a^2 + b^2.$$

Also the triangle P C D = the trapezium P M N D − the triangles P C M and D C N $= (x + x')\frac{y + y'}{2} - \frac{x y + x' y'}{2} = \frac{x' y + y' x}{2} = \frac{1}{2}\left\{ x' \frac{b x'}{a} + y' \frac{a y'}{b} \right\} = \frac{b^2 x'^2 + a^2 y'^2}{2 a b}$ $= \frac{a^2 b^2}{2 a b} = \frac{a b}{2}$, therefore the parallelogram P C D T = $a\,b$.

No notice has been taken of the positional value of the abscissa C N, since this is entirely a question of absolute values.

Putting the equation into the form

$$y^2 = \frac{b_1^2}{a_1^2}(a_1^2 - x^2) = \frac{b_1^2}{a_1^2}(a_1 - x)(a_1 + x),$$

we have the square upon the ordinate Q V : the rectangle P V, V P′ :: the square upon C D : the square upon C P.

137. The equation to the tangent at any point Q (x' y') found exactly as in (111.) is $a_1^2\, y\, y' + b_1^2\, x\, x' = a_1^2\, b_1^2$.

The points T and T′, where it cuts the new axes, are determined as in (114.); whence $CT = \frac{a_1^2}{x'}$, $CT' = \frac{b_1^2}{y'}$; and the tangents drawn at the two extremities of a chord meet in the diameter to that chord (114.).

138. Let the ellipse be now referred to its rectangular axes, and let the co-ordinates of P be x' y', then the equation to C P is $y = \frac{y'}{x'}x$, and the equation to C D is

$$y = x \tan.\theta' = -\frac{b^2}{a^2}\cot.\theta = -\frac{b^2\, x'}{a^2\, y'}\, x,$$

$$\text{or,} \quad a^2\, y\, y' + b^2\, x\, x' = 0.$$

But the equation to the tangent at P is

$$a^2\, y\, y' + b^2\, x\, x' = a^2\, b^2;$$

hence CD or the diameter conjugate to C P is parallel to the tangent at P.

From this circumstance the conjugate to any diameter is often defined to be the line drawn through the centre, and parallel to the tangent at the extremity of the diameter.

The equation to the conjugate diameter is readily remembered, since it is the same as that to the tangent without the last term, and therefore may be deduced from the equation to the curve, as at the end of article 111. The three equations are

$$a^2 y^2 + b^2 x^2 = a^2 b^2, \text{ to the curve,}$$

$$a^2\, y\, y' + b^2\, x\, x' = a^2\, b^2, \text{ to the tangent,}$$

$$a^2\, y\, y' + b^2\, x\, x' = 0, \text{ to the conjugate.}$$

The equation to the tangent D T passing through the point D, whose co-ordinates are $\frac{b\,x'}{a}$ and $-\frac{a\,y'}{b}$ (*note* 135), and parallel to C P, is

$$y - \frac{b\,x'}{a} = \frac{y'}{x'}\left(x + \frac{a\,y'}{b}\right)$$

or reducing

$$y\,x' - x\,y' = a\,b.$$

And the equation to C P is

$$y\,x' - x\,y' = 0.$$

These equations to the tangents and conjugate diameters, combined with the equation to the curve, will be found useful in the solution of problems relating to tangents.

139. Let x' and y' be rectangular co-ordinates of P; then, from the equation $a_1^2 + b_1^2 = a^2 + b^2$, we have $b_1^2 = a^2 + b^2 - a_1^2 = a^2 + b^2 - x'^2 - y'^2 = a^2 + b^2 - x'^2 - b^2 + \frac{b^2}{a^2} x'^2 = a^2 - \frac{a^2 - b^2}{a^2} x'^2 = a^2 - e^2 x'^2$ $= (a - ex')(a + ex') = r r'$.
That is, the square upon the conjugate diameter C D = the rectangle under the focal distances S P and H P.

140. Draw P F perpendicular upon the conjugate diameter C D, then by (135.) the rectangle P F, C D = ab,

$$\therefore \text{PF} = \frac{ab}{b_1} = \frac{ab}{\sqrt{a^2 + b^2 - a_1^2}} = \frac{ab}{\sqrt{rr'}}.$$ [*]

It was shown in (128.) that $\text{PG} = \frac{b}{a}\sqrt{rr'}$, and $\text{PG}' = \frac{a}{b}\sqrt{rr'}$;

hence, The rectangle P G, P F = The square on B C,
and The rectangle P G′, P F = The square on A C,
and The rectangle P G, P G′ = The square on C D.

SUPPLEMENTAL CHORDS.

141. Two straight lines drawn from a point on the curve to the extremities of a diameter are called supplemental chords. They are called principal supplemental chords if that diameter be the axis major.

Referring the ellipse to its axes, let P P′ be a diameter, Q P, Q P′ two supplemental chords; then, if x' y' be the co-ordinates of P, $-x'$, $-y'$ are those of P′; hence, the equation to Q P is $y - y' = \alpha(x - x')$,

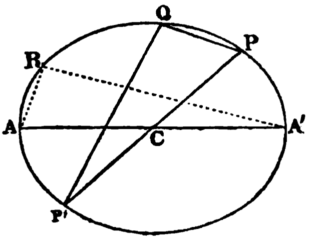
and the equation to Q P′ is $y + y' = \alpha'(x + x')$.

At the point of intersection, y and x are the same for both equations, being the co-ordinates of Q; hence, $y^2 - y'^2 = \alpha\alpha'(x^2 - x'^2)$;

but $a^2 y^2 + b^2 x^2 = a^2 b^2$ at Q,
and $a^2 y'^2 + b^2 x'^2 = a^2 b^2$ at P;

$$\therefore y^2 - y'^2 = -\frac{b^2}{a^2}(x^2 - x'^2);$$

$$\therefore \alpha\alpha' = -\frac{b^2}{a^2};$$



that is, The product of the tangents of the angles, which a pair of supplemental chords makes with the axis major, is constant.

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[*] If the distance C P = u, and p = the perpendicular from the centre on the tangent at P, this equation is

$$p^2 = \frac{a^2 b^2}{a^2 + \quad - u^2}.$$

If the curve was referred to any conjugate diameters, $2a_1$ and $2b_1$, we should find exactly in the same manner that the product of the tangents of the angles, which a pair of supplemental chords makes with any axis $2a_1$, is constant, and equal to $-\frac{b_1^2}{a_1^2}$.

The equation to a chord Q P being $y - y' = \alpha(x - x')$, the equation to its supplemental chord Q P′ is $y + y' = -\frac{b^2}{a^2\alpha}(x + x')$.

In the circle $b = a$ ∴ $\alpha\alpha' = -1$, which proves that in the circle the supplemental chords are at right angles to each other, a well-known property of that figure.

The converse of the proposition is thus proved.

Let A C A′ be any diameter, C the origin, and $\alpha\alpha' = -\frac{b_1^2}{a_1^2}$, then the equation to A R is $y = \alpha(x + a_1)$ (1), and the equation to A′R is $y = \alpha'(x - a_1) = -\frac{b^2}{a_1^2\alpha}(x - a_1)$ (2). To find the intersection of the lines A R and A′R, let y and x be the same for (1) and (2), and eliminate α by multiplication; hence,

$y^2 = -\frac{b_1^2}{a_1^2}(x^2 - a_1^2)$; or $a_1^2y^2 + b_1^2x^2 = a_1^2b_1^2$, and the locus of R is an ellipse whose axes are $2a_1$ and $2b_1$.

142. The equation $\alpha\alpha' = -\frac{b^2}{a^2}$ is remarkable, as showing that $\alpha\alpha'$ is the same, not only for different pairs of chords drawn to the extremities of the same diameter, but also for pairs of chords drawn to the extremities of any diameter; hence, if from the extremity of the axis major we can draw one chord A R parallel to Q P′, the supplemental chord R A′ will be parallel to Q P: this is possible in all cases, except when one chord is parallel or perpendicular to the axis.

143. To find the angle between two supplemental chords.

Let x, y be the co-ordinates of Q, and x' y' those of P,

Then tan. P Q P′ $= \frac{\alpha - \alpha'}{1 + \alpha\alpha'} = \frac{\frac{y - y'}{x - x'} - \frac{y + y'}{x + x'}}{1 - \frac{b^2}{a^2}} = \frac{2a^2}{a^2 - b^2}\,\frac{x'y - y'x}{x^2 - x'^2}$;

or, $= -\frac{2b^2}{a^2 - b^2}\,\frac{x'y - y'x}{y^2 - y'^2}$.

For the principal supplemental chords, we have $x' = a$, $y' = 0$;

$$\therefore \text{tan. A R A}' = -\frac{2b^2}{a^2 - b^2}\,\frac{a}{y}.$$

This value of the tangent being negative, the angle A R A′ is always obtuse, which is also evident, since all the points on the ellipse are within the circumscribing circle.

As y increases, the numerical value of the tangent decreases, or the angle increases (since the greater the obtuse angle, the less is its tangent); hence, the angle is a maximum when y is, that is, when $y = b$. This shows that the angle A B A′ is the greatest angle contained by the principal supplemental chords, and therefore by any supplemental chords. Also, its supplement B A B′ is the least angle contained by any supplemental chords. The angle between the chords being thus limited by the angles A′B A, B A B′, of which the former is greater, and the latter less, than a right angle, chords may be drawn containing any angle between these limits. This is done by describing a segment of a circle, containing the given angle, upon any diameter, except the axis, and joining the extremities of the diameter with the points of intersection of the ellipse and circle. Also, from the value of tan. P Q P′, it appears that, if the angle be a right angle, the two chords are perpendicular to the axes.

144. It was shown in (131.) that if θ and θ' were the angles which conjugate diameters make with the axis major, $\tan.\theta.\tan.\theta' = -\frac{b^2}{a^2}$, but α, α' being tangents of the angles which two supplemental chords make with the same axis, we have $\alpha\alpha' = -\frac{b^2}{a^2}$; $\therefore \tan.\theta.\tan.\theta' = \alpha\alpha'$; hence, if $\tan.\theta = \alpha$, we have $\tan.\theta' = \alpha'$; or if one diameter be parallel to any chord, the conjugate diameter is parallel to the supplemental chord.

145. Since supplemental chords can be drawn containing any angle within certain limits, conjugate diameters parallel to these chords may be drawn containing any given angle within the same limits.

Also, since the angle between the principal supplemental chords is always obtuse, the angle P C D between the conjugate diameters is also obtuse, and is the greatest when they are parallel to A B and A B′. In this case, being symmetrically situated with respect to the axes, they are equal to one another.

The magnitude of the equal conjugate diameters is found from the equation $a_1^2 + b_1^2 = a^2 + b^2$, $\therefore a_1^2 = \frac{a^2 + b^2}{2}$.

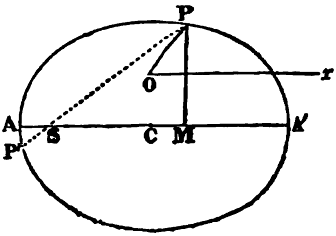
The equation to the ellipse referred to its equal conjugate diameters is $y^2 + x^2 = a_1^2$; however, this must not be confounded with the equation to the circle, which only assumes this form when referred to *rectangular* axes.

THE POLAR EQUATION.

146. Instead of an equation between rectangular co-ordinates x and y, we may obtain one between polar co-ordinates u and θ.

Let the curve be referred to the centre C, and to rectangular axes, and let the co-ordinates of the pole O be x' and y', θ the angle which the radius vector O P, or u, makes with a line O x parallel to the axis of x; then, by (61.), or by inspection of the figure, we have

$$y = y' + u \sin.\theta$$
$$x = x' + u \cos.\theta;$$



also $a^2 y^2 + b^2 x^2 = a^2 b^2$;

$\therefore$ by substitution, $a^2 (y' + u \sin.\theta)^2 + b^2 (x' + u \cos.\theta)^2 = a^2 b^2$;

Whence u may be found in terms of θ and constant quantities.

147. Let the centre be the pole:

$$\therefore x' = 0 \text{ and } y' = 0,$$

$$\therefore a^2 u^2 (\sin.\theta)^2 + b^2 u^2 (\cos.\theta)^2 = a^2 b^2;$$

$$u^2 = \frac{a^2 b^2}{a^2 (\sin.\theta)^2 + b^2 (\cos.\theta)^2} = \frac{a^2 b^2}{a^2 (\sin.\theta)^2 + (a^2 - a^2 e^2)(\cos.\theta)^2}$$

$$= \frac{a^2 b^2}{a^2 - a^2 e^2 (\cos.\theta)^2} = \frac{a^2 (1 - e^2)}{1 - e^2 (\cos.\theta)^2}.$$

148. Let the focus S be the pole:

$$\therefore y' = 0,\ x' = -ae = -c, \text{ and } u \text{ becomes } r;$$

hence the transformed equation (146.) becomes

$$a^2 (r \sin.\theta)^2 + b^2 (-c + r \cos.\theta)^2 = a^2 b^2;$$

$$\therefore a^2 r^2 (\sin.\theta)^2 + b^2 r^2 (\cos.\theta)^2 - 2 b^2 r c \cos.\theta + b^2 c^2 = a^2 b^2;$$

or, $a^2 r^2 (\sin.\theta)^2 + a^2 r^2 (\cos.\theta)^2 - c^2 r^2 (\cos.\theta)^2 - 2 b^2 r c \cos.\theta = a^2 b^2 - b^2 c^2 = b^4$ since $a^2 - b^2 = c^2$.

$$\text{or, } a^2 r^2 = c^2 r^2 (\cos.\theta)^2 + 2 b^2 r c \cos.\theta + b^4$$

$$= (c r \cos.\theta + b^2)^2;$$

$$\therefore a r = c r \cos.\theta + b^2$$

$$r = \frac{b^2}{a - c \cos.\theta} = \frac{a^2 (1 - e^2)}{a - a e \cos.\theta} = \frac{a (1 - e^2)}{1 - e \cos.\theta}.$$

149. Let any point on the curve be the pole:

Expanding the terms of the polar equation in (146.), and reducing by means of the equation $a^2 y'^2 + b^2 x'^2 = a^2 b^2$, we have

$$u = -2\,\frac{a^2 y' \sin.\theta + b^2 x' \cos.\theta}{a^2 (\text{siu}.\theta)^2 + b^2 (\cos.\theta)^2}.$$

If the pole is at A, we have $y' = 0$, and $x' = -a$,

$$\therefore u = \frac{2 b^2 a \cos.\theta}{a^2 (\sin.\theta)^2 + b^2 (\cos.\theta)^2} = \frac{2 a (1 - e^2) \cos.\theta}{1 - e^2 (\cos.\theta)^2}.$$

150. When the focus is the pole, the equation is often obtained directly from some known property of the curve.

Let $S P = r$, $C M = x$, and $A S P = \theta$,

$$\text{then } S P = a + e x \text{ (109.)}$$

$$= a + e (S M - S C)$$

$$= a + e (-r \cos.\theta - a e)$$

$$\therefore r + e r \cos.\theta = a - a e^2 \text{ and } r = \frac{a (1 - e^2)}{1 + e \cos.\theta}.$$

This is the equation generally used in astronomy, the focus S being the place of the sun, and the ellipse the approximate path of the planet.

Let $a\,(1 - e^2) = \frac{b^2}{a} = p$, where p is the parameter. (105.)

Then the last equation may be written under the following forms:

$$r = \frac{p}{2} \cdot \frac{1}{1 + e \cos.\theta} = \frac{p}{2} \cdot \frac{1}{1 - e + 2e\left(\cos.\frac{\theta}{2}\right)^2}$$

$$= \frac{p}{2} \cdot \frac{1}{(1+e)\left(\cos.\frac{\theta}{2}\right)^2 + (1-e)\left(\sin.\frac{\theta}{2}\right)^2}$$

If θ be measured, not from S A, but from a line passing through S, and making an angle α with S A, the polar equation is

$$r = \frac{p}{2} \cdot \frac{1}{1 + e \cos.\,(\theta - \alpha)}.$$

151. If P S meet the curve again P′, let S P′ $= r'$,

$$\text{then } r = \frac{p}{2} \cdot \frac{1}{1 + e \cos.\theta}$$

$$\text{and } r' = \frac{p}{2} \cdot \frac{1}{1 + e \cos.\,(\pi - \theta)} = \frac{p}{2} \cdot \frac{1}{1 - e \cos.\theta};$$

$$\therefore r + r' = \text{PP}' = \frac{p}{1 - e^2 (\cos.\theta)^2}$$

$$\text{and } r\,r' = \frac{p^2}{4} \cdot \frac{1}{1 - (e^2 \cos.\theta)^2} = \frac{p}{4}\,(r + r'),$$

or the rectangle S P, S P′ = ¼ of the rectangle under the principal parameter and focal chord.

152. Let C D, or b_1, be the semi-diameter parallel to S P, then (147.)

$$b_1^2 = \frac{a^2\,(1 - e^2)}{1 - e^2 (\cos.\theta)^2} = \frac{a}{2}\,\frac{p}{1 - e^2 (\cos.\theta)^2} = \frac{a}{2}\,(r + r'),$$

$$\therefore r + r = \frac{2\,b_1^2}{a};$$

that is, a focal chord at any point P, is a third proportional to the axis major and diameter to that chord.

CHAPTER IX.

THE HYPERBOLA.

153. In the discussion of the general equation of the second order, we observed that, referring the curve to the centre and rectangular axes, the equation to the hyperbola assumed the form

$$\left(\frac{a'}{-f'}\right)y^2 + \left(\frac{c'}{-f'}\right)x^2 = 1$$

where the co-efficients have different signs, 85. 86.

Let $\left(\frac{a'}{-f'}\right)$ be negative, then the equation becomes

$$-\left(\frac{a'}{-f'}\right)y^2 + \left(\frac{c'}{-f'}\right)x^2 = 1;$$

$$\text{or } P\,y^2 - Q\,x^2 = -1.$$

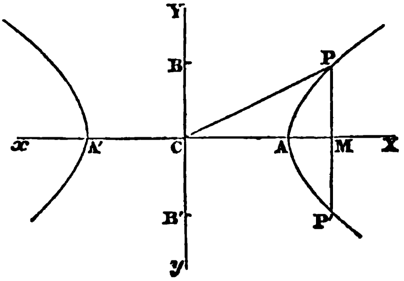
We now proceed to investigate this equation, and to deduce from it all the properties of the hyperbola.

154. Let the curve be referred to its centre C, and rectangular axes X x, Y y, meeting in C; C M $= x$, and M P $= y$; then, at the point where the curve cuts the axes, we have

$$y = 0,\ Q\,x^2 = 1,\ \therefore\ x = \pm\frac{1}{\sqrt{Q}},$$

$$x = 0,\ P\,y^2 = -1,\ \therefore\ y = \pm\sqrt{\frac{-1}{P}}$$

In the axis of x take C A $= \frac{1}{\sqrt{Q}}$, and CA′ $= -\frac{1}{\sqrt{Q}}$, and the curve cuts the axis X x in A and A: Since the value of y is impossible, the other axis never meets the curve; nevertheless we mark off two points, B and B′, in that axis, whose distances from C are C B $= +\frac{1}{\sqrt{P}}$ and C B′ $= -\frac{1}{\sqrt{P}}$.



Also if C A $= a$, and C B $= b$, we have $Q = \frac{1}{a^2}$, $P = \frac{1}{b^2}$; therefore the equation to the curve becomes

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = -1;$$

$$\text{or } a^2 y^2 - b^2 x^2 = - a^2 b^2;$$

$$\text{or } y^2 = \frac{b^2}{a^2}(x^2 - a^2)$$

155. From the last equation we have

$$y = \pm \frac{b}{a}\sqrt{x^2 - a^2}\ (1) \text{ and } x = \pm \frac{a}{b}\sqrt{y^2 + b^2}\ (2).$$

From (1) if x be less than $\pm a$, y is impossible; if, therefore, lines be drawn through A and A′, parallel to C Y, no part of the curve is found between these lines.

Again, for every value of x, greater than a, we have two real and equal values of y; that is, for any abscissa C M, greater than C A, we have two equal and opposite ordinates, M P, M P′.

Also as x increases from a to ∞, these values of y increase from 0 to $\pm\infty$; hence, we have two arcs A P, A P′, exactly equal and opposite to each other, and extending themselves indefinitely.

If x be negative, x^2 being positive, the same values of y must recur; hence, there are again two equal and opposite arcs which form another branch extending from A′ to ∞; thus the whole curve is divided into two equal parts by the axis of x.

From (2) it appears to be divided into two equal parts by the axis of y; hence it is symmetrical with respect to the axes; and its concavity is turned towards the axis of x, otherwise it might be cut by a straight line in more points than two, (71.)

156. If P be any point on the curve, we have

$$C\,P = \sqrt{x^2 + y^2} = \sqrt{x^2 + \frac{b^2}{a^2}(x^2 - a^2)} = \sqrt{\frac{a^2 + b^2}{a^2}x^2 - b^2};$$

hence C P is least when x is least, that is, when $x = a$, in which case C P becomes also equal to a; hence C A, or C A′, is the least line that can be drawn from the centre to the curve: thus, the axis A A′ is the least line that can be drawn through the centre to meet the curve. The other axis, B B′, never meets the curve.

In the equation $P y^2 - Q x^2 = -1$, the imaginary axis may be greater or less than the real one, according as Q is greater or less than P; hence the appellation of axis major cannot be generally applied to the real axis of the curve. In this treatise we shall call A A′ the transverse axis, and B B′ the conjugate axis.

157. The points A, A′ are called the vertices, or summits of the curve: either of these points may be taken for the origin by making proper substitutions.

Let A be the origin, A M $= x'$;

$$\text{Then } x = C\,M = C\,A + A\,M = a + x';$$

$$\therefore y^2 = \frac{b^2}{a^2}(x^2 - a^2) = \frac{b^2}{a^2}\{(a + x')^2 - a^2\} = \frac{b^2}{a^2}\{2\,a\,x' + x'^2\},$$

or, suppressing accents, $y^2 = \frac{b^2}{a^2}(2\,a\,x + x^2) = \frac{b^2}{a^2}\,x\,(2\,a + x)$.

This last equation is geometrically expressed by the following proportion:

The square upon M P : rectangle A M, M A′ :: the square upon B C : the square upon A C.

If the origin be at A′, the equation is $y^2 = \frac{b^2}{a^2}(x^2 - 2\,a\,x)$.

158. If $a = b$, the equation to the hyperbola becomes $y^2 - x^2 = -a^2$; this curve is called the equilateral hyperbola, and has, to the common hyperbola, the same relation that the circle has to the ellipse.

159. The analogy between the ellipse and hyperbola will be found to be very remarkable; the equations to the two curves differ only in the sign of b^2; for if, in the equation to the ellipse $a^2\,y^2 + b^2\,x^2 = a^2\,b^2$, we put $-\,b^2$ for b^2, we have the equation to the hyperbola: hence we might conclude that many of the algebraical results found in the one curve will be true for the other, upon changing b^2 into $-\,b^2$ in those results; and in fact this is the case, the same theorems are generally true for both, and may be proved in the same manner: for this reason we shall not enter at length into the demonstration of all the properties of the hyperbola, but merely put down the enunciations and results, with a reference at the end of each article to the corresponding one in the ellipse, except in those cases where there may be any modification required in the working. To prevent any doubt about the form of the figure, we shall insert figures in those places where they may be wanted: and, with this assistance, we trust that the present plan will offer no difficulty.

THE FOCUS.

160. The equation $y^2 = \frac{b^2}{a^2}(2\,a\,x + x^2)$ may be put under the form $y^2 = l\,x + \frac{l}{2\,a}\,x^2$, in which case the quantity $l = \frac{2\,b^2}{a}$ is called the principal parameter, or the Latus Rectum.

Since $l = \frac{2\,b^2}{a} = \frac{4\,b^2}{2\,a}$, the Latus Rectum is a third proportional to the transverse and conjugate axes.

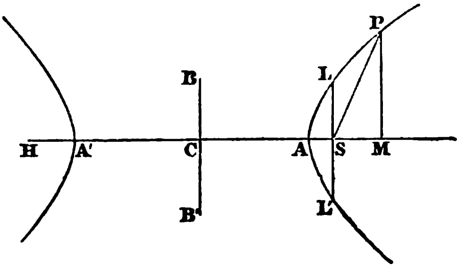
161. To find from what point in the transverse axis a double ordinate can be drawn equal to the Latus Rectum,

$$\text{Here } 4\,y^2 = l^2, \text{ or } \frac{4\,b^2}{a^2}(x^2 - a^2) = \frac{4\,b^4}{a^2};$$

$$\therefore x^2 - a^2 = b^2;$$

$$\text{or, } x^2 = a^2 + b^2;$$

$$\therefore x = \pm\sqrt{a^2 + b^2}.$$



Join A B, then $AB = \sqrt{a^2 + b^2}$; with centre C and radius A B describe a circle cutting the transverse axis in the points S and H, we have then $CS = \sqrt{a^2 + b^2}$, and $CH = -\sqrt{a^2 + b^2}$; thus S and H are the points through either of which, if an ordinate as L S L′ be drawn, it is equal to the Latus Rectum.

The two points S and H, thus determined, are called the foci.

162. The fraction $\dfrac{\sqrt{a^2 + b^2}}{a}$, which represents the ratio of C S to C A, is called the eccentricity: if this quantity, which is evidently greater than unity, be represented by the letter e, we have $\sqrt{a^2 + b^2} = a\,e$, whence $e^2 = \dfrac{a^2 + b^2}{a^2} = 1 + \dfrac{b^2}{a^2}$; $\therefore \dfrac{b^2}{a^2} = e^2 - 1$, and the equation to the hyperbola may be put under the form

$$y^2 = (e^2 - 1)(x^2 - a^2).$$

163. Since $a^2 + b^2 = a^2 e^2$, we have $b^2 = a^2 e^2 - a^2 = (a\,e - a)(a\,e + a)$;

Or the rectangle A S, S A′ = the square upon B C.

164. To find the distance from the focus to any point P in the curve, proceeding exactly as in (109.) we find

$$SP = e\,x - a,\ HP = e\,x + a;$$

Hence $HP - SP = 2a = AA'$, that is the difference of the distances of any point in the curve from the foci is equal to the transverse axis.

165. Conversely, To find the locus of a point, the difference of whose distances from two fixed points S and H is constant or equal $2a$.

If $SH = 2c$, the locus is an hyperbola, whose axes are $2a$ and $2\sqrt{a^2 + c^2}$, and whose foci are S and H. (110.)

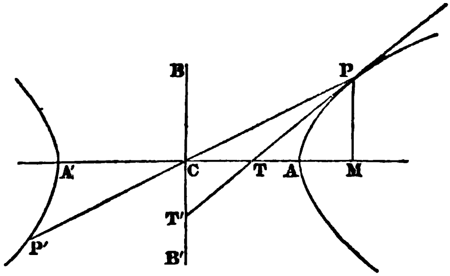
THE TANGENT.

166. To find the equation to the tangent at any point P $(x'\,y')$,

The required equation obtained as in (111.) is

$$a^2\,y\,y' - b^2\,x\,x' = -\,a^2\,b^2.$$

This form is easily recollected, since it may be obtained from the equation to the curve $a^2 y^2 - b^2 x^2 = -a^2 b^2$, by putting $y\,y'$ for y^2, and $x\,x'$ for x^2.



167. To find the points where the tangent cuts the axes;

Let $y = 0$, $\therefore\ x = \dfrac{a^2}{x'} = \text{C T}$; similarly $y = \text{C T}' = -\dfrac{b^2}{y'}$; hence we have

The rectangle C T, C M = the square upon A C;
and The rectangle C T′, M P = the square upon B C.

Since C T $\left(= \dfrac{a^2}{x'}\right)$ is always less than C A, the tangent to any point of the branch P A cuts the transverse axis between C and A.

The subtangent M T $= x' - \dfrac{a^2}{x'} = \dfrac{x'^2 - a^2}{x'}$. (115.)

The tangent at the extremity A of the transverse axis is perpendicular to that axis (116.).

If P C be produced to meet the curve again in P′, the tangents at P and P′ will be found to be parallel (116.).

168. To find the equation to the tangent at the extremity of the Latus Rectum,

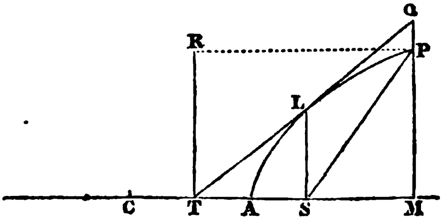
Generally the equation to the tangent is

$$a^2 y\,y' - b^2 x\,x' = -a^2 b^2;$$

at L, $x' = a\,e$, $y' = \dfrac{b^2}{a}$;

$$\therefore\ a^2 y \frac{b^2}{a} - b^2 x\,a\,e = -a^2 b^2,$$

$$y = e\,x - a.$$



Let the ordinate y, or M Q, cut the curve in P, then we have S P $= e x - a$ (164.).

$$\therefore \text{M Q} = \text{S P}:$$

Also C T $= \dfrac{a}{e}$, hence from T draw T R perpendicular to A C, and from P draw P R parallel to A C, then we have

$$\text{P R} = \text{M T} = \text{M C} - \text{C T} = x - \frac{a}{e} = \frac{e x - a}{e} = \frac{1}{e}.\text{S P}.$$

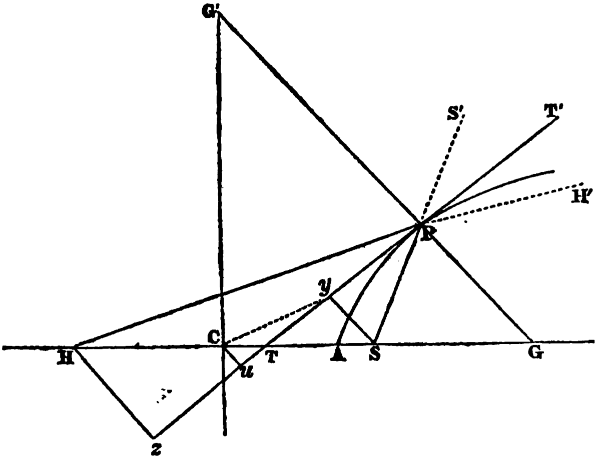
Consequently, the distances of any point P from S, and from the line T R, are in the constant ratio of $e : 1$.

The line T R is called the directrix.

If $x = 0$, we have $y = - a$; hence the tangent at the extremity of the Latus Rectum cuts the axis of y at the point where the circle on the transverse axis cuts the axis of y.

169. To find the length of the perpendicular from the focus on the tangent.

Let S y, H z be the perpendiculars on the tangent P T.



Taking the expression in (48.) we have

$$p = - \frac{y_1 - \alpha x_1 - d}{\sqrt{1 + \alpha^2}}$$

here $y_1 = 0$ and $x_1 = a e$ are co-ordinates of the point S, and $y = \alpha x + d$ is the equation to P y; but the equation to P y (166.) is also

$$y = \frac{b^2 x'}{a^2 y'} x - \frac{b^2}{y'} \therefore \alpha = \frac{b^2 x'}{a^2 y'} \text{ and } d = - \frac{b^2}{y'};$$

$$p = - \frac{- \dfrac{b^2 x'}{a^2 y'} a e + \dfrac{b^2}{y'}}{\sqrt{\left\{1 + \dfrac{b^4 x'^2}{a^4 y'^2}\right\}}} = + \frac{a b^2 (e x' - a)}{\sqrt{\{a^4 y'^2 + b^4 x'^2\}}}$$

$$= \frac{a b^2 (e x' - a)}{a b \sqrt{\{e^2 x'^2 - a^2\}}} = b \sqrt{\frac{e x' - a}{e x' + a}}.$$

Let $S P = r$, and $H P = 2a + r = r'$ $\therefore p = b\sqrt{\frac{r}{r'}}$, or $p^2 = b^2 \frac{r}{2a+r}$.

Similarly if $H z = p'$, we have $p'^2 = b^2 \frac{r'}{r}$.

By multiplication we have $p p' = b^2$; $\therefore$ hence

The rectangle $S y$, $H z$ = the square upon B C.

170. To find the locus of y or z in the last article.

The equation to the curve at P is $a^2 y'^2 - b^2 x'^2 = - a^2 b^2$.

The equation to the tangent at P is $a^2 y y' - b^2 x x' = - a^2 b^2$.

$$\text{The equation to } S y \text{ is } y = \frac{-a^2 y'}{b^2 x'} (x - c).$$

By eliminating x' and y', exactly as in (120.), we arrive at the equation

$$a^2 = y^2 + x^2;$$

Hence the locus of y is a circle described on the transverse axis as diameter.

171. To find the angle which the focal distance S P makes with the tangent P T.

The equation to the tangent is $y = \frac{b^2 x'}{a^2 y'} x - \frac{b^2}{y'}$, and the equation to S P is, $y - y' = \frac{y'}{x' - c} (x - x')$,

$$\text{hence tan. } S P T = \text{tan. } (P S X - P T X)$$

$$= \frac{\frac{y'}{x' - c} - \frac{b^2 x'}{a^2 y'}}{1 + \frac{b^2 x'}{a^2 y'} \frac{y'}{x' - c}} = \frac{a^2 y'^2 - b^2 x'^2 + b^2 c x'}{a^2 y' x' - a^2 c y' + b^2 x' y'}$$

$$= \frac{b^2 (c x' - a^2)}{y' c (c x' - a^2)} = \frac{b^2}{c y'}.$$

Similarly tan. $H P T = \frac{b^2}{c y'}$; $\therefore$ the angles S P T, H P T are equal; thus the tangent makes equal angles with the focal distances.

Produce S P to S′, then it is a property of light, that if a ray proceeding from H be reflected by the line T P T′, the angle S′ P T′ of the reflected ray will equal the angle H P T. Now, in the hyperbola, these angles are equal; hence if a light be placed at H, all rays which are incident on the curve will be reflected as if diverging from S; or if a body of rays proceeding to S be incident on the curve, they will converge to H. Hence these points S and H are called foci.

This important property of the curve is also thus proved from article (169.),

$$S y = p = b\sqrt{\frac{r}{r'}}, \text{ and } H z = p' = b\sqrt{\frac{r'}{r}};$$

$$\therefore S y : H z :: r \cdot\cdot r' :: S P : H P;$$

$\therefore$ angle $S P y = H P z$, and the tangent makes equal angles with the focal distances *.

172. To find the length of the perpendicular $C u$ from the centre on the tangent.

$$p = -\frac{y_1 - \alpha x_1 - d}{\sqrt{1 + \alpha^2}},$$

here $y_1 = 0$, $x_1 = 0$, $\alpha = \frac{b^2 x'}{a^2 y'}$, and $d = -\frac{b^2}{y'}$, $\therefore C u = \frac{a b}{\sqrt{r r'}}$

173. To find the locus of u.

The equation to $C u$ is $y = -\frac{a^2 y'}{b^2 x'} x$, eliminating $x' y'$ from this equation, and the equation to the tangent, we find, as in (123.), the resulting equation to be $a^2 x^2 - b^2 y^2 = (x^2 + y^2)^2$, which cannot be discussed at present.

174. From the equation to the tangent, and that to $C P$, we find, as in (124.),

$$\tan. C P T = \frac{a^2 b^2}{c^2 x' y'}.$$

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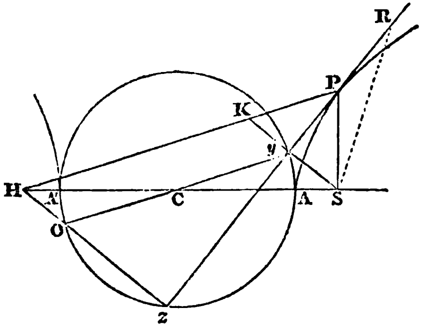
* The following geometrical method of drawing a tangent to the hyperbola, and proving that the locus of the perpendicular from the focus on the tangent is the circle on the transverse axis, will be found useful.



Let $A P$ be the hyperbola, P any point on it; join $S P$ and $H P$, and in $H P$ take $P K = P S$; bisect the angle $S P K$ by the line $P y z$, and join $S K$, cutting $P y$ in y.

1. $P y$ is a tangent to the hyperbola; for if R be any other point in the line $P y$, we have $H R - S R = H R - K R$ is less than $H K$ (Geom. i. 10) less than $2 a$, hence R, and every other point in $P y$, is without the curve.

2. The locus of y is the circle on the transverse axis: draw $H z$ parallel to $S y$, and join $C y$; then, because the triangles $S P y$, $K P y$ are equal, we have the angle $S y P$ a right angle, or $S y$ and $H z$ are perpendicular to the tangent. Also since $S y = K y$, and $S C = C H$, we have $C y$ parallel to $H K$, and $C y = \frac{1}{2} H K = \frac{1}{2} (H P - S P) = C A$.

3. The rectangle $S y$, $H z =$ the square on $B C$. Let $z H$ meet the circle again in O, and join $C O$; then the line $O C y$ is a straight line and a diameter, hence the triangles $C S y$, $C H O$ are equal, and the rectangle $S y$, $H z =$ the rectangle $H O$, $H z =$ the rectangle $H A'$, $H A =$ the square upon $B C$.

4. Let $S P = r$, $H P = 2 a + r$, $S y = p$ and $H z = p'$, then $p^2 = \frac{b^2 r}{2 a + r}$; for by similar triangles, $S y : S P :: H z : H P$, $\therefore p = \frac{r}{2 a + r} p'$, and, as above, $p p' = b^2$, $\therefore p^2 = \frac{b^2 r}{2 a + r}$.

From C u = C y sin. C y u, we have

$$\frac{a\,b}{\sqrt{r\,r'}} = a \text{ sin. C } y\,u \mathrel{.\dot{}.} \text{sin. C } y\,u = \frac{b}{\sqrt{r\,r'}}.$$

Also from H z = H P sin. H P z, we have

$$b\sqrt{\frac{r'}{r}} = r' \text{ sin. H P } z,\ \therefore \text{ sin. H P } z = \frac{b}{\sqrt{r\,r'}},$$

$\therefore$ angle C y u = angle H P z and C y is parallel to H P.

And if C E be drawn parallel to the tangent P T, and meeting H P in E, we have P E = C y = A C.

THE NORMAL.

175. The equation to the line passing through the point P $(x'\,y')$, and perpendicular to the tangent $\left(y = \frac{b^2\,x'}{a^2\,y'}\,x - \frac{b^2}{y'}\right)$ is

$$y - y' = -\frac{a^2\,y'}{b^2\,x'}\,(x - x').$$

To find where the normal P G cuts the axes.

$$\text{Let } y = 0 \mathrel{.\dot{}.} -y' = -\frac{a^2\,y'}{b^2\,x'}\,(x - x') \mathrel{.\dot{}.} x = x' + \frac{b^2\,x'}{a^2}$$

$$= \frac{a^2 + b^2}{a^2}\,x' = e^2\,x' = \text{C G}.$$

$$\text{Let } x = 0 \mathrel{.\dot{}.} y = y' + \frac{a^2\,y'}{b^2} = \frac{a^2 + b^2}{b^2}\,y' = \frac{a^2\,e^2}{b^2}\,y' = \text{C G}'.$$

Also the subnormal M G $= x - x' = \frac{b^2\,x'}{a^2}$; and S G = e. S P.

176. From the above values of C G, C G$'$, and M G$'$, we may demonstrate that P G $= \frac{b}{a}\sqrt{r\,r'}$, P G$'$ $= \frac{a}{b}\sqrt{r\,r'}$, and consequently that

The rectangle P G, P G$'$ = $r\,r'$ = the rectangle S P, H P.

Also S G$'$ $= \frac{a\,e}{b}\sqrt{r\,r'}$, G G$'$ $= \frac{a\,e^2}{b}\sqrt{r\,r'}$, and $\mathrel{.\dot{}.}$ G G$'$ = e.SG$'$.

177. Since the tangent makes equal angles with the focal distances, the normal, which is perpendicular to the tangent, also makes equal angles with the focal distances, one of them being first produced as to H$'$. This theorem may be directly proved from the above value of C G; for S G : H G :: $e^2\,x' - a\,e$: $e^2\,x' + a\,e$ :: $e\,x' - a$: $e\,x' + a$ :: S P : H P, hence the angle S P H$'$ is bisected by the line P G.

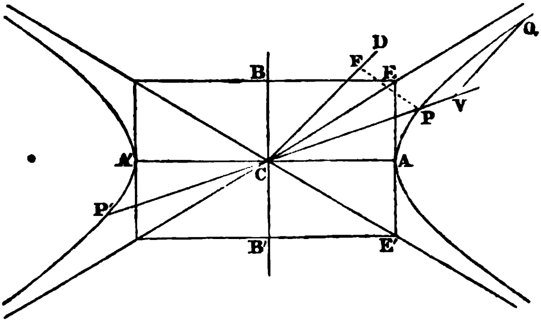
THE DIAMETERS.

178. It may be proved as for the ellipse (130.), that all the diameters of the hyperbola pass through the centre, and that any line through the

centre is a diameter. If $y = \alpha x + c$ be the equation to any chord, $a^2 \alpha y - b^2 x = 0$ is the equation to the diameter bisecting all chords parallel to $y = \alpha x + c$.

179. In the ellipse all the diameters must necessarily meet the curve; but this is not the case in the hyperbola, as will appear by finding the co-ordinates of intersection of the diameter and the curve.

Let $y = \beta x$ be the equation to a diameter C P, and substitute this value of y in the equation to the curve.



$$a^2 y^2 - b^2 x^2 = - a^2 b^2,$$

$$\therefore a^2 \beta^2 x^2 - b^2 x^2 = - a^2 b^2,$$

$$x^2 = \frac{a^2 b^2}{b^2 - a^2 \beta^2},$$

$$\therefore x = \pm \frac{a\, b}{\sqrt{b^2 - a^2 \beta^2}}.$$

These values are impossible, if $a^2 \beta^2$ is greater than b^2, that is, if β is greater than $\frac{b}{a}$; and if $\beta = \pm \frac{b}{a}$, the diameter meets the curve only at an infinite distance. The limits of the intersecting diameters are thus determined; through A, B and B′ draw lines parallel to the axes meeting in E and E′, then tan. E C A $= \frac{b}{a}$, and tan. E′ C A $= - \frac{b}{a}$, hence C E and C E′ produced are the lines required. Hence, in order that a diameter meet the curve, it must be drawn within the angle E C E′; thus the line C D never meets the curve.

The curve is symmetrical with respect to these lines C E, C E′, since the axis bisects the angle E C E′.

180. The hyperbola has an infinite number of pairs of conjugate diameters. This is proved by referring the equation to other axes by means of the formulas of transformation (57.)

$$y = x' \sin. \theta + y' \sin. \theta',$$

$$x = x' \cos. \theta + y' \cos. \theta';$$

hence the equation $a^2 y^2 - b^2 x^2 = - a^2 b^2$ becomes

$$\{a^2 (\sin. \theta')^2 - b^2 (\cos. \theta')^2\} y'^2 + \{a^2 (\sin. \theta)^2 - b^2 (\cos. \theta)^2\} x'^2$$
$$+ 2 \{a^2 \sin. \theta \sin. \theta' - b^2 \cos. \theta \cos. \theta'\} x' y' = - a^2 b^2.$$

In order that this equation be of the conjugate form, let the co-efficient of $y' x' = 0$,

$$\therefore\ a^2 \sin.\theta \sin.\theta' - b^2 \cos.\theta \cos.\theta' = 0,$$

$$\text{or,}\quad \tan.\theta \tan.\theta' = \frac{b^2}{a^2}.$$

Hence for any value of θ, we have a real value of θ', that is, there is an infinite number of pairs of axes to which, if the curve be referred, its equation is of the required conjugate form.

If $\tan.\theta$ be less than $\frac{b}{a}$, $\tan.\theta'$ must be greater than $\frac{b}{a}$, that is, if one diameter C P, in the last figure, meets the curve, the conjugate diameter C D does not; therefore in each system of conjugate diameters one is imaginary. Also, since the product of the tangents is positive, both angles are acute, or both obtuse; in the figure they are both acute, but for the opposite branch they must be both obtuse.

181. As in article (132.), it appears that there can be only one system of rectangular conjugate diameters.

182. The equation to the curve is now

$$\{a^2 (\sin.\theta')^2 - b^2 (\cos.\theta')^2\}\, y'^2 + \{a^2 (\sin.\theta)^2 - b^2 (\cos.\theta)^2\}\, x'^2 = -a^2 b^2.$$

If we successively make $y' = 0$, and $x' = 0$, we have the distances from the origin to the points in which the curve cuts the new axes; but as we already know (180.) that one of these new axes never meets the curve, we must represent one of these distances by an imaginary quantity.

Let the axis of x' meet the curve at a distance a_1 from the centre, and let the length of the other semi-axis be b_1 connected with the symbol $\sqrt{-1}$, that is, let the new conjugate diameters be $2\,a_1$ and $2\,b_1 \sqrt{-1}$, then we have

$$y = 0 \ \therefore\ \{a^2 (\sin.\theta)^2 - b^2 (\cos.\theta)^2\}\, a_1^2 = -a^2 b^2,$$

$$x = 0 \ \therefore\ \{a^2 (\sin.\theta')^2 - b^2 (\cos.\theta')^2\}\,(-b_1^2) = -a^2 b^2,$$

And the transformed equation becomes

$$\frac{a^2 b^2}{b_1^2}\, y'^2 - \frac{a^2 b^2}{a_1^2}\, x'^2 = -a^2 b^2,$$

$$\text{or,}\quad \frac{y'^2}{b_1^2} - \frac{x'^2}{a_1^2} = -1,$$

$$\text{or,}\quad a_1^2 y'^2 - b_1^2 x'^2 = -a_1^2 b_1^2.$$

183. From the transformation we obtain the three following equations:

$$a_1^2 \{a^2 (\sin.\theta)^2 - b^2 (\cos.\theta)^2\} = -a^2 b^2 \quad (1),$$

$$b_1^2 \{a^2 (\sin.\theta')^2 - b^2 (\cos.\theta')^2\} = +a^2 b^2 \quad (2),$$

$$\left.\begin{aligned} a^2 \sin.\theta \sin.\theta' - b^2 \cos.\theta \cos.\theta' &= 0, \\ \text{or,}\quad \tan.\theta \tan.\theta' &= \frac{b^2}{a^2} \end{aligned}\right\} \quad (3).$$

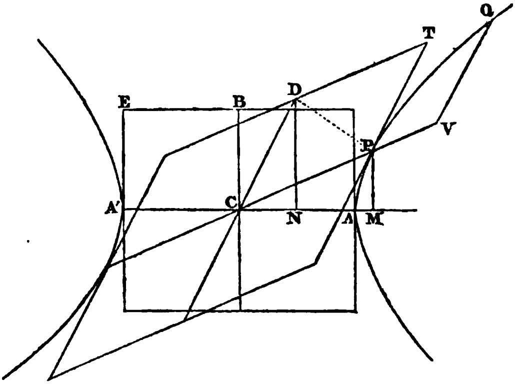
Following the steps exactly as in article (134.), or, which amounts to the same thing, putting $-b^2$ for b^2, and $-b_1^2$ for b_1^2 all through that article, we arrive at the result

$$a_1^2 - b_1^2 = a^2 - b^2,$$

or, the difference of the squares upon the conjugate diameters is equal to the difference of the squares upon the axes.

184. Again, multiplying (1) and (2) together, and (3) by itself, then subtracting the results, and reducing, as in the article (135.), we have

$$a_1\, b_1 \sin. (\theta' - \theta) = a\, b.$$



Now $\theta' - \theta$ is the angle P C D between the conjugate diameters C P and C D; hence, drawing straight lines at the extremities of the conjugate diameters, parallel to those diameters, we have, from the above equation, the parallelogram P C D T = the rectangle A′ C B E, and hence the whole parallelogram thus inscribed in the figure is equal to the rectangle contained by the axes *.

185. Returning to article (182.), the equation to the curve, suppressing the accents on x' and y', as no longer necessary, is

$$a_1^2\, y_1^2 - b_1^2\, x^2 = - a_1^2\, b_1^2;$$

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* The theorems in articles 183 and 184 may be proved also in the following manner:—

Referring the curve to its rectangular axes, as in art. (187.), let the co-ordinates of P be x' and y'; then the equation to C D is $a^2\, y\, y' - b^2\, x\, x' = 0$, and eliminating x and y between this equation and that to the curve ($a^2\, y^2 - b^2\, x^2 = - a^2\, b^2$) we have the co-ordinates C N and D N, independent of the sign $\sqrt{-1}$, with which they are both affected,

$$\text{C N} = x = \frac{a\, y'}{b}, \text{ and D N} = y = \frac{b\, x'}{a};$$

Hence we have

$$a_1^2 - b_1^2 = x'^2 + y'^2 - x^2 - y^2 = x'^2 + y'^2 - \frac{a^2\, y'^2}{b^2} - \frac{b^2\, x'^2}{a^2} = \frac{b^2\, x'^2 - a^2\, y'^2}{b^2}$$
$$+ \frac{a^2\, y'^2 - b^2\, x'^2}{a^2} = \frac{a^2\, b^2}{b^2} + \frac{- a^2\, b^2}{a^2} = a^2 - b^2.$$

Also the triangle P C D = the trapezium P M N D + the triangle D C N − the triangle P C M

$$= (x' - x)\, \frac{y + y'}{2} + \frac{x\, y - x'\, y'}{2} = \frac{x'\, y - y'\, x}{2} = \frac{1}{2} \left\{ x' \frac{b\, x'}{a} - y' \frac{a\, y'}{b} \right\}$$
$$= \frac{b^2\, x'^2 - a^2\, y'^2}{2\, a\, b} = \frac{a^2\, b^2}{2\, a\, b} = \frac{a\, b}{2},$$

therefore the parallelogram P C D T = $a\, b$.

In the last figure, $CP = a_1$, $CD = b_1$, $CV = x$ and $QV = y$: Putting the equation into the form

$$y^2 = \frac{b_1^2}{a_1^2}(x^2 - a_1^2) = \frac{b_1^2}{a_1^2}(x - a_1)(x + a_1),$$

we have the square upon Q V : the rectangle P V, V P′ :: the square upon C D : the square upon C P.

186. The equation to the tangent at any point Q $(x' y')$ is

$$a_1^2 y y' - b_1^2 x x' = - a_1^2 b_1^2.$$

187. Let the curve be referred to its axes C A, C B, and let the co-ordinates of P be $x' y'$, then the equation to C P being $y = \frac{y'}{x'} x$, the equation to CD is $y = x \tan. \theta' = x \frac{b^2}{a^2} \cot. \theta = \frac{b^2 x'}{a^2 y'} x$, or,

$$a^2 y y' - b^2 x x' = 0.$$

But the equation to the tangent at P is

$$a^2 y y' - b^2 x x' = - a^2 b^2;$$

hence C D, or the diameter conjugate to C P, is parallel to the tangent at P.

The equation to the conjugate diameter is the same as that to the tangent, omitting the last term $- a^2 b^2$.

188. Let x' and y' be the rectangular co-ordinates of P; then from the equation $a_1^2 - b_1^2 = a^2 - b^2$, we have

$$b_1^2 = a_1^2 - a^2 + b^2 = x'^2 + y'^2 - a^2 + b^2 = x'^2 + \left(\frac{b^2}{a^2} x'^2 - b^2\right) - a^2 + b^2$$

$$= \frac{a^2 + b^2}{a^2} x'^2 - a^2 = e^2 x'^2 - a^2 = (e x' - a)(e x' + a) = r r';$$

That is, the square upon the conjugate diameter C D = the rectangle under the focal distances S P and H P.

189. If P F be drawn perpendicular from P upon the conjugate C D, (see the last figure but one,) we have the rectangle P F, C D $= a b$, (184.).

$$\therefore PF = \frac{a b}{b_1} = \frac{a b}{\sqrt{a_1^2 - a^2 + b^2}} = \frac{a b}{\sqrt{r r'}};^*$$

Also $PG = \frac{b}{a} \sqrt{r r'}$, and $PG' = \frac{a}{b} \sqrt{r r'}$,

Hence the rectangle P G, P F = the square on B C;

And the rectangle P G′, P F = the square on A C;

And the rectangle P G, P G′ = the square on C D.

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* If the distance $CP = u$, and p = the perpendicular from the centre on the tangent, this equation is

$$p^2 = \frac{a^2 b^2}{u^2 - a^2 + b^2}.$$

SUPPLEMENTAL CHORDS.

190. Two straight lines drawn from a point on the curve to the extremities of a diameter are called supplemental chords; they are called principal supplemental chords if that diameter be the transverse axis.

The equations to a pair of chords are

$$y - y' = \alpha (x - x')$$
$$y + y' = \alpha' (x + x');$$

Whence $\alpha \alpha' = \frac{b^2}{a^2}$ as in (141.); hence the product of the tangents of the angles which a pair of supplemental chords makes with the transverse axis is constant; the converse is proved as in 141.

191. The angle between two supplemental chords is found from the expression

$$\tan. P Q P' = \frac{2 b^2}{a^2 + b^2} \frac{x' y - y' x}{y^2 - y'^2}.$$

And, if A R, A′ R be principal supplemental chords drawn to any point R on the curve,

$$\tan. A R A' = \frac{2 a b^2}{(a^2 + b^2) y}.$$

The angle A R A′ is always acute, and diminishes from a right angle to 0; the supplemental angle A A′ R′ increases at the same time from a right angle to 180°; hence, the angle between the supplemental chords may be any angle between 0 and 180°.

Chords may be drawn containing any angle between these limits, by describing on any diameter, except the axes, a segment of a circle containing the given angle, and then joining the extremities of the diameter with the point where the circle intersects the hyperbola. And therefore principal supplemental chords parallel to these may be drawn.

192. Conjugate diameters are parallel to supplemental chords (144.); and therefore they may be drawn containing any angle between 0 and 90°.

193. There are no equal conjugate diameters in the hyperbola, but in that particular curve where $b = a$, we have the equation

$$a_1^2 - b_1^2 = a^2 - b^2 = 0;$$

hence the conjugate diameters a_1 and b_1 are always equal to each other.

The equation to this curve, called the equilateral hyperbola, is

$$y^2 - x^2 = - a^2,$$

THE ASYMPTOTES.

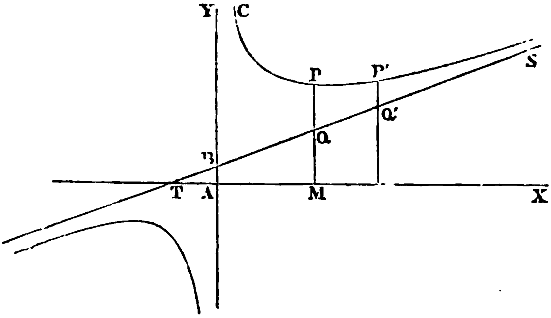
194. We have now shown that most of the properties of the ellipse apply to the hyperbola with a very slight variation: there is, however, a whole class of theorems quite peculiar to the latter curve, and these arise from the curious form of the branches extending to an infinite distance; it appears from the equation $\tan. \theta . \tan \theta' = \frac{b^2}{a^2}$ in (180.), that as $\tan. \theta$

approaches to $\frac{b}{a}$, tan. θ' approaches also to $\frac{b}{a}$, and thus, as a point P recedes along the curve from the origin, the conjugate diameters for that point approach towards a certain line C E, fig. (179.), and finally at an infinite distance come indefinitely near to that line.

We now proceed to show that the curve itself continually approaches to the same line C E, without ever actually coinciding with it. But as this species of line is not confined to the hyperbola, we shall state the theory generally.

195. Let C P P′ be a curve whose equation has been reduced to the form

$$y = a x + b + \frac{c}{x};$$



And let T B S be the line whose equation is

$$y = a x + b.$$

For any value of x we can find from this last equation a corresponding ordinate M Q, and by adding $\frac{c}{x}$ to M Q, we determine a point P in the curve: similarly we can determine any number of corresponding points (P′, Q′, &c.) in the curve and straight line.

Since $\frac{c}{x}$ decreases as x increases, the line P′ Q′ will be less than P Q, and the greater x becomes, the smaller does the corresponding P′ Q′ become; so that when x is infinitely great, P′ Q′ is infinitely small, or the curve approaches indefinitely near to the line T B S, but yet never actually meets it: hence T B S is called an asymptote to the curve, from three Greek words signifying " never coinciding."

The equation to the asymptote T B S is $y = a x + b$, or is the equation to the curve, with the exception of the term involving the inverse power of x.

196. The reasoning would have been as conclusive if there had been more inverse powers of x; and in general if the equation to a curve can be put into the form

$$y = \&c. + m x^3 + n x^2 + a x + b + \frac{c}{x} + \frac{d}{x^2} + \&c.$$

Then the equation to the curvilinear asymptote is

$$y = \&c. + m\,x^3 + n\,x^2 + a\,x + b$$

Also the equation $y = \&c. + m\,x^3 + n\,x^2 + a\,x + b + \frac{c}{x}$ gives a curve much more asymptotic than the preceding equation, and hence arises a series of curves, each " more nearly coinciding" with the original curve.

197. Let us apply this method to lines of the second order, whose general equation is (75.)

$$y = -\frac{b\,x + d}{2\,a} \pm \frac{1}{2\,a}\sqrt{\{(b^2 - 4\,a\,c)\,x^2 + 2\,(b\,d - 2\,a\,e)\,x + d^2 - 4\,a\,f\}}$$

$$= -\frac{b\,x + d}{2\,a} \pm \sqrt{\{m\,x^2 + n\,x + p\}}\text{, by substitution,}$$

$$= -\frac{b\,x + d}{2\,a} \pm x\sqrt{m}\left\{1 + \frac{n\,x + p}{m\,x^2}\right\}^{\frac{1}{2}},$$

$$= -\frac{b\,x + d}{2\,a} \pm x\sqrt{m}\left\{1 + \tfrac{1}{2}\left(\frac{n\,x + p}{m\,x^2}\right) - \tfrac{1}{8}\left(\frac{n\,x + p}{m\,x^2}\right)^2 + \&c.\right\}$$

$$= -\frac{b\,x + d}{2\,a} \pm \sqrt{m}\left\{x + \tfrac{1}{2}\,\frac{n}{m}\right\} \pm \frac{\text{constant terms}}{\text{powers of } x}.$$

Hence the equation to the asymptote is

$$y = -\frac{b\,x + d}{2\,a} \pm \sqrt{m}\left\{x + \tfrac{1}{2}\,\frac{n}{m}\right\}$$

$$= -\frac{b\,x + d}{2\,a} \pm \frac{\sqrt{b^2 - 4\,a\,c}}{2\,a}\left\{x + \frac{b\,d - 2\,a\,c}{b^2 - 4\,a\,c}\right\}$$

Now $b^2 - 4\,a\,c$ is negative in the ellipse, and therefore there is no locus to the above equation in this case; also if $b^2 - 4\,a\,c = 0$, the equation to the asymptote, found as above, will contain the term $\sqrt{x}$, and therefore will belong to a curvilinear asymptote; hence the hyperbola is the only one of the three curves which admits of a rectilinear asymptote.

It appears from the ± sign, that there are two asymptotes, and that the diameter $y = -\frac{b\,x + d}{2\,a}$ bisects them. Also these asymptotes pass through the centre; for giving to x the value $\frac{2\,a\,e - b\,d}{b^2 - 4\,a\,c}$, we have

$$y = -\frac{b\,x + d}{2\,a} = \frac{2\,c\,d - b\,e}{b^2 - 4\,a\,c},$$

and these values of x and y are the co-ordinates of the centre (80.).

198. If the equation want either of the terms x^2 or y^2, a slight operation will enable us to express the equation in a series of inverse powers of y or x; thus if the equation be

$$b\,x\,y + c\,x^2 + d\,y + e\,x + f = 0,$$

we have $y = -\frac{cx^2+ex+f}{bx+d} = -\frac{cx^2+ex+f}{bx\left(1+\frac{d}{bx}\right)}$

$$= -\frac{cx^2+ex+f}{bx}\left(1+\frac{d}{bx}\right)^{-1}$$

$$= -\left(\frac{cx}{b}+\frac{e}{b}+\frac{f}{bx}\right)\left(1-\frac{d}{bx}+\left(\frac{d}{bx}\right)^2-,\&c.\right)$$

Hence the equation to the asymptote, found by multiplying and neglecting inverse powers of x, is

$$y = -\frac{cx}{b}-\frac{e}{b}+\frac{cd}{b^2},$$

$$\text{or,}\quad by+cx=\frac{cd-be}{b}.$$

The other asymptote is determined by the consideration that if, for any finite value of x, we obtain a real infinite value of y, that value of x determines the position of an asymptote.

Here when $bx+d=0$, we have $y=\infty$; hence a line drawn parallel to the axis of y, and through the point $x=-\frac{d}{b}$, is the required asymptote.

If the equation be

$$ay^2+bxy+dy+ex+f=0,$$

the equations to the asymptotes are

$$ay+bx=\frac{ae-bd}{b},\ \text{and}\ by+e=0;$$

and the second asymptote is parallel to the axis of y.

If the equation be

$$bxy+dy+ex+f=0,$$

the equations to the asymptotes are

$$bx+d=0,\ \text{and}\ by+e=0;$$

the former asymptote being parallel to the axis of y, and the latter parallel to that of x.

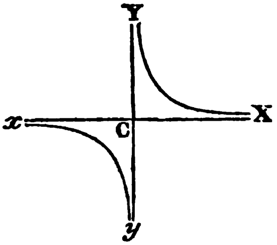
199. Lastly, if the equation be

$$bxy+f=0,$$

the asymptotes are then the axes themselves, and the curve is referred to its centre and asymptotes as axes.

The position of the curve in this case is directly obtained from the equation $y=-\frac{f}{bx}=\frac{m}{x}$, by substitution.

Let C X and C Y be the axes, then for $x=0$, $y=\infty$; as x increases y decreases, and when $x=\infty$, $y=0$; hence we have the branch Y X.



For x negative, y is negative; and as x increases from 0 to ∞, y decreases from ∞ to 0; hence another branch $y\,x$, equal and similar to the former.

200. To find the equation to the asymptotes from the equation to the hyperbola referred to its centre and axes,

$$y = \pm\frac{b}{a}\sqrt{x^2 - a^2} = \pm\frac{b}{a}x\sqrt{1 - \frac{a^2}{x^2}} = \pm\frac{b}{a}x\left\{1 - \tfrac{1}{2}\frac{a^2}{x^2} +, \&c.\right\}$$

Hence the equation to the asymptotes is

$$y = \pm\frac{b}{a}x.$$

To draw these lines, complete the parallelogram on the principal axes (see the figure, art. 179.); the diagonals of this parallelogram are the loci of the last equation, and therefore are the asymptotes required: thus C E and C E′, when produced, are the asymptotes.

The equation to the asymptotes, referred to the centre and rectangular axes, is readily remembered, since it is the same as the equation to the curve without the last term; the two equations are

$$a^2y^2 - b^2x^2 = -a^2b^2, \text{ to the curve,}$$

$$a^2y^2 - b^2x^2 = 0 \quad , \text{ to the asymptotes.}$$

If the curve be referred to conjugate axes, the equations are

$$a_1^2y^2 - b_1^2x^2 = -a_1^2b_1^2, \text{ to the curve,}$$

$$a_1^2y^2 - b_1^2x^2 = 0 \quad , \text{ to the asymptotes.}$$

201. If $b = a$, the equation to the hyperbola referred to its centre and rectangular axes is $y^2 - x^2 = -a^2$, therefore the equation to the asymptotes is $y^2 - x^2 = 0$, or $y = \pm x$; hence these asymptotes cut the axes at an angle of 45°, or the angle between them is 90°; hence the equilateral hyperbola is also called the rectangular hyperbola.

202. If the curve be referred to its vertex A and rectangular axes, the equation to the curve is

$$y = \pm\frac{b}{a}(x^2 - 2ax)^{\frac{1}{2}} = \pm\frac{b}{a}x\left(1 + \frac{2a}{x}\right)^{\frac{1}{2}},$$

and, expanding and neglecting inverse powers of x, the equation to the asymptotes is

$$y = \pm\left(\frac{b}{a}x + b\right).$$

203. If we take the equation to any line $\left(y = \frac{b}{a}x + c\right)$ parallel to the asymptote, and eliminate y between this equation and the equation to the curve, we find only one value of x; and thus a straight line parallel to the asymptote cuts the hyperbola only in one point.

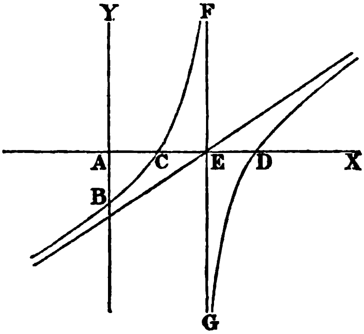
204. In article (77.) it was stated that, in some cases, the form of the curve could not be readily ascertained: thus, when the curve cuts neither diameter, there might be some difficulty in ascertaining its correct position: the asymptotes will, however, be found very useful in this respect: for example, if the equation is $xy = x^2 + bx + c^2$, or $y = x + b + \frac{c^2}{x}$, we have for $x = 0$, $y = \infty$; and when x becomes very great, y approximates to $x + b$; hence the lines A Y and T B S, in figure (194), will represent the asymptotes of the curve; and since the curve never cuts the axes, its course is entirely confined within the angle Y B S and the opposite angle T B A; hence the position of the curve is at once determined, as in figure (194).

Ex. 2. $y(x - 2) = (x - 1)(x - 3)$, or $y = \frac{(x-1)(x-3)}{x-2}$

In the first place we ascertain that the curve is an hyperbola by the test $b^2 - 4ac$ being positive; then draw the rectangular axes A X, A Y: to find the points where the curve cuts the axes,

$$\text{Let } x = 0, \quad \therefore y = -\tfrac{3}{2} = AB,$$
$$\text{Let } y = 0, \quad \therefore x = 1 = AC,$$
$$\text{also } x = 3 = AD,$$

thus the curve passes through the points B, C, and D.



Again, to find the asymptotes, we have $y = \infty$ for $x = 2$; hence, if A E = 2, the line F E G, drawn perpendicular to A X, is one asymptote.

To find the other, we have

$$y = \frac{(x-1)(x-3)}{x-2} = \frac{(x-1)(x-3)}{x\left(1 - \frac{2}{x}\right)} = \frac{(x-1)(x-3)}{x}\left(1 - \frac{2}{x}\right)^{-1}$$

$$= \frac{x^2 - 4x + 3}{x}\left\{1 + \frac{2}{x} +, \&c.\right\} = \left(x - 4 + \frac{3}{x}\right)$$

$$\left\{1 + \frac{2}{x} +, \&c.\right\} = x - 4 + 2 + \frac{a}{x} +, \&c.;$$

hence the equation to the asymptote is $y = x - 2$, and therefore this line must be drawn through the point E, making an angle of 45° with A X.

We can now trace the course of the curve completely; for all values of x less than 1, y is negative, hence the branch B C; for x greater than 1, but less than 2, y is positive and increases from 0 to ∞, hence the branch C F; for x greater than 2, but less than 3, y is negative, hence the branch G D; and for x greater than 3, y is positive and approximating to $x - 2$, hence the branch from D extending to the second asymptote.

For negative values of x, y is negative, and increases from $-\frac{3}{2}$ to ∞, approximating also to the value $-x-2$; hence the curve extends downwards from B towards the asymptote.

Ex. 3. $y(x-a) = x(x-2a)$. Here $x = a$ and $y = x - a$, are the equations to the asymptotes. The figure is like the last, supposing that A and C coincide.

Ex. 4. $y^{\frac{1}{2}} = a\,x^{-\frac{1}{2}} + x^{\frac{1}{2}}$. The axis of y is one asymptote, since $x = 0$ gives $y = \infty$: Also

$$y = x\left(1 + \frac{a}{x}\right)^2 = x + 2a + \frac{a^2}{x};$$

hence $y = x + 2a$ gives the other asymptote.

205. In order to discuss an equation of the second order completely, we have given, in Chapter VII., a general method of reducing that equation to its more simple forms.

In that chapter we showed that the equation, when belonging to an hyperbola, could be reduced to the form $a y^2 + c x^2 + f = 0$. (84.)

Now the same equation can be reduced also to the form $x y = k^2$; and as this form is of use in all discussions about asymptotes, we shall proceed to its investigation.

206. Let the general equation be referred to rectangular axes, and let it be

$$a y^2 + b x y + c x^2 + d y + e x + f = 0.$$

Let $x = x' + m$, and $y = y' + n$, and then, as in article (80.), put the co-efficients of x' and y' each $= 0$; by this means the curve is referred to its centre, and its equation is reduced to the form

$$a y'^2 + b x' y' + c x'^2 + f' = 0.$$

Again, to destroy the co-efficients of x'^2 and y'^2, take the formulas of transformation from rectangular to oblique co-ordinates (57.).

$$y' = x'' \sin.\theta + y'' \sin.\theta',$$

$$x' = x'' \cos.\theta + y'' \cos.\theta';$$

then, by substituting and arranging, the central equation becomes

$$y''^2 \{a(\sin.\theta')^2 + b \sin.\theta' \cos.\theta' + c(\cos.\theta')^2\}$$

$$+ x''^2 \{a(\sin.\theta)^2 + b \sin.\theta \cos.\theta + c(\cos.\theta)^2\}$$

$$+ x'' y'' \{2\,a \sin.\theta' \sin.\theta + b\,(\sin.\theta \cos.\theta' + \sin.\theta' \cos.\theta) + 2\,c \cos.\theta' \cos.\theta\} + f' = 0.$$

There are two new indeterminate quantities θ and θ' introduced; therefore we may make two suppositions respecting the co-efficients in the transformed equation; hence, letting the co-efficients of x''^2 and $y''^2 = 0$, we have

$$a\,(\sin.\theta)^2 + b \sin.\theta \cos.\theta + c\,(\cos.\theta)^2 = 0 \qquad (1),$$

$$a\,(\sin.\theta')^2 + b\,(\sin.\theta')\,(\cos.\theta') + c\,(\cos.\theta')^2 = 0 \quad (2).$$

Dividing the first of these two equations by $(\cos.\theta)^2$, we have

$$a\,(\tan.\theta)^2 + b \tan.\theta + c = 0;$$

$$\text{hence} \quad \tan.\theta = \frac{b \pm \sqrt{b^2 - 4\,a\,c}}{2\,a}.$$

From the similarity of the equation (1) and (2), it is evident that we shall arrive at the same value for $\tan.\theta'$; hence, letting one of the above values refer to θ, the other will refer to θ'; or both the new axes are determined in position from the above values of $\tan.\theta$.

The equation is now reduced to the form

$$b'\,x''\,y'' + f' = 0.$$

207. To find the value of b', we have

$$b' = 2\,a \sin.\theta' \sin.\theta + b\,(\sin.\theta \cos.\theta' + \sin.\theta' \cos.\theta) + 2\,c \cos.\theta' \cos.\theta,$$

$$= \cos.\theta' \cos.\theta\,\{2\,a \tan.\theta' \tan.\theta + b\,(\tan.\theta' + \tan.\theta) + 2\,c\}.$$

From the eauation involving $\tan.\theta$, we have

$$\tan.\theta\,.\,\tan.\theta' = \frac{c}{a},\ \tan.\theta + \tan.\theta' = -\frac{b}{a},$$

$$\text{and therefore} \quad \cos.\theta \cos.\theta' = \frac{a}{\sqrt{(a-c)^2 + b^2}};$$

$$\therefore\ b' = \frac{a}{\sqrt{(a-c)^2 + b^2}}\left\{2\,c - \frac{b^2}{a} + 2\,e\right\} = -\frac{b^2 - 4\,a\,c}{\sqrt{(a-c)^2 + b^2}}.$$

$$\text{Also} \quad f' = \frac{a\,e^2 - c\,d^2 - b\,d\,e}{b^2 - 4\,a\,c} + f \quad (80.).$$

Hence the final equation is

$$-\frac{b^2 - 4\,a\,c}{\sqrt{(a-c)^2 + b^2}}\,x''\,y'' + \frac{a\,e^2 + c\,d^2 - b\,d\,e}{b^2 - 4\,a\,c} + f = 0.$$

208. If the original axes are oblique we must take the formulas in (56.), and then, following the above process, we find

$$\tan.\theta = \frac{\pm\sqrt{b^2 - 4\,a\,c} - b + 2\,c \cos.\omega}{2\,(a + c\,(\cos.\omega)^2 - b \cos.\omega)},$$

$$b' = \frac{-\,(b^2 - 4\,a\,c)}{\sqrt{\{(a + c - b \cos.\omega)^2 + (b^2 - 4\,a\,c)\,(\sin.\omega)^2\}}}.$$

209. The following examples relate to the reduction of the general equation referred to rectangular axes, to another equation referred to the asymptotes.

Ex 1. $y^2 - 10xy + x^2 + y + x + 1 = 0$,

$$m = \frac{1}{8},\ n = \frac{1}{8},\ f' = \frac{9}{8};\ \text{tan. } \theta = 5 \pm 2\sqrt{6},\ b' = -\frac{48}{5};$$

$$\therefore\ -\frac{48}{5} x'' y'' + \frac{9}{8} = 0,$$

$$\text{or, } x'' y'' = \frac{15}{128}.$$

Ex. 2. $4y^2 - 8xy - 4x^2 - 4y + 28x - 15 = 0$,

$$b' = -8\sqrt{2},\ f' = 2,$$

$$\therefore\ -8\sqrt{2}\, x'' y' + 2 = 0,$$

$$\text{or, } x'' y'' = \frac{1}{4\sqrt{2}}.$$

Ex. 3. $\dfrac{a}{x} + \dfrac{b}{y} = 1$, or $xy = ay + bx$.

The axes are here parallel to the asymptotes (198.): in order to transfer the origin to the centre, let $y = y' + n$ and $x = x' + m$, hence we have $m = a$, $n = b$, and the reduced equation is

$$x' y' = ab.$$

210. If θ and θ' be the angles which the asymptotes make with the original rectangular axes, we have from the equation (206.),

$$a\,(\text{tan. } \theta)^2 + b\,\text{tan. } \theta + c = 0,$$

$$\therefore\ \text{tan. } \theta.\ \text{tan.} \quad = \frac{c}{a}$$

Now when $c = -a$, this equation becomes $\text{tan. } \theta.\ \text{tan. } \theta' = -1$, or, $\text{tan. } \theta.\ \text{tan. } \theta' + 1 = 0$; hence by (47.), the angle between the asymptotes is in this case $= 90°$; and thus whenever, in the general hyperbolic equation, we have $c = -a$, the curve is a rectangular hyperbola.

Ex. 4. $y^2 - x^2 = \sqrt{2}$.

The curve is a rectangular hyperbola, and is referred to its centre and rectangular axes; also taking the two values of tan. θ in (206.), we have $\text{tan. } \theta = 1$, and $\text{tan. } \theta' = -1$; hence $\theta = 45°$ and $\theta' = -45°$, and the formulas of transformation become

$$y = \frac{x' - y'}{\sqrt{2}},\ x = \frac{x' + y'}{\sqrt{2}};$$

$$\therefore\ \left(\frac{x' - y'}{\sqrt{2}}\right)^2 - \left(\frac{x' + y'}{\sqrt{2}}\right)^2 = \sqrt{2},$$

$$\text{or } \quad -2x'y' = \sqrt{2},$$

$$\text{and } x'y' = -\frac{1}{\sqrt{2}}.$$

In this example the curve is placed as in the next figure, and at first was referred to the axes C X and C Y, but now is referred to the asymptotes C x and C y, supposing C y and C x to change places, and the angle x C y = 90°.

211. Conversely given the equation $x\,y = k^2$, to find the equation referred to the rectangular axes, and thence to deduce the lengths of the axes.

For this purpose we use the formulas of transformation from oblique to rectangular axes (56.).

$$y = \frac{x' \sin.\,\theta + y' \cos.\,\theta}{\sin.\,\omega},$$

$$x = \frac{x' \sin.\,(\omega - \theta) - y' \cos.\,(\omega - \theta)}{\sin.\,\omega};$$

substituting these values in the equation $x\,y = k^2$, we have

$$x'^2 \sin.\,\theta \sin\,(\omega - \theta) - y'^2 \cos.\,\theta \cos.\,(\omega - \theta)$$
$$+ x' y' \{\cos.\,\theta \sin.\,(\omega - \theta) - \sin.\,\theta \cos.\,(\omega - \theta)\} = k^2 (\sin.\,\omega)^2.$$

Let the co-efficient of $x'\,y' = 0$,

$\therefore$ $\cos.\,\theta \sin.\,(\omega - \theta) - \sin.\,\theta \cos.\,(\omega - \theta)$, or $\sin.\,(\omega - 2\,\theta) = 0$;

$$\therefore\ \omega = 2\,\theta, \text{ and } \theta = \frac{\omega}{2};$$

hence the new rectangular axis of x, determined by the angle θ, bisects the angle ω between the asymptotes; this agrees with the remark at the end of (179.).

The transformed equation, putting $\theta = \frac{\omega}{2}$, is

$$x'^2 \left(\sin.\,\frac{\omega}{2}\right)^2 - y'^2 \left(\cos.\,\frac{\omega}{2}\right)^2 = k^2 (\sin.\,\omega)^2,$$

or, putting $2 \sin.\,\frac{\omega}{2} \cos.\,\frac{\omega}{2}$ for $\sin.\,\omega$, and dividing

$$\frac{y'^2}{4\,k^2 \left(\sin.\,\frac{\omega}{2}\right)^2} - \frac{x'^2}{4\,k^2 \left(\cos.\,\frac{\omega}{2}\right)^2} = -1;$$

Comparing this with the equation $\frac{y^2}{b^2} - \frac{x^2}{a^2} = -1$, we have

$$a = 2\,k \cos.\,\frac{\omega}{2}, \text{ and } b = 2\,k \sin.\,\frac{\omega}{2};$$

hence the lengths of the semi-axes are determined.

If the equation had been $x y + a x + b y + c = 0$, first refer the curve to its centre, and then proceed as above.

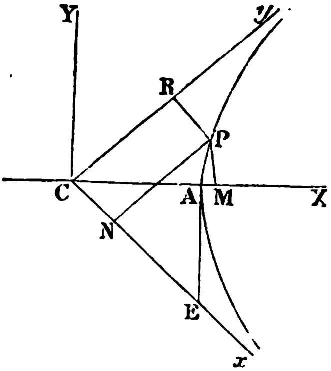
212. To deduce the equation $x\,y = k^2$ from the equation to the curve referred to the centre and rectangular axes.

Let C X, C Y be the rectangular axes,

C x, C y the asymptotes, or the new axes,

$\left.\begin{matrix} \text{C M} = x \\ \text{M P} = y \end{matrix}\right\}$ the original co-ordinates of P,

$\left.\begin{matrix} \text{C N} = x' \\ \text{N P} = y' \end{matrix}\right\}$ the new co-ordinates of P.



Then taking the formulas of transformation from rectangular to oblique axes (57),

$$y = x' \sin.\theta + y' \sin.\theta',$$
$$x = x' \cos.\theta + y' \cos.\theta',$$

and substituting in the equation $a^2 y^2 - b^2 x^2 = - a^2 b^2$, we have

$$a^2 (x' \sin.\theta + y' \sin.\theta')^2 - b^2 (x' \cos.\theta + y' \cos.\theta')^2 = - a^2 b^2,$$
$$\text{or, } \{a^2 (\sin.\theta')^2 - b^2 (\cos.\theta')^2\} y'^2 + \{a^2 (\sin.\theta)^2 - b^2 (\cos.\theta)^2\} x'^2,$$
$$+ 2 \{ a^2 \sin.\theta \sin.\theta' - b^2 \cos.\theta \cos.\theta' \} x' y' = - a^2 b^2.$$

In order that this equation may be of the required form, it must not contain the terms in x'^2 and y'^2; but since we have introduced two indeterminate quantities, we can make the two suppositions that the co-efficients of these terms shall $= 0$;

$$\therefore\ a^2 (\sin.\theta')^2 - b^2 (\cos.\theta')^2 = 0,$$
$$a^2 (\sin.\theta)^2 - b^2 (\cos.\theta)^2 = 0,$$

From the last of these equations we have $\tan.\theta = \pm \dfrac{b}{a}$, and as we obtain from the other equation the same value of $\tan.\theta'$, it follows that the values of θ and θ' are both contained in the equation $\tan.\theta = \pm \dfrac{b}{a}$, that is, if $\tan.\theta \left(= - \dfrac{b}{a}\right)$ refers to the axis of x, then $\tan.\theta \left(= + \dfrac{b}{a}\right)$ refers to the axis of y, (we have chosen $\tan.\theta = - \dfrac{b}{a}$ for the axis of x, in order to agree with the figure).

The equation to the curve referred to its asymptotes is now

$$2 \{a^2 \sin.\theta \sin.\theta' - b^2 \cos.\theta \cos.\theta'\} x' y' = - a^2 b^2,$$
$$\text{or, } \quad 2 \cos.\theta \cos.\theta' \{a^2 \tan.\theta \tan.\theta' - b^2\} x' y' = - a^2 b^2;$$

but since $\tan.\theta = \pm \dfrac{b}{a}$, we have

$$\cos. \theta = \frac{1}{\sqrt{1 + (\tan. \theta)^2}} = \frac{a}{\sqrt{a^2 + b^2}} = \cos. \theta';$$

$$\therefore 2 \frac{a^2}{a^2 + b^2} \left\{ - a^2 \frac{b^2}{a^2} - b^2 \right\} x' y' = - a^2 b^2,$$

$$\text{or,} \quad - \frac{4 a^2 b^2}{a^2 + b^2} x' y' = - a^2 b^2;$$

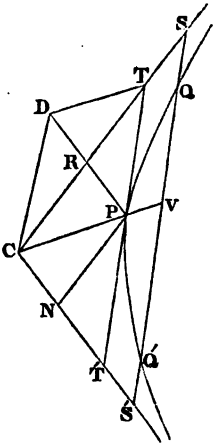
$$\therefore x' y' = \frac{a^2 + b^2}{4}.$$

If $b = a$, or the curve be the rectangular hyperbola, the equation referred to the asymptotes is $x y = \frac{a^2}{2}$.

213. The angle between the asymptotes is 2θ; if therefore P R be drawn parallel to C N, the area $\text{PNCR} = x y \sin. 2 \theta = x y \,.\, 2 \sin. \theta \cos. \theta = \frac{a^2 + b^2}{4} \,.\, 2 \,.\, \frac{b}{\sqrt{a^2 + b^2}} \,.\, \frac{a}{\sqrt{a^2 + b^2}} = \frac{a b}{2}$.

Thus all the parallelograms constructed upon co-ordinates parallel to the asymptotes are equal to each other, and to half the rectangle in the semi-axes.

214. Let C S, C S′ be the asymptotes to the curve referred to conjugate diameters C P, C D $(a_1 b_1)$, then if P T be parallel to C D, it is a tangent at P (187.); T P T′ is also a double ordinate to the asymptote, for the equation to C S is $y = \pm \frac{b_1}{a_1} x$, and when $x = a_1, y = \pm b_1$. Hence P T = P T′, or the parts of the tangent contained between the point of contact and the asymptotes are equal to each other, and to the conjugate diameter.



215. Join D T, then D P is a parallelogram; also because C D is equal and parallel to P T′, we have the line D P parallel to the asymptote

C S′. Hence, if the conjugate diameters be given, the asymptotes may always be found by completing the parallelogram upon the conjugate diameters, and then drawing the diagonals. Also, if the asymptotes be given, a conjugate diameter to C P may be found by drawing P R parallel to C S′ and taking P D double of P R.

If the asymptotes be given, a tangent may be drawn by taking C T double of C R, and joining P T.

If the position of the focus is known, the length of the conjugate axis is equal to the perpendicular, from the focus on the asymptote.

216. To find the equation to the tangent P T, when referred to the asymptotes as axes,

Let x', y' be the co-ordinates of P, and x'' y'' co-ordinates of another point on the curve.

$$\therefore\ y' = \frac{k^2}{x'},\ \text{and}\ y'' = \frac{k^2}{x''}$$

$$\therefore\ \frac{y' - y''}{x' - x''} = -\ \frac{k^2}{x'\,x''} = -\ \frac{y'}{x''}$$

$$\therefore\ y - y' = -\ \frac{y'}{x''}\,(x - x')\ \text{is the equation to a secant.}$$

When $x'' = x'$ we have the equation to the tangent

$$y - y' = -\ \frac{y'}{x'}\,(x - x') = -\,\frac{y'}{x'}\,x + y'$$

$$\therefore\ x'y + y'\,x = 2\,x'y' = 2\,k^2.$$

This equation to the tangent is readily obtained from the equation to the curve ($xy = k^2$ or $xy + xy = 2k^2$) by putting $x'y$ and xy' successively for xy, and then adding the results.

Let $y = 0$ $\therefore$ C T′ $= 2\,x' = 2$ C N; and C T $= 2\,y' = 2$ N P;

The triangle C T T′ $= \frac{1}{2}.\ 2\,x'.\ 2\,y'$ sin. T C T′ $=\ 2\,x'\,y'$ sin. $2\,\theta = a\,b$, (213.)

217. The two parts S Q, S′Q′ of any secant S Q Q′S comprised between the curve and its asymptote are equal; for if the diameter C P V and its conjugate C D be drawn, we have V Q = V Q′ from the equation to the curve $\left(y = \pm\,\frac{b_1}{a_1}\,\sqrt{x^2 - a_1^2}\right)$, and from the equation to the asymptotes $\left(y = \pm\,\frac{b_1}{a_1}\,x,\right)$ we have V S = V S′ $\therefore$ S Q = S′ Q′.

218. If Y and y are the ordinates V S, V Q respectively, we have

$$Y^2 - y^2 = \frac{b_1^2}{a_1^2}\,x^2 - \frac{b_1^2}{a_1^2}\,(\,x^2 - a_1^2)$$

$$\text{or}\ (Y - y)\,(Y + y) = b_1^2.$$

Thus the rectangle S Q, Q S′ = the square upon C D.

THE POLAR EQUATION.

219. Let the curve be referred to the centre C, and to rectangular axes C A, C B, and let the co-ordinates of the pole O be x' and y', O being situated anywhere in the plane of the curve and P any point on the curve, as in (146.), θ the angle which the radius vector O P or u makes with a line parallel to the axis of x. Then we have by (61.)

$$y = y' + u \sin.\theta$$
$$x = x' + u \cos.\theta$$
$$\text{also,}\quad a^2 y^2 - b^2 x^2 = -a^2 b^2$$
$$\therefore\ a^2 (y' + u \sin.\theta)^2 - b^2 (x' + u \cos.\theta)^2 = -a^2 b^2$$

220. Let the centre be the pole, $\therefore\ x' = 0$, and $y' = 0$,

$$\therefore\ u^2 = \frac{-a^2 b^2}{a^2 (\sin.\theta)^2 - b^2 (\cos.\theta)^2} = \frac{a^2 (e^2 - 1)}{e^2 (\cos.\theta)^2 - 1.}$$

221. Let the focus S be the pole,

$$\therefore\ y' = 0, x' = ae \text{ and } u \text{ becomes } r,$$

Substituting these values, and following the steps in (148.), we find

$$r = \frac{b^2}{a - c\cos.\theta} = \frac{a(e^2 - 1)}{1 - e\cos.\theta}$$

If the angle A S P $= \theta$, we have

$$r = \frac{a(e^2 - 1)}{1 + e\cos.\theta}$$

This is the equation generally used. It may easily be obtained from the equation $r = ex - a$, fig. (161) $= e(ae - r\cos.\theta) - a$,

$$\therefore\ r = \frac{a(e^2 - 1)}{1 + e\cos.\theta}.$$

222. If $\frac{p}{2} = a(e^2 - 1)$ we have $r = \frac{p}{2} \cdot \frac{1}{1 + e\cos.\theta} = \text{S P}$, and if P S meet the curve again in P$'$, we have the rectangle S P, S P$' = \frac{p}{4}$ (P S + S P$'$) $= \frac{p}{4}$ P P$'$.

The length of the chord through the focus $= 2\frac{b_{,}^2}{a}$ where $b_{,}$ is the diameter to that chord.

THE CONJUGATE HYPERBOLA.

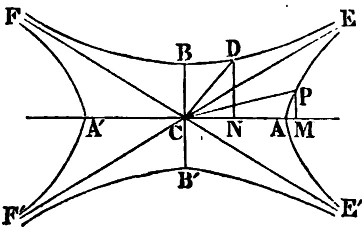
223. There is another equation to the hyperbola, not yet investigated. If $\left(\frac{c'}{-f'}\right)$ be negative in article 153, the equation is $Py^2 - Qx^2 = 1$,

or $a^2 y^2 - b^2 x^2 = a^2 b^2$ if $P = \frac{1}{b^2}$, and $Q = \frac{1}{a^2}$. If we examine the course of this curve, we shall find that $BB' = 2b$ is the real or transverse axis, and AA', or $2a$, is the conjugate axis, and that the curve extends indefinitely from B to B′, so that it is, in form, like the hyperbola already investigated, but only placed in a different manner.

Both curves are represented in the next figure; the real axis of the one being the conjugate or imaginary axis of the other.

It is evident from the form of the equations that both curves have got common asymptotes E C E′, F C F′.

224. Let C P and C D be two conjugate diameters to the original hyperbola A P E, it is required to find the locus of D.



Let $CM = x'$, $MP = y'$, $CN = x$, $ND = y$,

$$\text{then } a_1^2 - b_1^2 = a^2 - b^2;$$

$$\therefore x'^2 + y'^2 = x^2 + y^2 + a^2 - b^2$$

but the equation to C D is

$$a^2 y y' - b^2 x x' = 0 \quad \therefore x' = \frac{a^2 y y}{b^2 x},$$

$$\therefore x'^2 + y'^2 = \frac{a^4 y^2 + b^4 x^2}{b^4 x^2} y'^2 = x^2 + y^2 + a^2 - b^2,$$

$$\therefore y'^2 = \frac{b^4 x^2}{a^4 y^2 + b^4 x^2}(x^2 + y^2 + a^2 - b^2),$$

$$\text{and } x'^2 = \frac{a^4 y^2}{a^4 y^2 + b^4 x^2}(x^2 + y^2 + a^2 - b^2).$$

Substituting these values in the equation $a^2 y'^2 - b^2 x'^2 = - a^2 b^2$, and reducing, we have $a^2 y^2 - b^2 x^2 = a^2 b^2$, hence the locus of D is the conjugate hyperbola, and hence arises its name.

By changing the sign of the constant term in the equation to any hyperbola, referred to its centre, we directly obtain the equation to its conjugate, referred to the same axes of x and y. Both curves are comprised in the form

$$(a^2 y^2 - b^2 x^2)^2 = a^4 b^4, \text{ or } x^2 y^2 = k^4.$$

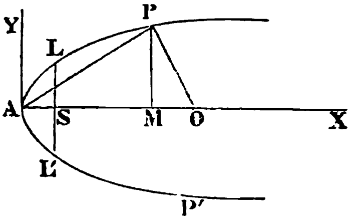
CHAPTER X.

THE PARABOLA.

225. THE equation to the parabola, referred to rectangular axes, has been reduced to the form $a' y^2 + e' x = 0$ (94.).

From this equation we proceed now to deduce all the important properties of the parabola.

Let $\frac{e'}{-a'} = p, \quad \therefore y^2 = p x.$



Let A be the origin; A X, A Y the axes; then for $x = 0$ we have $y = 0$, and the curve passes through the origin A.

For each positive value of x there are two equal and opposite values of y, which increase from 0 to ∞, according as x increases from 0 to ∞; hence there are two equal arcs, A P and A P', proceeding from A, without any limit. This curve is symmetrical with respect to its axis A X, and its concavity is turned towards that axis, otherwise it could be cut by a straight line in more points than one.

For every negative value of x, y is imaginary.

226. The point A is called the vertex of the parabola; A X, A Y the principal axes; but, generally speaking, A X alone is called the **Axis** of the parabola. Thus the equation to the curve referred to its axis and vertex is $y^2 = p x$.

From this equation we have The square upon the ordinate = The rectangle under the abscissa and a constant quantity; or the square upon the ordinate varies as the abscissa.

227. The last property of this curve points out the difference between the figures of the hyperbola and parabola; both have branches extending to infinity, but of a very different nature; for the equation to the hyperbola is $y^2 = \frac{b^2}{a^2}(x^2 - a^2) = \frac{b^2}{a^2} x^2 \left(1 - \frac{a^2}{x^2}\right)$, and therefore, for large values of x, the values of y^2 increase nearly as the corresponding values of x^2 or y varies nearly as x; hence the hyperbolic branch rises much more rapidly than that of the parabola, whose ordinate varies only as $\sqrt{x}$. When x is very great, the former takes nearly the course of the line $y = \frac{b}{a} x$, but in the parabola, y is not much increased by an increase of x, and therefore the curve tends rather towards parallelism with the axis of x.

228. The equation to the parabola may be derived from that of the ellipse by considering the axis major of the ellipse to be infinite.

Let C be the centre, and S the focus of an ellipse whose equation is

$$y^2 = \frac{2\,b^2}{a}\,x - \frac{b^2}{a^2}\,x^2. \quad (105.)$$

$$\text{Let } m = \text{A S} = \text{A C} - \text{S C} = a - \sqrt{a^2 - b^2}, \text{ (fig. 106.)}$$

$$\therefore b^2 = 2\,a\,m - m^2;$$

$$\therefore y^2 = \left(4\,m - \frac{2\,m^2}{a}\right)x - \left(\frac{2\,m}{a} - \frac{m^2}{a^2}\right)x^2.$$

Now if a be considered to vary, this will be the equation to a series of ellipses, in which the distance A S, or m, is the same for all, but the axis major different for each; thus giving to a any particular value, we have a corresponding ellipse. Let now a be infinite, then, since all the other terms vanish, the equation becomes $y^2 = 4\,m\,x$; hence the ellipse has gradually approached to the parabolic form, as its axes enlarged, and finally coincided with it when the axis major was infinite *.

In the same manner the equation to the parabola may be derived from that to the hyperbola.

THE FOCUS.

229. The quantity p, which is the co-efficient of x in the equation to the parabola, is called the principal parameter, or Latus Rectum of the parabola.

Since $p = \frac{y^2}{x}$, the principal parameter is a third proportional to any abscissa and its corresponding ordinate.

In article (228.) we have used the equation $y^2 = 4\,m\,x$ for the parabola, merely to avoid fractions with numerical denominators; it appears that many of the operations in this chapter are similarly shortened, without losing any generality, by merely putting $4\,m$ for p; hence we shall use the equation $y^2 = 4\,m\,x$ in most of the following articles, recollecting that all the results can be expressed in terms of the principal parameter, by putting $\frac{p}{4}$ for m wherever m occurs.

230. To find the position of the double ordinate which is equal to the Latus Rectum.

Let $2\,y = 4\,m$, $\quad\therefore 4\,y^2 = 16\,m^2$, or $16\,m\,x = 16\,m^2$, and $x = m$.

In A X take A S $= m$, then the ordinate L S L′ drawn through S, is the Latus Rectum.

The point S is called the focus.

The situation of the focus S may be also thus determined:

Let A M $= x$, M P $= y$, join A P, and draw P O perpendicular to A P,

$$\text{Then A M} : \text{M,P} :: \text{M P} : \text{M O} = \frac{y^2}{x} = 4\,m, \therefore \text{A S} = m = \tfrac{1}{4}\,\text{M O}.$$

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* If x is very small when compared with a, the equation to the ellipse is very nearly that to a parabola; and this is the reason that the path of a comet near its perihelion appears to be a portion of a parabola.

231. To find the distance of any point P in the curve from the focus:

Let $SP = r$, $AM = x$, $MP = y$; also at s, $y' = 0$, and $x' = m$,

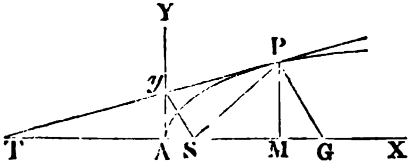
$$\therefore r^2 = (y - y')^2 + (x - x')^2 = y^2 + (x - m)^2 = 4mx + (x - m)^2$$
$$= (x + m)^2;$$
$$\therefore r = SP = x + m.$$

THE TANGENT.

232. To find the equation to the tangent at any point P (x', y') of the parabola.

The equation to a secant through two points on the curve (x', y') $(x'' y'')$ is

$$y - y' = \frac{y' - y''}{x' - x''}(x - x'),$$



Also $y'^2 = 4mx'$, and $y''^2 = 4mx''$;

$$\therefore y'^2 - y''^2 = 4m(x' - x''),$$

and $$\frac{y' - y''}{x' - x''} = \frac{4m}{y' + y''}.$$

Thus the equation to the secant becomes

$$y - y' = \frac{4m}{y' + y''}(x - x');$$

but when the two points coincide $y'' = y'$, and the secant becomes a tangent,

$$\therefore y - y' = \frac{4m}{2y'}(x - x'),$$

$$\text{or } yy' - y'^2 = 2m(x - x'),$$

$$\therefore yy' = y'^2 + 2m(x - x') = 4mx' + 2m(x - x'$$

$$\therefore yy' = 2m(x + x').$$

This equation is immediately deduced from that to the curve

$$(y^2 = 4mx = 2m(x + x))$$

by writing yy' for y^2, and $x + x'$ for $x + x$.

233. To find the points where the tangent cuts the axes.

Let $y = 0$, $\therefore x + x' = 0$, $\therefore x = -x'$, or $AT = -AM$;
Hence the absolute value of the sub-tangent MT is 2 AM.

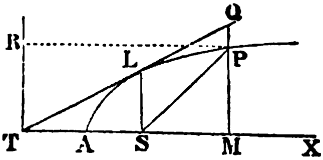
Let $x = 0$, $\therefore\ y = 2m\dfrac{x'}{y'} = \dfrac{y'^2}{2y'} = \dfrac{y'}{2}$, $\quad \therefore\ \text{A}y = \frac{1}{2}\,\text{MP}$.

234. The equation to the tangent being $yy' = 2m(x + x')$, we have at the vertex A, x' and y' each $= 0$, therefore the equation to the tangent becomes $2mx = 0$, or $x = 0$;

But $x = 0$ is the equation to the axis A Y;

Hence the tangent at the vertex of the parabola coincides with the axis of y.

235. To find the equation to the tangent at the extremity of the principal parameter.



$$yy' = 2m(x + x')$$

At L we have $x' = m$, and $y' = 2m$,

$$\therefore\ 2my = 2m(x + m),$$

$$\therefore\ y = x + m,$$

If the ordinate y or M Q cut the parabola in P, we have $\text{SP} = x + m$ (231.), $\therefore\ \text{MQ} = \text{SP}$.

236. To find the point where this particular tangent cuts the axis of x.

$$\text{Let } y = 0,\ \therefore\ x = \text{AT} = -m = -\text{AS}.$$

From T draw T R perpendicular to A X, and from P draw P R parallel to A X, then taking the absolute value of A T, we have

$$\text{PR} = \text{AT} + \text{AM} = m + x = \text{SP}.$$

Consequently the distances of any point P from S, and from the line T R, are equal to one another.

This line, T R, is called the directrix; for knowing the position of this line and of the focus, a parabola may be described.

This tangent cuts the axis at an angle of 45°. (36 Ex. 3.)

237. To find the length of the perpendicular S y from the focus on the tangent.

Taking the expression in (48.) we have

$$\text{S}y = -\frac{y_1 - \alpha x_1 - b}{\sqrt{1 + \alpha^2}}.$$

But from the figure 232, we have $y_1 = 0$, and $x_1 = m$ for the coordinates of the point S, and $y = \alpha x + b$ is the equation to the line P T; also the equation to P T is

$$y = \frac{2m}{y'}(x + x'),$$

$$\therefore\ \alpha = \frac{2m}{y'},\ \text{and } b = \frac{2mx'}{y'},$$

$$\therefore\ \text{S}y = \frac{\dfrac{2m}{y'}m + \dfrac{2mx'}{y'}}{\sqrt{\left\{1 + \dfrac{4m^2}{y'^2}\right\}}} = \frac{2m(m + x')}{\sqrt{\{y'^2 + 4m^2\}}} = \frac{2m(m + x')}{\sqrt{\{4mx' + 4m^2\}}}$$

$$= \sqrt{m(m + x')} = \sqrt{mr},\ \text{if SP} = r;$$

Hence the square on S y = the rectangle S P, S A;

or, S P : S y :: S y : S A.

238. To find the locus of y in the last article.

The equation to the tangent P T, fig. 232, is $y = \frac{2m}{y'}(x + x')$;

Hence the equation to S y passing through the point $(m, 0)$, and perpendicular to P T, is

$$y = -\frac{y'}{2m}(x - m).$$

To find where this line cuts the axis of y, put $x = 0$, $\therefore y = \frac{y'}{2}$, but this is the point where the tangent at P cuts the same axis (233.); hence the tangent and the perpendicular on it from the focus meet in the axis A Y, or the locus of y is the axis A Y.

239. Again, to find where the perpendicular S y cuts the directrix, put

$$x = -m, \therefore y = -\frac{y'}{2m}(x - m) = -\frac{y'}{2m}(-m - m) = y',$$

but this is the ordinate M P; hence a tangent being drawn at any point P, the perpendicular on it from the focus cuts the directrix in the point where the perpendicular from P on the directrix meets that directrix.

240. To find the angle which the tangent makes with the focal distance.

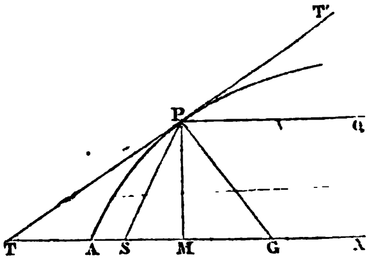
The equation to the tangent P T is $y = \frac{2m}{y'}(x + x')$.

The equation to the focal distance S P through the points S $(= 0, m)$ and P $(= x', y')$ is

$$y = \frac{y'}{x' - m}(x - m),$$

And tan. S P T = tan. (P S X − P T X)

$$= \frac{\frac{y'}{x' - m} - \frac{2m}{y'}}{1 + \frac{y'}{x' - m}\cdot\frac{2m}{y'}}$$



$$= \frac{y'^2 - 2m(x' - m)}{y'(x' - m) + 2my'} = \frac{4mx' - 2mx' + 2m^2}{y'(x' + m)} = \frac{2m(x' + m)}{y'(x' + m)}$$

$$= \frac{2m}{y'} = \frac{y'}{2x'}, \text{ since } 2m = \frac{y'^2}{2x'}:$$

But M P = M T tan. P T M, $\therefore$ tan. P T M $= \frac{y'}{2x'}$, $\therefore$ tan. S P T

$=$ tan. S T P $=$ tan. T′ P Q, if P Q be drawn parallel to the axis of x. Thus the tangent at P makes equal angles with the focal distance, and with a parallel to the axis through P.

This important theorem may also be deduced from the property in article 233. It is there proved that the absolute value of A T is A M, hence we have $\mathrm{S\,T} = \mathrm{S\,A} + \mathrm{A\,T} = m + x = \mathrm{S\,P}$, and therefore the angle S P T $=$ angle S T P $=$ angle Q P T′.

If a ray of light, proceeding in the direction Q P, be incident on the parabola at P, it will be reflected to S, on account of the equal angles Q P T′, S P T: similarly all rays coming in a direction parallel to the axis, and incident on the curve, would converge to S; and if a portion of the curve revolve round its axis, so as to form a hollow concave mirror, all rays from a distant luminous point in the direction of the axis would be concentrated in S. Thus, if a parabolic mirror be held with its axis pointing to the sun, a very powerful heat will be found at the focus.

Again, if a brilliant light be placed in the focus of such a mirror, all the rays, instead of being lost in every direction, will proceed in a mass parallel to the axis, and thus illuminate a very distant point in the direction of that axis. This property of the curve has led to the adoption of parabolic mirrors in many light-houses.

THE NORMAL.

241. To find the equation to the normal P G, at a point P $(x'y')$.

The equation to a straight line, through P, is $y - y' = \alpha\,(x - x')$, and as this line must be perpendicular to the tangent whose equation is $y = \frac{2\,m}{y'}\,(x + x')$, we have $\alpha = -\frac{y'}{2\,m}$, hence the equation to the normal is $y - y' = -\frac{y'}{2\,m}\,(x - x')$.

242. To find the point where the normal cuts the axis of x.

Let $y = 0 \quad \therefore\ x - x' = 2m$, or the subnormal M G is constant and equal to half the principal parameter.

Hence $\mathrm{S\,G} = \mathrm{S\,M} + \mathrm{M\,G} = x' - m + 2\,m = x' + m = \mathrm{S\,P}$.

And $\mathrm{P\,G} = \sqrt{y'^2 + 4\,m^2} = \sqrt{4\,m\,x' + 4\,m^2} = \sqrt{4\,m\,(x' + m)} = \sqrt{4\,m\,r}$.

Hence the normal P G is a mean proportional between the principal parameter and the distance S P.

THE DIAMETERS.

243. It was shown in article 81, that the parabola has no centre.

Since for every positive value of x there are two equal and opposite values of y, the axis of x is a diameter, but that of y is not; hence the axes cannot be called conjugate axes. The parabola has an infinite number of diameters, all parallel to the axis; to prove this,

Let $y = \alpha\,x + b$ be the equation to any chord,

$y^2 = p\,x$ the equation to the curve.

Transfer the origin to the bisecting point $x'y'$ of the chord, then the equations become $y = \alpha\,x$, and $(y + y')^2 = p\,(x + x')$:

To find where the chord intersects the curve, put αx for y in the second equation.

$$\therefore\ (\alpha x + y')^2 = p(x + x')$$

$$\text{or } \alpha^2 x^2 + (2\alpha y' - p)x + y'^2 - p x' = 0$$

But since the origin is at the bisection of the chord, the two values of x must be equal to one another, and have opposite signs; hence the second term of the last equation must $= 0$, $\therefore\ 2\alpha y' - p = 0$.

This equation gives the value of y', and since it is independent of b, it will be the same for any chord parallel to $y = \alpha x + b$; hence $y = \dfrac{p}{2\alpha}$ is the equation to the locus of all the middle points of a system of parallel chords, and this equation is evidently that to a straight line parallel to the axis; and conversely.

244. To transform the equation into another referred to a new origin and to new axes, and so that it shall preserve the same form,

$$\text{Let } x = a + x' \cos.\theta + y' \cos.\theta',$$

$$\text{and}\quad y = b + x' \sin.\theta + y' \sin.\theta', \quad (57.)$$

Substituting these values in the equation $y^2 = p x$ and arranging, we have

$$y'^2 (\sin.\theta')^2 + x'^2 (\sin.\theta)^2 + 2x'y' \sin.\theta \sin.\theta' + y'(2b \sin.\theta' - p\cos.\theta')$$
$$+ x'(2b \sin.\theta - p\cos.\theta) + b^2 - ap = 0.$$

And as this equation must be of the form $y^2 = p x$, we must have

$$(\sin.\theta)^2 = 0 \quad \ldots \quad (1),$$
$$2\sin.\theta.\sin.\theta' = 0 \quad \ldots \quad (2),$$
$$2b\sin.\theta' - p\cos.\theta' = 0 \quad (3),$$
$$b^2 - ap = 0 \quad \ldots \quad (4).$$

Hence the equation becomes

$$y'^2 (\sin.\theta')^2 + (2b\sin.\theta - p\cos.\theta)x' = 0;$$

$$\text{or since } \theta = 0,\quad y'^2(\sin.\theta')^2 - p x' = 0.$$

245. On the examination of the equations (1) (2) (3) and (4), it appears from (1) that the new axis of x' is parallel to the original axis of x; and θ being 0 from (1), of course (2) is destroyed, and thus the equations of condition are reduced to three: but there are four unknown quantities, hence there are an infinite number of points to which, if the origin be transferred, the equation may be reduced to the same simple form.

We may take the remaining three quantities a, b and θ', in any order, and arrive at the same results. Suppose a is known, then from (4), $b^2 = p\,a$, this equation shows that a must be taken in a positive direction from A, and also that the new origin must be taken on the curve itself, or the new origin is at some point P on the curve, as in the next figure.

From (3) we have $\tan.\theta = \dfrac{p}{2b} = \dfrac{b}{2a}$;

but this is exactly the value of the tangent of the angle which a tangent PT to the curve makes with the Axis (240.): hence the new axis of y is a tangent to the curve at the new origin P.

The results are therefore these,—the new origin is at any point P on the curve (see the next figure). The axes are one (P X′) parallel to the axis A X, and the other (P Y′) is a tangent at the new origin P. Lastly, from the form of the equation, the new axis of x is a diameter.

246. The equation is $y'^2 = \frac{p}{(\sin.\,\theta')^2}\,x' = p'x$ where $p' = \frac{p}{(\sin\,\theta')^2}$

$= p\,(\text{cosec.}\,\theta')^2 = p\,(1 + \cot.\,\theta')^2 = p\left(1 + \frac{4\,a^2}{b^2}\right) = p + 4\,a$

$= 4\left(\frac{p}{4} + a\right) = 4\,S\,P$ (231.)

Hence the new parameter at P is four times the focal distance S P.

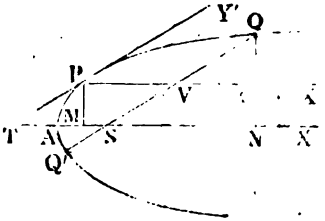
247. The equation to the parabola, when we know the position and direction of the new axes, is readily obtained from the original equation referred to rectangular co-ordinates.

Let the point P be the new origin, P X′, P Y′ the new axes, angle Y′ P X′ $= \theta$.

Also, let A N $= x$, N Q $= y$ be the rectangular co-ordinates of Q.

And A M $= a$, M P $= b$ . . . . . . P.

P V $= x'$, V Q $= y'$ be the new co-ordinates of Q.



Then $y = $ Q N $=$ M P $+$ O Q $= b + y'\sin.\,\theta$,

$x = $ A N $=$ A M $+$ P V $+$ V O $= a + x' + y'\cos.\,\theta$.

Substituting these values in the equation $y^2 = p\,x$,

we have $(b + y'\sin.\,\theta)^2 = p\,(a + x' + y'\cos\,\theta)$;

$\therefore\ y'^2\,(\sin.\,\theta)^2 + (2\,b\sin.\,\theta - p\cos.\,\theta)\,y' + b^2 = p\,a + p\,x'$;

but $b^2 = p\,a$, and $\tan.\,\theta = \tan.\,PTM = \frac{b}{2\,a} = \frac{b}{2\,\frac{b^2}{p}} = \frac{p}{2\,b}$;

$$\therefore\ 2\,b\sin.\,\theta - p\cos.\,\theta = 0,$$

and the equation is reduced to the form

$$y'^2\,(\sin.\,\theta)^2 = p\,x'.$$

Also from $(\tan.\,\theta)^2 = \frac{b^2}{4\,a^2}$ we have $(\cos.\,\theta)^2 = \frac{4\,a^2}{4\,a^2 + b^2}$.

and $(\sin.\,\theta)^2 = \frac{b^2}{4\,a^2 + b^2} = \frac{p}{4\,a + p}$;

$$\therefore y'^2 \frac{p}{4a+p} = px'$$

$$y'^2 = (4a+p)\,x' = 4\left(a + \frac{p}{4}\right)x' = p'\,x', \text{ where } p' = 4\,\mathrm{S\,P}.$$

Hence the square upon the ordinate = the rectangle under the abscissa and parameter.

248. To find the length of the ordinate which passes through the focus:

Here, $x = \mathrm{PV} = \mathrm{ST} = \mathrm{SP} = r \;\therefore\; y^2 = px = 4r\,.\,r = 4r^2$
$\therefore y = 2r.$

Hence, $\mathrm{Q\,Q'} = 4\,\mathrm{S\,P}$.

Thus the ordinate through the focus is equal to four times the focal distance S P, is equal to the parameter at the point P.

Hence, generally, if the origin of co-ordinates be at any point P on the parabola, and if the axes be a diameter and a tangent at P, the parameter to the point P is that chord which passes through the focus.

249. The equation to a tangent at any point Q $(x'\ y')$, referred to the new axes P X′, P Y′, is

$$y\,y' = \frac{p'}{2}\left(x + x'\right)$$

Let $y = 0 \;\therefore\; x = -x'$, hence the sub-tangent = twice the abscissa.

Let $x = 0 \;\therefore\; y = \dfrac{p'\,x'}{2y} = \dfrac{y'}{2} = \frac{1}{2}$ the ordinate.

For y put $-y$, then we have the equation to the tangent at the other extremity Q′ of the ordinate Q V Q′; hence it may be proved that tangents at the two extremities of a chord meet in a diameter to that chord.

250. If the chord Q V Q′ pass through the focus, as in the figure, the co-ordinates of Q are $y' = 2\,\mathrm{S\,P} = 2\,r$, and $x' = \mathrm{P\,V} = \mathrm{S\,P} = r$, also $p' = 4\,r$; hence the equation to the tangent at Q, or $y\,y' = \dfrac{p'}{2}\,(x + x')$ becomes $y = x + r$, and similarly the equation to the tangent at Q′ is $-y = x + r$, and these lines meet the axis P X′ at a distance $-r$ from P, that is, tangents at the extremity of any parameter meet in the directrix.

Also, the angle between these tangents is determined from the equation

$$\tan.\,\theta = \frac{(\alpha - \alpha')\sin.\,\omega}{1 + \alpha\alpha' + (\alpha + \alpha')\cos.\,\omega} \qquad (51.)$$

$$= \frac{(1+1)\sin.\,\omega}{1 - 1 + 0} \text{ (since } \alpha = 1 \text{ and } \alpha' = -1)$$

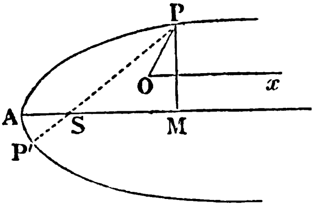
$$= \frac{1}{0} = \tan.\,90^\circ.$$

Hence, pairs of tangents drawn at the extremities of any parameter meet in the directrix at right-angles.

THE POLAR EQUATION.

251. To find the polar equation to the curve.

Let the co-ordinates of any point O be x' and y', and let θ be measured from a line O x, which is parallel to the axis of the curve:



Then by (61.), or by inspection of the figure, we have

$$y = y' + u \sin. \theta$$

$$x = x' + u \cos. \theta$$

Substituting these values of x and y in the equation $y^2 = p\,x$, we have

$$(y' + u \sin. \theta)^2 = p\,(x' + u \cos. \theta)$$

252. Let the pole be at any point on the curve,

$$\therefore\ y'^2 + 2\,u\,y' \sin. \theta + u^2 (\sin. \theta)^2 = p\,x' + p\,u \cos. \theta;$$

$$\text{or, } u\,(\sin. \theta)^2 = p \cos. \theta - 2\,y' \sin. \theta, \quad \text{since } y'^2 = p\,x';$$

$$\therefore\ u = \frac{p \cos. \theta - 2\,y' \sin. \theta}{(\sin. \theta)^2},$$

And if the vertex be the pole, we have $y' = 0$;

$$\therefore\ u = \frac{p \cos. \theta}{(\sin. \theta)^2}.$$

253. Let the focus S be the pole, $\therefore\ y' = 0,\ x' = \frac{p}{4}$ and u becomes r; hence the general equation $(y' + u \sin. \theta)^2 = p\,(x' + u \cos. \theta)$ becomes

$$r^2 (\sin. \theta)^2 = \frac{p^2}{4} + p\,r \cos. \theta,$$

$$\text{or } r^2 (\sin. \theta)^2 + r^2 (\cos. \theta)^2 = \frac{p^2}{4} + p\,r \cos. \theta + r^2 (\cos. \theta)^2$$

$$\therefore\ r^2 = \left(\frac{p}{2} + r \cos. \theta\right)^2$$

$$r = \frac{p}{2} + r \cos. \theta; \quad \text{or } r = \frac{p}{2} \cdot \frac{1}{1 - \cos. \theta}.$$

The polar equation in this case is also easily deduced from article (231). Let angle A S P $= \theta$,

then $r = \text{S P} = \text{A M} + \text{A S} = 2\,\text{A S} + \text{S M} = \frac{p}{2} - r \cos. \theta$;

$$\therefore\ r = \frac{p}{2}\,\frac{1}{1 + \cos. \theta} = \frac{p}{\left(\cos. \frac{\theta}{2}\right)^2}$$

254. If P S meet the curve again in P′, we have S P′

$$\frac{p}{2} \cdot \frac{1}{1 - \cos.(\pi - \theta)} = \frac{p}{2} \; \frac{1}{1 + \cos.\theta};$$

hence the rectangle

$$\text{P S, S P}' = \frac{p}{4} \; \frac{p}{1 - (\cos.\theta)^2} = \frac{p}{4} (\text{S P} + \text{S P}') = \frac{p}{4}, \text{P P}'.$$

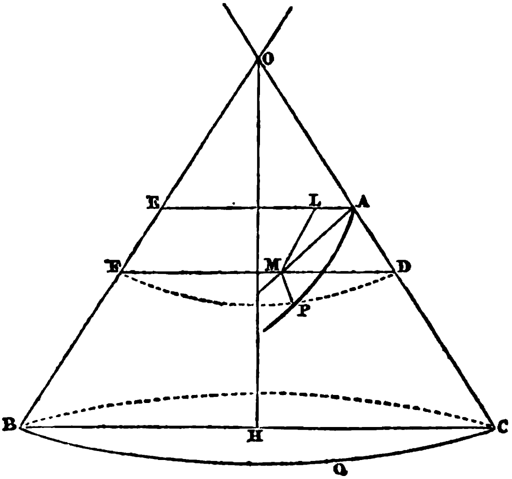
CHAPTER XI.

THE SECTIONS OF A CONE.

255. It is well known that the three curves, the ellipse, the hyperbola, and parabola, were originally obtained from the section of a cone, and that hence they were called the conic sections. We shall now show the manner in which a cone must be cut by a plane, in order that the section may be one of these curves.

A right cone is the solid generated by the revolution of a right-angled triangle about one of its perpendicular sides.

The fixed side, O H, about which the triangle revolves, is called the axis; and the point O, where the hypothenuse of the triangle meets the axis, is called the vertex of the cone. If the revolving hypothenuse be produced above the vertex, it will describe another cone, having the same axis and vertex. Any point in the hypothenuse of the triangle describes a circle; hence, the base of the triangle describes a circular area called the base of the cone.



Section made by planes which pass through the vertex and along the axis are called vertical sections; these are, evidently, triangles.

If a plane pass through the cone in any direction, the intersection of it with the surface of the cone is called a conic section. The nature of the line thus traced will be found to be different, according to the various positions of the cutting plane. It is our purpose to show, generally, to what class of curves a section must necessarily belong; and, afterwards, to point out the particular species of curve due to a given position of the cutting plane.

256. Let O B Q C be a right cone, O the vertex, O H the axis, B C Q the circular base, P A the line in which the cutting plane meets the surface of the cone; A being the point in the curve nearest to the vertex O. Let O B H C A be a vertical plane passing through the axis O H and perpendicular to the cutting plane P A M.

A M, the intersection of these planes, is a straight line, and is called the axis of the conic section, the curve being symmetrically placed with regard to it.

Let F P D be a section parallel to the base, it is therefore a circle, and F M D, its intersection with the vertical plane O B H C A, is a diameter.

Since both this last plane F P D and the cutting plane P A M are perpendicular to the vertical plane O B H C, M P the intersection of the two former is perpendicular to the vertical plane, (Euc. xi. 19, or Geometry iv. 18,) and, therefore, to all lines meeting it in that plane. Hence M P is perpendicular to F D and to A M.

Let the angle O A M, which is the inclination of the cutting plane to the side of the cone, $= \alpha$, and let the $\angle$ A O B $= \beta$, draw A E parallel to B H and M L parallel to O B.

Let A M $= x$, M P $= y$, and A O $= a$.

Then by the property of the circle

The square on M P = the rectangle F M, M D;

$$\text{and } MD = \frac{MA \sin. MAD}{\sin. MDA} = x \frac{\sin. \alpha}{\cos. \frac{\beta}{2}};$$

$$\text{Also, } FM = EA - AL = \frac{AO \sin. AOE}{\sin. OEA} - \frac{AM \sin. AML}{\sin. ALM};$$

But angle O E A $= 90^\circ - \frac{\beta}{2}$, angle A L M $= 90^\circ + \frac{\beta}{2}$, and if we produce M L to meet O A, we shall find that the angle A M L $= 180^\circ - (\alpha + \beta)$;

$$\text{hence } FM = a\frac{\sin. \beta}{\cos. \frac{\beta}{2}} - x\frac{\sin. (\alpha+\beta)}{\cos. \frac{\beta}{2}};$$

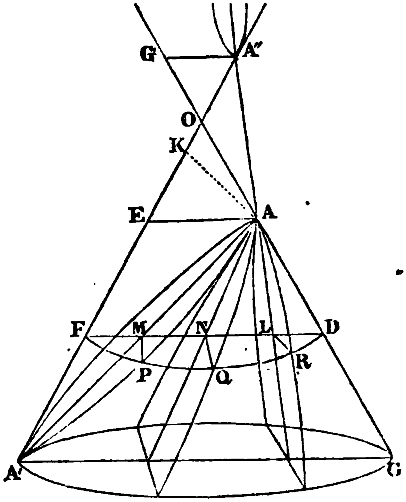
$$\therefore\ y^2 = x\frac{\sin. \alpha}{\cos. \frac{\beta}{2}}\left\{a\frac{\sin. \beta}{\cos. \frac{\beta}{2}} - x\frac{\sin. (\alpha+\beta)}{\cos. \frac{\beta}{2}}\right\}$$

$$\text{or } y^2 = \frac{\sin. \alpha}{\left(\cos. \frac{\beta}{2}\right)^2}\{a \sin. \beta . x - \sin. (\alpha+\beta)\, x^2\};$$

which equation being of the second degree, it follows that the sections of the cone are curves of the second degree.

Comparing this with the equation $y^2 = p x + q x^2$, which represents an ellipse, a parabola, or an hyperbola, according as q is negative, nothing, or positive; we observe that the section is an ellipse, a parabola, or an hyperbola, according as sin. $(\alpha + \beta)$ is positive, nothing, or negative. To investigate these various cases, we shall suppose the cutting plane to move about A, so that α may take all values from 0 to 180°.

257. Let $\alpha = 0$, $\therefore y^2 = 0$, and $y = 0$; this is the equation to the straight line which is the axis of x.



And this appears, also, from the figure; for when $\alpha = 0$, the cutting plane just touches the cone, and hence the line of intersection A M is in the position A O.

258. Let $\alpha + \beta$ be less than 180°. The curve is an ellipse. In the figure the angles A O E and O A M being together less than 180°, the lines O E and A M meet in A′, or the sectional plane cuts both sides of the cone.

259. Let M be the centre of the ellipse, then F M = $\frac{1}{2}$ A E and M D = $\frac{1}{2}$ A′ G;

$\therefore$ The square on the axis minor = The rectangle A E, A′ G.

Also by drawing perpendiculars from A and E upon A′ G, it may be proved that

The square on the axis major = The square on A G + The rectangle A E, A′ G.

And $\therefore$ The distance between the foci = A G.

If the straight line A K be drawn making the angle E A K = the angle E A A′, then A K is the latus rectum of the section.

And if a circle be inscribed in the triangle A′ A O, it will touch the line A A′ in the focus of the section. (Geometry, Appendix, prop. 21.)

260. Let $\alpha = 90^\circ - \frac{\beta}{2}$, then $\frac{\sin.\alpha \sin.(\alpha + \beta)}{\left(\cos.\frac{\beta}{2}\right)^2} = 1$,

and the equation is that to a circle, the cutting plane being parallel to the base.

261. Let $\alpha + \beta = 180^\circ$, $\therefore \sin.(\alpha + \beta) = 0$, and the curve is a parabola. The plane, continuing to turn, has now come into the position A N Q, the axis A N being parallel to O F, or the cutting plane parallel to a side of the cone.

The equation to the parabola is $y^2 = 4'a\left(\sin.\frac{\beta}{2}\right)^2 x$.

If A K be drawn making the angle E A K = the angle A O K, then A K is the latus rectum of the section, and the circle which touches A O, A N and O F, will touch A N in the focus of the parabola.

262. Let $\alpha + \beta$ be greater than 180° $\therefore$ $\sin.(\alpha + \beta)$ is negative, and the curve is an hyperbola; The cutting plane is now in the position A L R; in this case the lines A L, E O must meet if produced backwards, or the plane cuts both cones, and the curve consists of two branches, one on the surface of each cone.

As in the ellipse, it may be proved that the square on the conjugate axis = the rectangle A E, A″ G′; that A G′ is the distance between the foci, that A K is the latus rectum, and that the circle touching A′ O, O A and A L touches A L at the focus.

263. We may also suppose a to have different values, or the cutting plane to meet the cone in some other point than A, for example:

Let $a = 0$ $\therefore$ $y^2 = -\frac{\sin.\alpha \sin.(\alpha + \beta)}{\left(\cos.\frac{\beta}{2}\right)^2} x^2$;

Since $\sin.\alpha$ and $\left(\cos.\frac{\beta}{2}\right)^2$ are positive, the rationality of this equation will depend upon $\sin.(\alpha + \beta)$.

If $\alpha + \beta$ is less than 180° the radical quantity is impossible, and the only solution of the equation is $x = 0$ and $y = 0$, or the section is a point; this is the case when the cutting plane passes through the vertex O, and is parallel to any elliptic section A P A′.

If $\alpha + \beta$ is greater than 180° we have two straight lines which cut each other at the origin. In this case the cutting plane is drawn through O, parallel to A L R, and the intersection with the cone is two straight lines meeting in O.

264. We may conclude from this discussion, that the conic sections are seven: a point, a straight line, two straight lines which intersect, a circle, an ellipse, an hyperbola, and a parabola or all the curves of the second degree and their varieties, with the exception of two parallel lines, which is a variety of the parabola.

The three latter sections, the ellipse, hyperbola, and parabola, are those which are usually termed "conic sections," and which have been the study and delight of mathematicians since the time of Plato. In his school they were first discovered; and his disciples, excited, no doubt, by the many beautiful properties of these curves, examined them with such industry, that in a very short time several complete treatises on the conic sections were published. Of these, the best still extant is that of Apollonius of Perga. It is in eight books, four of which are elementary; and four on the abstruser properties of these curves. The whole work is well worth attention, as showing how much could be done by the ancient analysis, and as giving a very high opinion of the geometrical genius of the age.

Apollonius gave the names of ellipse and hyperbola to those curves—Hyperbola, because the square on the ordinate is equal to a figure "exceeding" ("ὑπερβαλλον") the rectangle under the abscissa and latus rectum by another rectangle.—B. i. p. 13.

Ellipse, because the square on the ordinate is "defective" ("ελλειπον") with regard to the same rectangle.—p. 14.

It is not known who gave the name of parabola to that curve—probably Archimedes, because the square of the ordinate is equal ("παραβαλλον") to the rectangle of the abscissa and latus rectum.

Thus, the ancients viewed these curves geometrically, in the same manner as we are accustomed to express them by the equations:

$$y^2 = px + \frac{p}{2a} x^2,$$

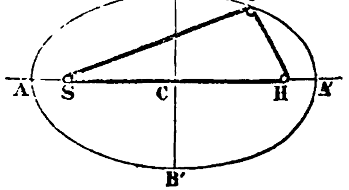
$$y^2 = px - \frac{p}{2a} x^2,$$

$$y^2 = px.$$

DESCRIPTION OF THE CONIC SECTIONS BY CONTINUED MOTION.

265. The conic sections being curves of great importance, not only from their mathematical properties, but also from their usefulness in the arts and sciences, it becomes necessary that we should be able to describe these curves with accuracy. Now, a curve may be drawn in two ways, either by "mechanical description" or by "points." As an instance of the first method we may mention the circle, described by the compasses, or by means of a string fastened at one end to the centre, and the other carried round by the hand, the hand tracing the curve. This mechanical method, or, as it is sometimes called, "that by continued motion," is not always practicable: no curve is so simple, in this respect, as the circle; hence we are often obliged to have recourse to the second method, or that by points: this is done by taking the equation to the curve and from some property expressed geometrically, finding a number of points, all of which belong to the curve, and then neatly joining these points with a pen or other instrument. We shall commence with the mechanical description of these curves.

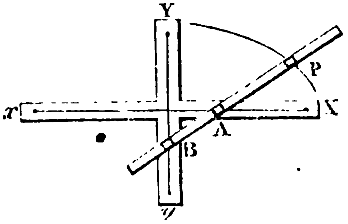
266. To trace an ellipse of which the axes are given:



Let A A′, B B′ be the axes: with centre B and radius A C describe a circle cutting A A′ in S and H, these points are the foci. Place pegs at S and H. Let one extremity of a string be held at A, and pass the string round H back again to A, and there join its two ends by a knot, so that its length shall be just double of A H; place a pen or other pointed instrument within this string, and move it round the points S and H, so that the string be always stretched; the pen will trace out the required ellipse. For if P be one of its positions, we have

$$\mathrm{SP} + \mathrm{PH} + \mathrm{HS} = 2\,\mathrm{AH} = \mathrm{AA}' + \mathrm{HS};$$
$$\therefore \mathrm{SP} + \mathrm{PH} = \mathrm{AA}'.$$

267. Another method is by means of an instrument called the elliptic compasses, or the trammel.



Let X x and Y y be two rulers with grooves in them, and fastened at right angles to each other. Let B P be a third ruler, on which take B P equal to the semi-axis major, and P A the semi-axis minor. At B a peg is so fixed that the point B with the peg can move along Y y; a similar peg is fixed at A. By turning the ruler B P round, a pen placed at P will trace out the curve. Suppose C to be the point where the axes meet, C M $= x$ and M P $= y$, the rectangular co-ordinates of P, and suppose that B N is drawn parallel to C M and meeting P M in N, then A M $= \frac{b}{a}$ B N, and

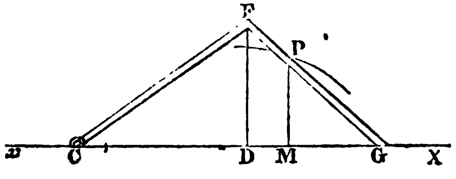
The square on A P = the square on P M + the square on A M;

$$\text{or } b^2 = y^2 + \frac{b^2}{a^2}x^2,$$
$$\therefore a^2y^2 + b^2x^2 = a^2b^2.$$

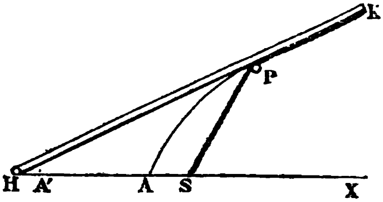
268. The following is also a very simple method of describing the ellipse. X x is a ruler of any length, C F, F G are two rulers, each equal to half the sum of the semi-axes. These rulers are fastened together by a moveable joint at F, and F C turns round a pivot at C; F P is taken equal to half the difference of the semi-axes. Let the point G slide along the line X x, then the point P will trace out the curve. Draw F D and P M perpendicular to C X, and let C M $= x$, and M P $=$ y, then

The square on F G = the square on F D + the square on D G;

$$\text{or} \left(\frac{a+b}{2}\right)^2 = \left(\frac{a+b}{2}\frac{y}{b}\right)^2 + \left(\frac{a+b}{2}\frac{x}{a}\right)^2$$
$$\therefore \frac{y^2}{b^2} + \frac{x^2}{a^2} = 1.$$

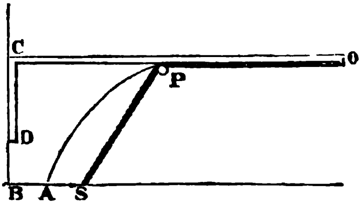


For a description of the Elliptograph, and other instruments for describing ellipses, we must refer our readers to the treatise on Practical Geometry, where an extremely good account is given of all the instruments, and also the advantages and disadvantages of each are well exhibited.



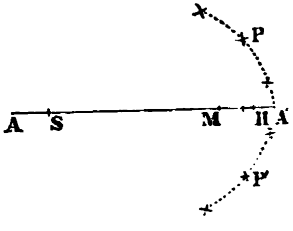
269. To trace the hyperbola by continued motion, let A A′ be the transverse axis, S H the distance between the foci, H P K a ruler movable about H. A string, whose length is less than H K by A A′ is fastened to K and S; when the ruler is moved round H, keep the string stretched, and in part attached to the ruler by a pencil as at P; then, since the difference of H P and P S is constantly the same, the point P will trace out the curve.

If the length of the string be H K, a straight line perpendicular to H S will be traced out; and if the string be greater than H K, the opposite branch, or that round H, will be described.

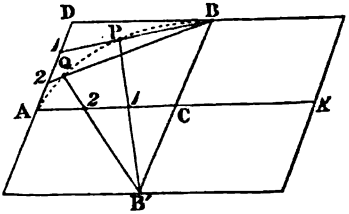


270. To trace the parabola by continued motion. Let S be the focus, and B C the directrix. Apply a carpenter's square O C D to the ruler B C, fasten one end of a thread whose length is C O to O, and the other end to the focus S; slide the square D C O along B C, keeping the thread tight by means of a pencil P, and in part attached to the square. Then since S P = P C, the point P will describe a parabola.

Description of the Conic Sections by Points.



271. Given the axes of an ellipse to describe the curve. Let A A′ be the axis major, S and H the foci. With centre S, and any radius A M less than A A′, describe a circle, and with centre H and radius A′ M describe a second circle, cutting the former in two points P and P′; then since $SP + PH = AM + MA' = AA'$, P is a point in the required curve; and thus any number of points may be found, and the curve described.



272. Given a pair of conjugate diameters to describe the curve. Let A A′, B B′, be the conjugate diameters. Through B draw B D parallel to A C, and through A draw A D parallel to B C. Divide A D and A C into the same number of equal parts as three. From B draw lines to the dividing points in A D, and from B′ draw lines to the dividing points in A C; the intersections P, Q, of these lines are points in the ellipse.

For let C be the origin; $CA = a_1$, $CB = b_1$,

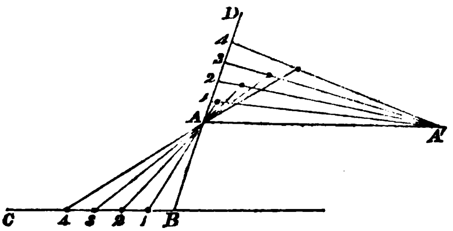
Then the equation to B P is $y - b_1 = \frac{b_1}{3a_1}x$;

and the equation to B′P is $y + b_1 = -\frac{3b_1}{a_1}x$.

Hence the product of the tangents of the angles which these lines B P, B′P make with the axis of $x = -\frac{b_1}{3a_1}\frac{3b_1}{a_1} = -\frac{b_1^2}{a_1^2}$ and is constant; therefore P is a point in the curve (141).

Innumerable points may be thus found in the four quadrants of the figure.

273. The following is perhaps the best method of tracing the ellipse by points:



Let A A′ be a diameter and A B equal and parallel to the conjugate diameter. Through B draw B C parallel to A A′ and equal to any multiple of A A′. In B A produced, take A D the same multiple of A B. Divide B C into any number of equal parts, and A D into the same number of equal parts. Through A draw lines to the points of division in B C, and through A′ draw lines to the points of division in A D; the intersections of corresponding lines will give points in an ellipse whose conjugate diameters are A A′ and A B. The proof is the same as in the last case.

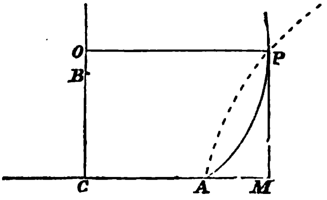
274. Given the axes of an hyperbola to trace the curve.

Let A A′ be the transverse axis, S and H the foci, which are given points; with centre S and any radius A M greater than A A′, describe a circle, and with centre H and radius A′ M describe a second circle, cutting the former in two points P and P, these are points in the required curve.

The proof is much the same as that for the ellipse (271.)

Again, if, in article 273, B C was taken to the right of B instead of the left, as in the figure, the intersections of the corresponding lines will give an hyperbola.

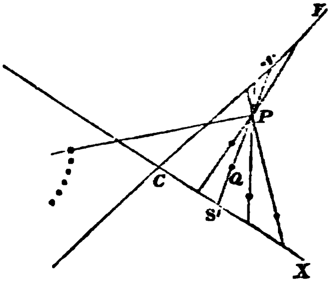
275. To describe the rectangular hyperbola by points.



Let C A, C B be the equal semi-axes with any centre O in C B produced and with radius O A, describe a circle; draw O P perpendicular to C O meeting the circle in P, then P is a point in the curve; Let $CM = x$, $MP = y$; then the square on C O = the square on O A − the square on C A;

$$\text{or } y^2 = x^2 - a^2.$$

276. Given the asymptotes C X, C Y of an hyperbola, and one point P in the curve, to describe the curve by points.

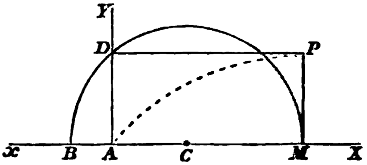


Through P draw any line S P S′ terminated by the asymptotes; in it take S′Q = S P; then Q is a point in the curve (217), and similarly any number of points may be found.

Together with the asymptotes, another condition must always be given to enable us to trace the curve, for the position of the asymptotes only gives us the ratio of the axes, and not the axes themselves.

277. To describe the parabola by points, when the principal parameter p is given.

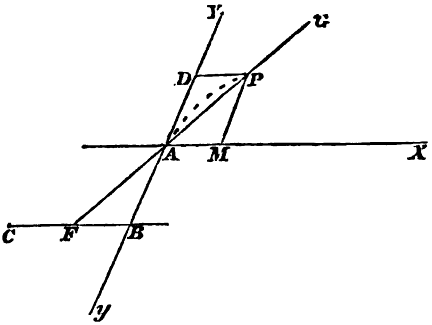
Let A X, A Y be the rectangular axes; in A x take A B = p; with any centre C in A X and radius C B describe a circle B D M, cutting A Y in D and A X in M, draw D P and M P perpendicular to A Y and AX respectively; then P is a point in the curve.



Let A M = x, M P = y; then the square on A D = the rectangle B A, A M,

$$\text{or } y^2 = p\,x.$$

278. Given the angle between the axes and any parameter p' to describe the curve.



Let A X, Y A y be the axes, A B the parameter. Through B draw C B parallel to A X. Through A draw any line F A G, meeting B C in F; in A Y take A D = B F, and draw D P parallel to A X, cutting A G in P, then P is a point in the curve.

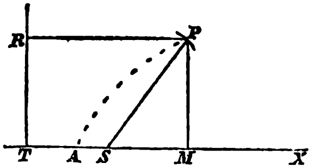
Draw M P parallel to A Y, and let

$$\text{A M} = x, \text{ and M P} = y,$$

$$\text{then M P : M A :: A B : F B},$$

$$\text{or } y : x :: p' : y; \quad \therefore y^2 = p'\,x.$$

279. Given the position of the directrix T R and the focus S, to trace any of the conic sections by points.



Draw S T perpendicular to T R, then T S produced will be the axis of the curve.

Let $e : 1$ be the ratio of the distance of any point P in the curve from the focus and from the directrix; hence if A S : A T :: e : 1; A is a point in the curve. Take any point M in A X, and with centre S and radius equal to e times T M, describe a circle; draw M P perpendicular to A X, and meeting the circle in P, then P is a point in the curve.

Let A be the origin of rectangular co-ordinates, $AM = x$, $MP = y$, $AS = m$, and $\therefore AT = \frac{m}{e}$;

$$\text{then } SP = e \cdot TM = e \cdot PR$$

$$\therefore \quad y^2 + (x - m)^2 = e^2\left(x + \frac{m}{e}\right)^2$$

$$\text{or } y^2 + x^2 - 2mx + m^2 = e^2 x^2 + 2emx + m^2;$$

$$\therefore \quad y^2 + (1 - e^2)x^2 - 2mx(1 + e) = 0;$$

which is the equation to the curves of the second order.

Let e be less than unity, $\therefore y^2 = (1 - e^2)\left\{\frac{2m}{1-e}x - x^2\right\}$.

Comparing this equation with that to the ellipse $y^2 = \frac{b^2}{a^2}(2ax - x^2)$, we have

$$2a = \frac{2m}{1 - e} \text{ and } \frac{b^2}{a^2} = 1 - e^2,$$

$$\therefore \quad b^2 = \frac{m^2}{(1 - e)^2}(1 - e^2) = m^2\frac{1 + e}{1 - e};$$

hence the curve is an ellipse whose axes are $\frac{2m}{1-e}$ and $2m\sqrt{\frac{1+e}{1-e}}$.

Let e be greater than unity, $\therefore y^2 = (e^2 - 1)\left\{\frac{2m}{e-1}x + x^2\right\}$;

and the curve is an hyperbola, whose axes are $\frac{2m}{e-1}$ and $2m\sqrt{\frac{e+1}{e-1}}$.

Let e be equal to unity, $\therefore y^2 = 4mx$;

the curve is a parabola, whose principal parameter is $4m$.

260. The general equation to all the conic sections being

$$y^2 + (1 - e^2)x^2 - 2mx(1 + e) = 0,$$

it follows that if we find any property of the ellipse from this equation, it will be true for the hyperbola and parabola, making the necessary changes in the value of e:

Thus the equation to the tangent is

$$yy' + (1 - e^2)xx' - m(1 + e)(x + x') = 0, \text{ for the ellipse,}$$

$$yy' - (e^2 - 1)xx' - m(1 + e)(x + x') = 0, \text{ for the hyperbola,}$$

$$\text{and } yy' - 2m(x + x') = 0, \text{ for the parabola.}$$

Also most of the results found in Chapter VIII. for the ellipse will be true for the hyperbola, by putting $-b^2$ for b^2; and will be true for the parabola by transferring the origin to the vertex of the ellipse, by then putting $\frac{m}{1-e}$ for a, and $m^2\,\frac{1+e}{1-e}$ for b^2; and then making $e=1$. Thus the equation to the tangent at the extremity of the Latus Rectum in the ellipse, when the origin is at the vertex, is

$$y = a + e\,(x - a)\ (117),$$

$$\text{or,}\quad y = a\,(1 - e) + e\,x,$$

$$\text{for } a \text{ put } \frac{m}{1-e}, \text{ and then let } e = 1;$$

$$\therefore\ y = m + x, \text{ as in (235).}$$

281. If S P $= r$, and A S P $= \theta$, the polar equation to the curve thus traced is easily found:

$$\text{S P} = e.\,\text{P R} = e\,(\text{T S} + \text{S M}),$$

$$\text{or}\quad r = e\,(m\,\frac{1+e}{e} - r\cos.\theta);$$

$$\therefore\ r = \frac{m\,(1+e)}{1 + e\cos.\theta}.$$

Or since $m\,(1+e) = \frac{b^2}{a}$ for the ellipse and hyperbola, and $= 2\,m$ for the parabola, we have (putting p for the principal parameter) the general polar equation to the three curves,

$$r = \frac{p}{2}\,\frac{1}{1 + e\cos.\theta}\ (150).$$

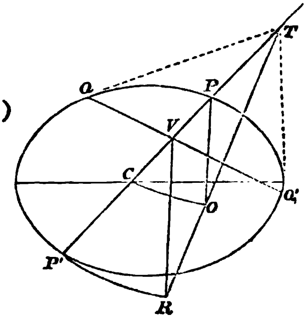
282. To draw a tangent at a given point P on the ellipse.

Draw the ordinate M P, and produce it to meet the circumscribing circle in Q, from Q draw a tangent to the circle meeting the axis major produced in T, join P T; this line is a tangent to the ellipse (114).

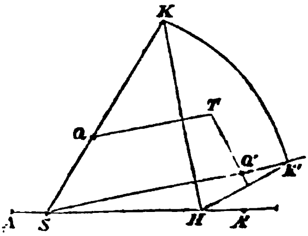
Again, taking the figure in the note appended to Art. 121, join S P, H P, and produce H P to K, so that P K $=$ P H; join S K; the line P y bisecting S K is a tangent.

283. To draw a tangent to the ellipse from a point T without the curve.

(1)



(2)



Draw the line T P C P′ through the centre, fig. 1.; draw a conjugate diameter to C P: then the question is reduced to finding a point V in C P, through which a chord Q V Q′ is to be drawn, so that T Q and T Q′ may be tangents.

Take C V a third proportional to C T and C P, then V is the required point (136).

Again, with centre T and radii T P′, T C describe circles C O, P′ R, draw any line T O R, cutting these circles in O and R; join P O, and draw R V parallel to P O: then it may be proved by similar triangles that C V is a third proportional to C T and C P, and therefore V is the required point.

284. If the axes, and not the ellipse, are given, take S and H the foci, fig. 2, with centre S and radius A A′ describe a circle, and with centre T and radius T H describe another circle, cutting the former in the points K and K′; join S K and SK′, H K and H K′; from T draw the lines T Q and TQ′ perpendicular to H K and H K′, these lines meet S K and SK′ in the required points Q and Q′. The proof will readily appear upon joining H Q and H Q′, and referring to the note, page 77.

285. To draw a tangent to the hyperbola at a given point P on the curve.

Join S P and H P, note, page 97; in H P take PK = S P, and join S K; the line P Y bisecting S K is the required tangent.

286. To draw a tangent from a given point T without the curve.

The two methods given (283) for the ellipse will apply, with the necessary alteration of figure, to the hyperbola.

287. To draw a tangent to the parabola at a given point P on the curve.

Draw an ordinate P M to the axis, fig. 232, and in the axis produced take A T = A M, join P T; this line is a tangent (233); or take S T = S P, and join P T.

288. To draw a tangent to a parabola from a given point T without the curve.

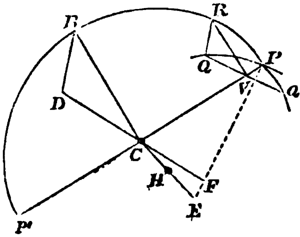
Draw a diameter T P V parallel to the axis, and cutting the curve in P, take P V = P T, and draw an ordinate Q V Q′ to the abscissa P V, then T Q and T Q′ are the required tangents (249).

If the directrix and focus be given, but not the curve; with centre T and radius T S describe a circle, cutting the directrix in the points R and R′, join R S and R′ S; draw R Q and R′ Q′ parallel to the axis, and then TQ and T Q′ perpendicular to R S and R′ S (239).

289. An arc Q P Q′ of a conic section, being traced on a plane to find to which of the curves it belongs; and also the axes and focus of the section.

Draw a line L through the middle of two parallel chords, and another line L′ through the middle of other two parallel chords, if the lines L, L′ are parallel, the curve is a parabola, if they meet on the concave side of the curve it is an ellipse, if on the convex side it is an hyperbola. (130. 243.)

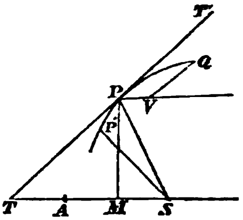
290. Let the curve be an ellipse, the point where the lines L L′ meet is the centre C; let P P′ be a diameter, its conjugate C D is thus found; describe a circle on P P′ as diameter, and draw V R, C B perpendicular to P P′; join R Q, and draw B D parallel to R Q, meeting a line parallel to Q V, passing through C; then C D is the conjugate diameter (136).



To find the length and position of the axes; draw P F perpendicular on C D, and produce it to E, making P E = C D, join C E, and bisect C E in H; join P H; then from the triangle C P E we have the side C E in terms of C P and C D $= \sqrt{\{a_1^2 + b_1^2 - 2\,a_1 b_1 \text{ sin. } (\theta' - \theta)\}} = \sqrt{\{a^2 + b^2 - 2\,a\,b\}} = a - b$ ∴; C H $= \dfrac{a-b}{2}$; also from the same triangle we have P H $= \dfrac{a+b}{2}$; hence P H + H E is the semi-axis major, and P H − H E is the semi-axis minor.

In H P take H K = H E, then C K is the direction of the axis-major.

291. If the arc Q P Q′ be an hyperbola, the conjugate diameter may be found by a process somewhat similar to that for the ellipse; the asymptotes may then be drawn by Art. 215. The direction of the axes bisects the angle of the asymptote, and their length is determined by drawing a tangent P T, and perpendicular P M, to the axis, and taking C A a mean proportional between C M and C T (167).



292. If the arc be a portion of a parabola, draw T P T′ parallel to Q V, and then draw P S, making the angle S P T = the angle T′ P V; repeat this construction for another point P′, then the junction of PS and P′S determines the focus S (240); the axis is parallel to P V, and the vertex is found by drawing a perpendicular on the axis, and then bisecting T M (233).

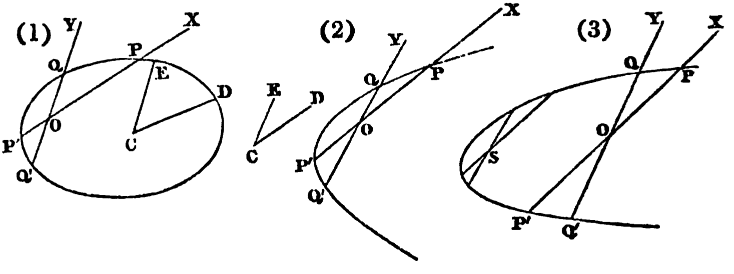
293. We shall conclude the subject of conic sections with the following theorem.

If through any point within or without a conic section two straight lines making a given angle with each other, be drawn to meet the curve, the rectangle contained by the segments of the one will be in a constant ratio to the rectangle contained by the segments of the other.

Case 1. The ellipse and hyperbola.

Let C D, C E be two semi-diameters parallel to the chords P O P′, Q O Q′; then, wherever chords parallel to these be drawn, we shall always have the following proportion:

The rectangle P O, O P′ : the rectangle Q O, O Q′ :: the square on C D : the square on C E.



Let O be the origin of oblique axes O X, O Y: then the equation to the curve will be of the form

$$ay^2+bxy+cx^2+dy+ex+f=0.$$

Let $x=0$; $\therefore\ ay^2+dy+f=0$, and the product of the roots being $\frac{f}{a}$, we have

$$\text{The rectangle } QO,\ OQ'=\frac{f}{a};$$

$$\text{Similarly the rectangle } PO,\ OP'=\frac{f}{c};$$

$\therefore$ the rectangle QO, OQ' : the rectangle PO, OP' $::\frac{f}{a}:\frac{f}{c}::c:a$

Now, let the origin be transferred to the centre without changing the direction of the axes, then the form of the equation is

$$ay^2+bxy+cx^2+f'=0 \quad (81).$$

Let $x=0$; $\therefore$ the square on $CE=\frac{-f'}{a}$; and the square on $CD=\frac{-f'}{c}$; $\therefore$ the square on CE : the square on CD; $::c:a$;

$\therefore$ the rectangle QO, OQ' : the rectangle PO, OP' :: the square on CE : the square on CD.

In the hyperbola fig. (2), CE and CD do not meet the curve; but in order to show that these lines are semi-diameters, let the axis of y be carried round till it becomes conjugate to CD, then the formulas for transformation in (55) become for $\theta=0$,

$$y=y'\frac{\sin.\theta'}{\sin.\omega},\ x=x'+y'\frac{\sin.(\omega-\theta')}{\sin.\omega}.$$

If these values of x and y be substituted in the general central equation above, and it be reduced to the conjugate form by putting $b'=0$, the transformed equation is of the form $a'y^2+cx^2+f'=0$, where c and f' are not changed, and $-\frac{f'}{c}$ is the square on the semi-diameter along the axis of x (86); hence the theorem is true for the hyperbola.

Case 2. The Parabola fig. (3.)

As before, we have the rectangle PO, OP' : the rectangle QO, $OQ'::c:a$.

Let P and Q be the parameters to the chords POP' and QOQ'; transfer the origin to the focus, the axes remaining parallel to PO, and QO, by which transformation c and a are not altered.

Now in this case, the chords passing through the focus, we have the rectangle PS, SP' : the rectangle QS, SQ'; $::\frac{p}{4}P:\frac{p}{4}Q$ (254) and also as $c:a$; hence the rectangle PO, OP' : the rectangle QO, $OQ'::c:a::P:Q$.

294. If the point O be without the curves, and the points PP' coincide as well as Q and Q', or the lines become tangents, we have for the ellipse and hyperbola,

The square on O P : the square on O Q :: the square on C D : the square on C E;

or O P : O Q :: C D : CE.

For the parabola;

The square on O P : the square on O Q :: S P : S Q;

hence it may be proved that, if a polygon circumscribe an ellipse, the algebraical product of its alternate segments are equal. And the same theorem will apply to tangents about an hyperbola; the tangents commencing from any point in the asymptotes.

CHAPTER XII.

ON CURVES OF THE HIGHER ORDERS.

295. Having completed the discussion of lines of the second order, we should naturally proceed to the investigation of the higher orders; but the bare mention of the number of those in the next or third order (for they amount to eighty) is quite sufficient to show that their complete investigation would far exceed the limits of an elementary treatise like the present. Nor is it requisite: we have examined the sections of the cone at great length, because, from their connexion with the system of the wòrld, every property of these curves may be useful; but it is not so with the higher orders; generally speaking they possess but few important qualities, and may be considered more as objects of mathematical curiosity than of practical utility.

The third order is chiefly remarkable from its investigation having been first undertaken by Newton. Of the eighty species now known, seventy-two were examined by him; eight others, which escaped his searching eye, have since been discovered.

Those who wish to study these curves, may refer to Newton's "Enumeratio Linearum tertii Ordinis;" or to the work of Stirling upon the same subject.

Of the fourth order there are above five thousand species, and the number in the higher orders is so enormous as to preclude the possibility of their general investigation in the present state of analysis.

A systematic examination of curves being thus impossible, all that we can do is to give a selection, taking care that amongst them shall be found all the algebraical or transcendental curves which are most remarkable either for their utility or history.

We shall generally introduce them as examples of indeterminate problems, that is, of problems leading to final equations, containing two variables. We shall then trace the loci of those equations, and explain, when necessary, anything relating to the construction or properties of the curves.

It would be useless to give any general rules for the working of these questions; those given for determinate problems will here serve equally well; but, in both cases, experience is the only sure guide. In the solution of these problems we shall not always follow the same, nor even the easiest, method; but we shall endeavour to vary the manner, so that an attentive observer may learn how to act in any particular case.

We commence with problems leading to loci of the second order.

296. Given the straight line A B ($= a$) to find the point P without A B, so that A P : P B :: m : 1.

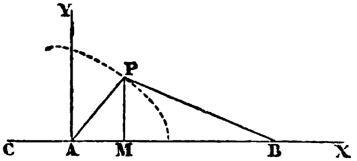
Let A be the origin of rectangular co-ordinates, A X and A Y the axes, A M $= x$, M P $= y$, and ∴ M B $= a - x$,

$$\text{then } AP : PB :: m : 1$$

$$\text{or } \sqrt{x^2 + y^2} : \sqrt{(a-x)^2 + y^2} :: m : 1;$$

$$\therefore x^2 + y^2 = m^2 (a-x)^2 + m^2 y^2,$$

$$\text{or } (1 - m^2) y^2 + (1 - m^2) x^2 + 2 m^2 a x - m^2 a^2 = 0,$$



$$\text{or } y^2 + \left(x + \frac{m^2 a}{1 - m^2}\right)^2 = \frac{m^2 a^2}{(1 - m^2)^2}.$$

This equation shows that there are an infinite number of points satisfying the conditions of the problem, all situated on the circumference of a circle (66).

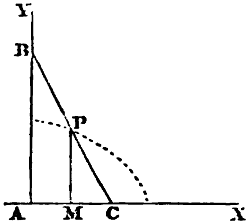
To draw this circle; in A x take A C $= \frac{m^2 a}{1 - m^2}$, and with centre C and radius $\frac{m a}{1 - m^2}$ describe a circle; this is the required locus.

If $m = 1$, reverting to the original equation we have $x = \frac{a}{2}$, which is the equation to a straight line drawn through the bisection of A B, and parallel to A Y.

297. If perpendiculars be drawn to two lines given in position from a point P, and the distance between the feet of the perpendiculars be a constant quantity a, required the locus of P.

Let the intersection of the given lines be the origin of rectangular axes, take one of the lines for the axis of x, and let $y = \alpha x$ be the equation to the other; then the equation to the line passing through P $(x' y')$, and perpendicular to the line $y = \alpha x$, is $y - y' = -\frac{1}{\alpha}(x - x')$; then from these two equations the co-ordinates of the point where their loci meet, that is, the co-ordinates of the foot of the perpendicular are readily obtained; and then the final equation found, by art. 29, is $y'^2 + x'^2 = a^2 \frac{(1 + \alpha^2)}{\alpha^2}$, which belongs to a circle whose centre is at the intersection of the lines.

298. A given straight line B C moves between two straight lines, A B, A C, so that its extremities B C are constantly on those lines; to find the curve traced out by any given point P in B C.



Let the lines A B, A C be the axes of y and x,

$AM = x$, $BP = a$,
$MP = y$, $PC = b$,

and let B A C be a right angle;

then $AM : BP :: MC : PC$,

$$x : a :: \sqrt{b^2 - y^2} : b$$
$$\therefore b^2 x^2 = a^2 b^2 - a^2 y^2,$$
$$\text{or } a^2 y^2 + b^2 x^2 = a^2 b^2,$$

which is the equation to an ellipse whose centre is A and axes $2a$, $2b$. If a ladder be placed against a wall, and its foot drawn along the ground at right angles to the wall, any step will trace out a quarter of an ellipse, and the middle step will trace out a quadrant of a circle.

If the co-ordinate axes be inclined at an angle θ, we have

$$AB = \frac{a+b}{b}\, y, \text{ and } AC = \frac{a+b}{a}\, x,$$

$$\text{Whence } a^2 y^2 + b^2 x^2 - 2\, a\, b \cos.\theta \,.\, x\, y - a^2 b^2 = 0,$$

which is the equation to an ellipse (76).

It is easy to see that from this problem arises a very simple mechanical method of describing the ellipse.

If a straight line B C of variable length move between two straight lines A B, A C, so that the triangle A B C is constant, the curve traced out by a point P which divides B C in a given ratio is an hyperbola.

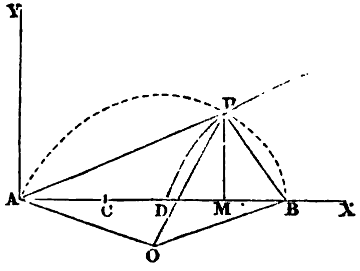
299. Given the line A B ($= c$) to find a point P without A B, such that drawing P A and P B, the angle P B A may be double of P A B.

Let A be the origin; A X, A Y the rectangular axes:

The equation to A P is $y = \alpha x$, (1)

and that to B P is $y = \alpha' (x - c)$;

but $\alpha' = \tan.\, PBX = -\tan.\, PBA = -\tan.\, 2PAB = -\frac{2\alpha}{1 - \alpha^2}$;



$$\therefore y = \frac{-2\alpha}{1 - \alpha^2}(x - c) \quad (2)$$

Eliminating α between the equations (1) and (2), we have $y^2 = 3x^2 - 2cx$, hence the locus of P is an hyperbola; comparing its equation with the equation $y^2 = \frac{b^2}{a^2}(x^2 - 2ax)$, we find the axes to be $\frac{2c}{3}$ and

$\frac{2c}{\sqrt{3}}$, and the centre at C where $AC = \frac{c}{3}$.

By this hyperbola, a circular arc may be trisected; for if A P B be the arc to be trisected, describe the hyperbola D P as above, and let the curves intersect in P; then if 0 be the centre of the circle, the angle $AOP = 2ABP = 4PAB = 2POB$, or the arc P B is one-third of B P A.

This problem may also be thus solved:

Let $AM = x$, $MP = y$, and angle $PAB = \theta$;

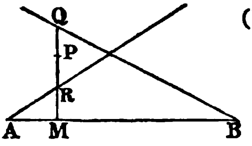
Then $\tan.\theta = \frac{y}{x}$, and $\tan.2\theta = \frac{y}{c-x}$ but $\tan.2\theta = \frac{2\tan.\theta}{1-(\tan.\theta)^2}$,

$$\therefore \frac{y}{c-x} = \frac{\frac{2y}{x}}{1-\frac{y^2}{x^2}}, \text{ or } y^2 = 3x^2 - 2cx.$$

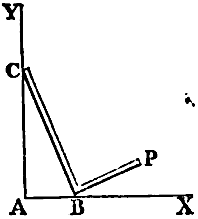
On examination it will be seen that the above two methods of solution are identical.

300. The following problems give loci of the second order.

1. From the given points A and B, (fig. 1,) two straight lines given in position are drawn, M R Q is a common ordinate to these lines, and M P is taken in M R Q a mean proportional to M Q and M R; required the locus of P.



(1)



(2)

2. A common carpenter's square C B P, (fig. 2,) moves in the right angle X A Y, so that the point C is always in A Y, and the right angle B in the line A X; required the locus of P.

3. If the base and difference of the angles at the base of a triangle be given, the locus of the vertex is an equilateral hyperbola.

4. To find a point P, from which, drawing perpendiculars on two given straight lines, the enclosed quadrilateral shall be equal to a given square.

301. Let AQA' be an ellipse, AA' the axis major, QQ' any ordinate, join AQ and $A'Q'$; required the locus of their intersection P.

Let C be the origin of rectangular co-ordinates.

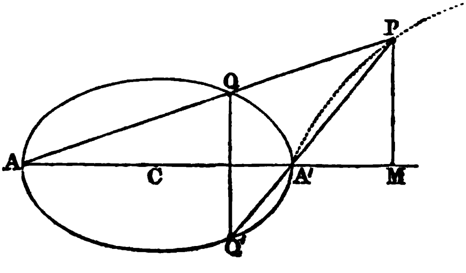
$$CM = x,\ MP = y,\ CN = x',\ \text{and } NQ = y'.$$

Then the equation to A Q is $y = \alpha x + c$

which at A is $0 = -\alpha a + c$;

$$\therefore\ y = \alpha(x + a),$$

At Q it becomes $y' = \alpha\,(x' + a)$ $\therefore\ \alpha = \dfrac{y'}{x' + a}$;



Hence the equation to A Q is $y = \dfrac{y'}{x' + a}\,(x + a)$, (1)

And similarly that to A′Q′ is $y = \dfrac{-\,y'}{x' - a}\,(x - a)$, (2)

Also $a^2\,y'^2 + b^2\,x'^2 = a^2\,b^2$; (3)

Eliminating x' and y' between (1) and (2), we have

$$x' = \frac{a^2}{x} \text{ and } y' = \frac{a\,y}{x}.$$

Substituting in (3) we obtain the final equation

$$a^2\,\frac{a^2\,y^2}{x^2} + \frac{b^2\,a^4}{x^2} = a^2\,b^2,$$

$$\text{or } a^2\,y^2 - b^2\,c^2 = -\,a^2\,b^2,$$

which is the equation to an hyperbola, whose centre is C, and transverse axis 2 a.

The method of elimination used in this problem is of great use; the principle admits of a clear explanation. We have the equations to A Q and A′ Q′; putting x and y the same for both equations intimates that x and y are the co-ordinates C M and M P in one particular case of intersection; but the elimination of x' and y' intimates that x and y are also always the co-ordinates of intersection, and therefore that the resulting equation belongs to the locus of their intersection.

302. To find the locus of the centres of all the circles drawn tangential to a given line A X, and whose circumferences pass through a given point Q (a b).

Let S Q M be one of these circles, referred to rectangular axes A x, A y.

x, y the co-ordinates of its centre P,

x', y' . . . . . . any point on its circumference.

Then the equation to S Q M is

$$(y' - y)^2 + (x' - x)^2 = r^2; \quad (65)$$

but passing through Q, it becomes

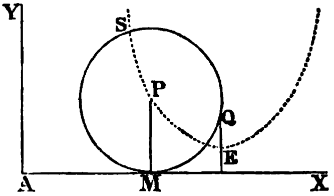
$$(b - y)^2 + (a - x)^2 = r^2,$$

and, being tangential to A X, we have $r = y$,

$$\therefore\ (b - y)^2 + (a - x)^2 = y^2,$$

$$\text{or } x^2 - 2\,a\,x - 2\,b\,y + a^2 + b^2 = 0.$$

This is the equation to a parabola (78).



It may be put in the form $(x - a)^2 = 2\,b\left(y - \frac{b}{2}\right)$. Hence if we transfer the origin to the point E $\left(a,\ \frac{b}{2}\right)$, we have the equation $x^2 = 2\,b\,y$, and the curve is referred to its vertex E, which is the centre of the least circle.

If, instead of the circle passing through a given point, it touch a given circle, a parabola is again the locus of P.

303. Let A B, B C, C D, and D A (fig. 1, p. 148) be four straight lines given in position, to find the locus of a point P, such, that drawing the lines P E, P F, P G, and P H making given angles with A B, B C, C D, and D A, we may have the rectangle P E, P F = the rectangle P G, P H.

Let O be the origin of rectangular axes O X, O Y; x and y the co-ordinates of P; β, β', β'' and β''' the cosecants of the angles which the lines P E, P F, &c., make with A B, B C, &c. Then the equation

to A B being $y' = \alpha\,x' + b$ we have $\text{P E} = \frac{y - \alpha\,x - b}{\sqrt{1 + \alpha^2}}\,\beta$ (49)

to B C . . $y' = \alpha'\,x' + b'$. . . $\text{P F} = \frac{y - \alpha'\,x - b'}{\sqrt{1 + \alpha'^2}}\,\beta'$

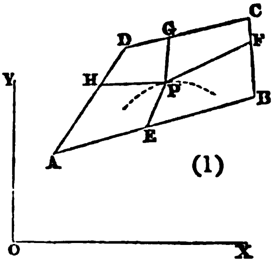
to D C . . $y' = \alpha''\,x' + b''$ . . $\text{P G} = \frac{y - \alpha''\,x - b''}{\sqrt{1 + \alpha''^2}}\,\beta''$

to A D . . $y' = \alpha'''\,x' + b'''$ . . $\text{P H} = \frac{y - \alpha'''\,x - b'''}{\sqrt{1 + \alpha'''^2}}\,\beta'''$;

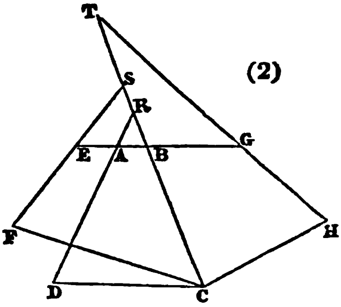
$$\therefore \text{ by the question } \frac{y - \alpha\,x - b}{\sqrt{1 + \alpha^2}}\;\frac{y - \alpha'\,x - b'}{\sqrt{1 + \alpha'^2}}\;\beta\,\beta'$$

$$= \frac{y - \alpha''\,x - b''}{\sqrt{1 + \alpha''^2}}\;\frac{y - \alpha'''x - b'''}{\sqrt{1 + \alpha'''^2}}\;\beta''\,\beta'''.$$

This equation being evidently of two dimensions, the locus of P is a conic section, the particular species of which depends on the situation of the given lines.



(1)



(2)

This problem may be expressed much more generally. Suppose 3, 4, 5 or a greater number of lines to be given in position, required a point from which, drawing lines to the given lines, each making a given angle with them, the rectangle of two lines thus drawn from the given point may have a given ratio to the square on the third, if there are three; or to the rectangle of the two others, if there are four: or again, if there are five lines, that the parallelopiped composed of three lines may have a given ratio to the parallelopiped of the two remaining lines, together with a third given line, or to the parallelopiped composed of the three others, if there are six: or again, if there are seven, that the algebraical product of four may have a given ratio to the algebraical product of the three others and a given line, or to the four others, if there are eight, and so on.

This was a problem which very much perplexed the ancient geometricians. Pappus says, that neither Euclid nor Apollonius could give a solution. He himself knew that when there are only three or four lines the locus was a conic section, but he could not describe it, much less could he tell what the curve would be when the number of lines were more than four. When the number of lines were seven or eight, the ancients could scarcely enunciate the problem, for there are no figures beyond solids, and without the aid of algebra, it is impossible to conceive what the product of four lines can mean.

It was this problem which Descartes successfully attacked, and which, most probably, led him to apply algebra generally to geometry. The following solution is that given by Descartes, with a few abbreviations:

A B, A D, E F and G H (fig. 2) are the given lines, C the required point from which are drawn the lines C B, C D, C F and C H making given angles C B A, C D A, C F E, and C H G. A B $(= x)$ and B C $(= y)$ are the principal lines to which all the others will be referred. Suppose the given lines to meet C B in the points R, S, T, and A B in the points A, E and G. Let A E $= c$ and A G $= d$.

Then since all the angles of the triangle A B R are known, we have B R $= \alpha . $A B $= \alpha x$; $\therefore$ C R $= \alpha x + y$ and C D $= \beta\,(\alpha x + y)$; also B S $= \alpha' . $B E $= \alpha' (c + x)$; $\therefore$ C S $= y + \alpha' (c + x)$ and C F $= \beta' \{y + \alpha' (c + x)\}$; also B T $= \alpha'' . $B G $= \alpha'' (d - x)$: $\therefore$ C T $= y + \alpha'' (d - x)$ and C H $= \beta'' \{y + \alpha'' (d - x)\}$; then since the rectangle C B, C F = the rectangle C D, C H, we have the equation

$$y\beta'\{y + \alpha'(c + x)\} = \beta(\alpha x + y)\beta''\{y + \alpha''(d - x)\}.$$

This equation Descartes showed to belong to a conic section which he described. He also gave the following numerical example:

Let $EA = 3$, $AG = 5$, $AB = BR$, $BS = \frac{1}{2}BE$, $GB = BT$, $CD = \frac{3}{2}CR$, $CF = 2CS$, $CH = \frac{2}{3}CT$, the angle $ABR = 60°$, and the rectangle C B, C F = the rectangle C D, C H. By the above method he found the equation to be

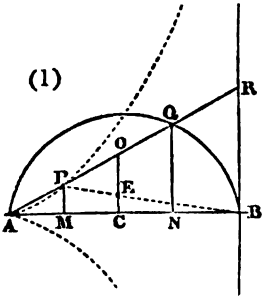
$$y^2 + xy + x^2 - 2y - 5x = 0;$$

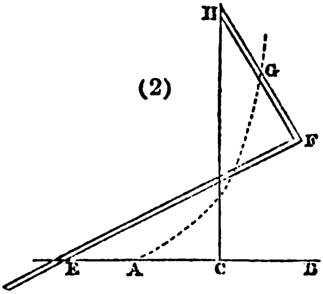
which he showed belonged to a circle. Taking the expressions in art. (72) we have the co-ordinates of the centre $\frac{8}{3}$ and $-\frac{1}{3}$, and the radius $= \frac{\sqrt{19}}{3}$.

304. Let A Q B be a semi-circle of which A B is the diameter, B R an indefinite straight line perpendicular to A B, A Q R a straight line meeting the circle in Q and B R in R; take A P = Q R; required the locus of P.

Let A be the origin of rectangular axes, and A B the axis of x.

$AB = 2a$, $AM = x$, $MP = y$, and draw QN parallel to MP;


(1)


(2)

then since A P = Q R, we have A M = B N,

and $AM : MP :: AN : NQ$;

that is, $x : y :: (2a - x) : \sqrt{(2a - x)x}$; (65)

$$\therefore y^2 = \frac{x^3}{2a - x} \text{ and } y = \pm \sqrt{\frac{x^3}{2a - x}}.$$

The following table gives the corresponding values of x and y:

	1	2	3	4	5	6
Values of x	0	a	$< 2a$	$2a$	$> 2a$	$-$
Values of y	0	a	possible	∞	impos.	impos.

From (1) the curve passes through the origin, from (2) it bisects the semi-circular arc A Q B, from (3) there are possible values of y for all values of x less than $2a$, from (4) there is an infinite ordinate at B, or B R is an asymptote to the curve: from these values we thus obtain an infinite arc proceeding from A to meet the asymptote B R. Again, from (5) for any value of x greater than $2a$, y is impossible, or no part of the curve is found to the right of the asymptote; and from (6) no part of the curve is on the left of A. Also, for every value of x there are two of y equal and opposite; hence there is a branch below A B similar to the one above it.

Diocles, a mathematician of the sixth century, invented this curve, which he called the Cissoid, from a Greek word signifying "ivy," because this curve climbs up its asymptote like ivy up a tree. He employed it in solving the celebrated problem of the insertion of two mean proportionals between given extremes.

Before his time, Pappus had reduced the problem to this case:

Let B C, C E be the two extremes, and A Q B a circle whose centre is C and radius C B; draw an indefinite straight line B E P through E and then draw the straight line A P O Q meeting B E and C E produced, and also meeting the circle at Q in such a manner that O Q = O P, then C O will be the first of the two mean proportionals. But the point P could not be directly found: hence, Diocles invented this curve to determine a series of points which will solve the problem for any length of C E: for example, suppose that B C, C E and the cissoid be drawn, join B E meeting the curve in P, then since O R = O A and Q R = A P we have O Q = O P.

From the definition of the curve it can be readily described by points; but as this is only a tentative process at best, and therefore not geometrically correct, Newton invented a very simple instrument for describing the curve by continued motion:

Let C H (fig. 2, p. 149) be a straight line parallel to B R; take A E = A C and let E F H be a common carpenter's square, the side F E being of indefinite length, and F H = A B; move this square so that the longer leg F E always passing through E, the extremity H of the other slides along C H, the middle point G of F H traces out the cissoid.

To obtain the polar equation to this curve:

$$\text{Let } y = r \sin.\ \theta \text{ and } x = r \cos.\ \theta;$$

Substitute these values in the equation $y^2 = \frac{x^3}{2a-x}$,

$$\therefore r^2 (\sin.\theta)^2 = \frac{r^3 (\cos.\theta)^3}{2a - r\cos.\theta}; \text{ whence } r = 2a \sin.\theta \, . \tan\theta .$$

Ex. If a perpendicular be drawn from the vertex of a parabola to a tangent, the locus of their intersection is the cissoid.

305. If C be a point in the diameter A B of the circle A Q B, and M Q any ordinate, join B Q, and draw C P parallel to B Q, meeting M Q in P; required the locus of P.

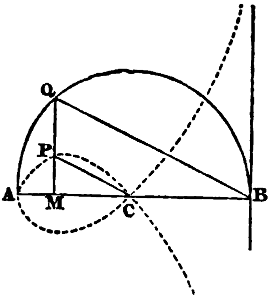
Let $AM = x$,

$MP = y$,

$AB = a$,

$AC = b$;

then $BM : MQ :: CM : MP$,



or $(a - x) : \sqrt{ax - x^2} :: (b - x) : y$,

$$\therefore y = \pm (b - x)\sqrt{\frac{x}{a - x}}.$$

Hence the following table of values:

	1	2	3	4	5	6
Values of x	0	b	a	$< a$	$> a$	$-$
Values of y	0	0	$\pm\infty$	pos.	imp.	imp.

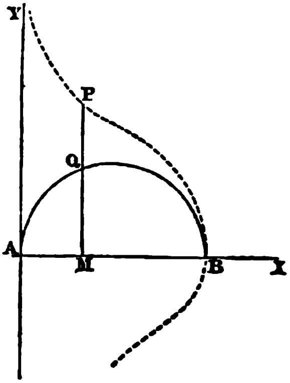
From (1) and (2) the curve passes through A and C; from (3) the ordinate at B is an asymptote to the curve; from (4) there are two arcs between A and C, also two between C and B; from (5) and (6) no part of the curve extends to the right of B or the left of A.

If $b = 0$, the oval between A and C disappears, and the curve is the cissoid of Diocles.

If b is negative, or the point C on the left of A, the curve consists of two branches proceeding from A to the asymptote through B, and the point C, though not on the curve, yet essentially belongs to it. This insulated point is called a conjugate point. The theory of such points will be fully explained in the treatise on the Differential Calculus.

Ex. A point Q is taken in the ordinate M P of the parabola, always equidistant from P, and from the vertex of the parabola; required the locus of Q.

306. M Q is an ordinate to the semicircle A Q B, and M Q is produced to P, so that M P : M Q :: A B : A M to find the locus of P.



Let A B X and A Y be the rectangular axes.

$$AM = x,$$

$$MP = y,$$

$$AB = 2a;$$

Then $MP : MQ :: AB : AM$,

$$\text{or } y : \sqrt{2ax - x^2} :: 2a : x,$$

$$yx = \pm 2a\sqrt{2ax - x^2},$$

$$\therefore y = \pm 2a\sqrt{\frac{2a - x}{x}};$$

	1	2	3	4	5
Values of x	0	$2a$	$< 2a$	$> 2a$	neg.
Values of y	$\pm\infty$	0	pos.	imp.	imp.

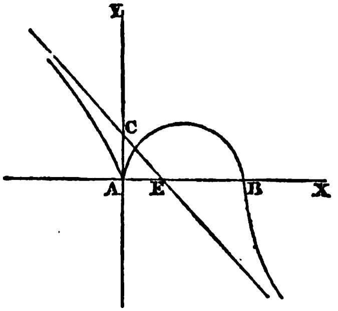
From (1) we have the ordinate at the origin infinite, and therefore an asymptote to the curve; from (2) the curve cuts the axis at B; from (3) the curve extends between A and B; from (4) no part of the curve is beyond B; from (5) no part is to the left of A.

This curve is called the Witch, and is the invention of an Italian lady, Maria Gaetana Agnesi, Professor of Mathematics in the University of Bologna, A.D. 1748.

307. In the circle the square on the ordinate is equal to the rectangle under the segments of the diameter; required the form of the curve on which the curve upon the ordinate is equal to the parallelopiped, of which the base is the square on one segment, and the altitude is the other segment, or $y^3 = x^2(2a - x)$.

Let A be the origin, A X, A Y the rectangular axes, and $AB = 2a$.

Let $x = 0$ or $= 2\,a$, $\therefore\ y = 0$; hence the curve passes through A and B; for $x < 2\,a$, y is positive; but when x is $> 2\,a$, y increases negatively to infinity, since the third root of a negative quantity is negative and possible. Again, y is positive for all negative values of x, and increases to ∞; also for each value of x, there is only one real value of y, the other two roots of an equation $y^3 \pm 1 = 0$, being always impossible.



Expanding the equation we have

$$y = -x\sqrt[3]{1 - \frac{2\,a}{x}} = -x\left\{1 - \frac{1}{3}\,\frac{2\,a}{x} - \frac{4\,a^2}{9\,x^2} + \&c.\right\};$$

$\therefore$ the equation to the asymptote is $y = -x + \dfrac{2\,a}{3}$ (195).

In A Y take A C $= \dfrac{2\,a}{3}$, and in A X take A E $= \dfrac{2\,a}{3}$, join C E, this line produced is an asymptote to the curve.

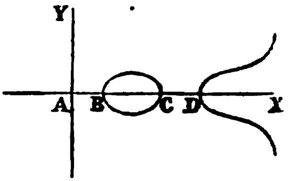
Ex. Find the locus of the equation, $y^3 + x^3 = a^3$; and of the equation $y^3 = a^2\,x - x^3$.

308. To trace the curve whose equation is $a\,y^2 = x^3 + m\,x^2 + n\,x + p$.

Case (1). Suppose the roots of this equation to be real and unequal, and to be represented by the letters a, b, and c, of which a is less than b and b less than c, then the equation is of the form

$$y = \pm\sqrt{\left\{\left(\frac{x-a}{a}\right)(x-b)\;(x-c)\right\}}.$$

	1	2	3	4	5	6	7	8	9	10
Values of x	0	a	$< a$	$> a < b$	b	$> b < c$	—	$> c$	∞	—
Values of y	imp.	0	imp.	pos.	0	imp.	0	pos.	$\pm\infty$	imp.



Let A be the origin, A X, A Y, the axes; A B $= a$, A C $= b$, and A D $= c$;

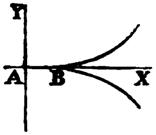
From (2) (5) and (7) the curve passes through B, C, and D; from (3) and (6) no part of the curve is found between A and B, or C and D; from (4) there are two branches between B and C; from (8) and (9) the curve proceeds from D to ∞, and from (10) no part of the curve is on the left of A.

If the roots had been negative, the curve would have the same form, but would be rather differently situated with regard to the origin.

Case (2). If two roots be equal, the equation is $y = (x-c)\sqrt{\dfrac{x-a}{a}}$, or $y = \pm (x-a)\sqrt{\dfrac{x-c}{a}}$; in the former case the figure is nearly the same as above, when the points C and D coincide; in the latter, supposing the points B and C to coincide, or the oval to become a conjugate point.

Case (3). If two of the roots be impossible, we have only the bell-shaped part of the curve from D.

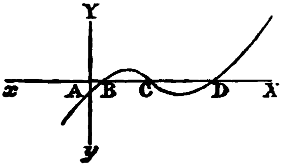
Case (4). If the three roots be equal, the equation is $a y^2 = (x - a)^3$.



The figure now consists of two branches proceeding from B with their convexity towards the axis. This curve is called the semi-cubical parabola; its equation is the most simple when the origin is at the vertex B; that is, putting x instead of $x - a$, when $a y^2 = x^3$.

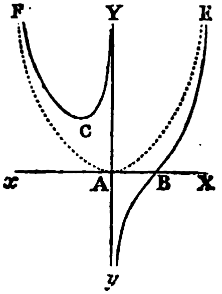
This curve is remarkable as being the first curve which was rectified, that is, the length of any portion of it was shown to be equal to a number of the common rectilinear unit.

309. The equation $a^2 y = x^3 + m x^2 + n x + p$, can be traced exactly as in the last article: the accompanying figure applies to the case when the three roots are positive, real, and unequal. If two of them be equal, one of the semi-ovals disappears; if three are equal, both disappear: in this case the equation is of the form $a^2 y = (x - a)^3$, or $a^2 y = x^3$, if the origin be transferred to B; the curve is then called the cubical parabola.



310. If the equation be $a x y = x^3 + m x^2 + n x + p$, the axis of y is an asymptote, and there is a branch in the angle Y A x; the rest

of the curve is like that in the last figure, supposing the lower branch from B to come to A y as the asymptote, the form will vary as the roots vary. We shall take the case where $y = \frac{x^3 - a^3}{a x}$.



	1	2	3	4	5	6	7
Values of x	0	a	$< a$	$> a$	∞	$-$	$-\infty$
Values of y	∞	0	$-$	$+$	∞	$+$	$+\infty$

From (1) Ay is an asymptote; from (2) the curve cuts the axis at B ($AB = a$); from (3, 4, 5) it is below the axis of x from A to B, and above from B to ∞; from (6) and (7) we have the branch F C Y.

This curve is called the trident, from its form. This curve enables us to point out the difference between what are called parabolic and hyperbolic branches of a curve: B y and C Y are hyperbolic, because they admit of a straight line Y y for the asymptote; but B E and C F are parabolic, because they admit of a parabolic asymptote, represented by the dotted curve F A E,

whose equation is $y = \frac{x^2}{a}$ (196).

Ex. Find the locus of the equation $x^2 y - y - x^3 - x = 0$.

If the equation be $x y^2 + a^2 y = x^3 + m x^2 + n x + p$, the form of the curve will depend on the nature of the roots of the equation $x^4 + m x^3 + n x^2 + p x + \frac{a^4}{4} = 0$; there will be no difficulty in any particular case. Generally the equation to the asymptotes is $y = \pm (x + \frac{1}{2} m)$; and the axis of y is an asymptote.

311. If the terms x^3 and $m x^2$ are wanting, the equation is

$$x y^2 + a^2 y = n x + p,$$

$$\therefore\ y = \frac{-a^2 \pm \sqrt{\{a^4 + 4 p x + 4 n x^2\}}}{2 x}$$

If the denominator of this expression had been constant, the equation would have belonged to an ellipse, hyperbola, or parabola, according as n was negative, positive or nothing; hence if such constant quantity be replaced by the variable quantity $2x$, the conic section becomes "hyperbolized" by having an infinite branch proceeding to the axis of y as an asymptote.

For the nine figures corresponding to the values of p, see Newton, Enum. Lin. Tert. Ord.

From the last article it appears that all curves of the third order have infinite branches; and this must necessarily be the case, for every equation of an odd degree has at least one real root, so that there is always one real value of y corresponding to any real value of x.

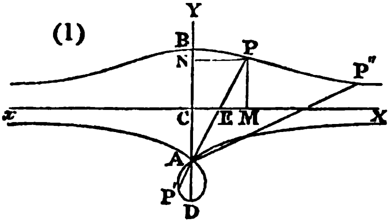
312. The conchoid of Nicomedes.

Let X x (fig. 1) be an indefinite straight line, A a given point, from which draw the straight line A C B perpendicular to X x, and also any number of straight lines A E P, A E P″, &c.; take E P always equal to C B, then the locus of P is the conchoid.

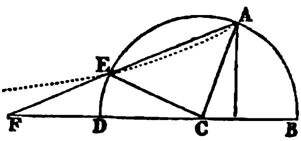
If in E A we take E P′ = E P the locus of P′ is called the inferior conchoid; both conchoids form but one curve, that is, both are expressed by the same equation.

C B is called the modulus, and X x the base or rule;

(1)



(2)



$$\text{Let } A C = a, \qquad C M = x,$$

$$C B = b, \qquad M P = y,$$

then $\quad$ E P : P M : : A P : A N,

or $\quad b : y :: \sqrt{x^2 + (a+y)^2} : a + y,$

$$\therefore\ y^2 x^2 + y^2 (a+y)^2 = b^2 (a+y)^2,$$

$$\therefore\ x^2 = (b^2 - y^2)\left(\frac{a+y}{y}\right)^2$$

$$\therefore\ x = \pm \frac{a+y}{y}\sqrt{b^2 - y^2}.$$

We have three cases according as b is $> a$, $= a$, or $< a$.

Case 1. $b > a$.

	1	2	3	4	5	6	7	8
Values of y	0	b	$< b$	$> b$	$-a$	$-b$	$< -a$	$> -a, < -b$
Values of x	∞	0	pos.	imp.	0	0	pos.	pos.

From (1) X x is an asymptote; from (2) the curve passes through B; from (3) and (4) the curve extends from the asymptote upwards to B and no higher; hence the branch B P P″. Again from (5) and (6) the curve passes through A and D if C D $= b$; from (7) there is a branch A × extending from A to the asymptote; and from (8) the curve exists between A and D; the double value of x gives the same results along C x.

Case 2. $b = a$; in the table of values put $b = a$, and omit (8); thus the figure will be the same as the preceding, with the exception of the oval A P′ D, which vanishes by the coincidence of A and D.

Case 3. $b < a$; in the table of values put b for a in (7), and for (8) write "if y is $> -b$, x is impossible;" the upper part of the curve is not altered, but the point D falls between A and C; from (8) no part of the curve is between D and A; but from (5) A is a point not on the curve, but belonging to it, and called a conjugate point. In this case the lower curve is similar to the upper one.

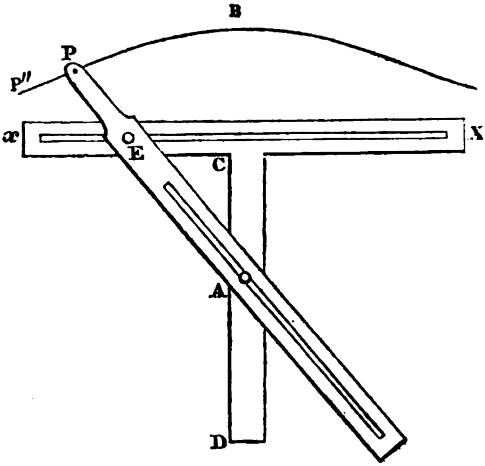
The generation of the conchoid gives a good idea of the nature of an asymptote, for the line E P must always be equal to C B, and this condition manifestly brings the curve continually nearer to C X, as at P″, so that the curve, though never actually coinciding with C X, approaches nearer to it than by any finite distance.

This curve was invented by Nicomedes, a Greek geometrician, who flourished about 200 years B.C. He called it the Conchoid, from a Greek word signifying "a shell;" it was employed by him in solving the problems of the duplication of the cube, and the trisection of an angle.

To show how the curve may be applied to the latter problem, let B C A (fig. 2) be the angle to be trisected; draw A E F meeting the circle in E, and the diameter produced in F, and so that the part E F equal the radius C A, then it is directly seen that the arc D E is one-third of B A.

Now it is not possible by the common geometry, that is, with the straight line and circle alone, to draw the line A E F, so that E F shall be equal to C A (the tentative process, though easy, being never considered geometrically correct), and for a long time the ancient geometricians would not hear of any other mathematical instruments than the ruler and compasses; hence the problem was quite insuperable: finding at last that this was the case, they began to invent some curves to assist in the solution of this and other problems: of these curves, the most celebrated is the conchoid of Nicomedes. It may be thus applied to the present problem. Let A be the pole of the inferior conchoid, B F the asymptote or base, and A C the modulus, the intersection of the curve with the circle evidently gives the required point E. The superior conchoid may also be used for the same purpose.

Unless the curve could be described by continued motion, the solution would be incomplete. Nicomedes therefore invented the following simple machine for describing it. Let x X be a straight ruler with a groove cut in it; C D is another ruler fixed at right angles to x X; at A there is a fixed pin, which is inserted in the groove of a third ruler A E P; in A P is a fixed pin at E, which is inserted into the groove of x X; P E is any given length; then, by the constrained motion of the ruler P E A, a pencil at P will trace out a conchoid, and another pencil fixed in E A would trace out the inferior curve.



This curve was formerly used by architects; the contour of the shaft of a column being the portion B P P'' of a conchoid.

The polar equation to the conchoid is thus found:

Let A (fig. 1, page 156) be the pole, A P $= r$, P A B $= \theta$;

$$\therefore\ y + a = r \cos.\ \theta, \text{ and } x = r \sin.\ \theta.$$

Substituting these values in the equation, and reducing, we arrive at the polar equation $r = a \sec.\ \theta + b$.

The polar equation may, however, be much more easily obtained from the definition of the curve. We have

$$r = \text{A P} = \text{A E} + \text{E P} = \text{A C sec. C A E} + \text{C B} = a \sec.\ \theta + b.$$

313. The following method of obtaining the equation to the conchoid will be found applicable to many similar problems.

Let any number of lines, A E P, fig. 1, be drawn cutting C X in different points E, &c.; from each of these points E as centre, and with radius b describe a circle cutting the line A E P in P and P′; the locus of the point P is the conchoid.

Let A be the origin of the rectangular co-ordinates.

A B the axis of y, and A X parallel to C X in the figure.

Let the general equation to the line A E P be $y = \alpha x$, where α is indeterminate;

Then $y' = a$, and $x' = \frac{a}{\alpha}$ are the equations to the point E;

The equation to the circle which has the point E for its centre and radius b, is

$$(y - y')^2 + (x - x')^2 = b^2,$$

$$\text{or} \quad (y - a)^2 + \left(x - \frac{a}{\alpha}\right)^2 = b^2.$$

And eliminating α between this equation, and that to the line A E P, we have the final equation to the curve,

$$(y - a)^2 + \left(x - \frac{a\,x}{y}\right)^2 = b^2,$$

$$\text{or } (y - a)^2 \frac{x^2 + y^2}{y^2} = b^2.$$

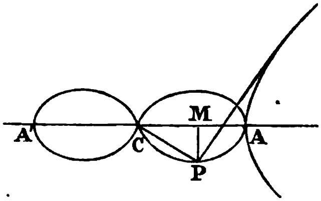
In general if the line C X be a curve whose equation is $y = f(x)$, the co-ordinates of the point E are found by eliminating x and y from the equations $y = \alpha\, x$, and $y = f(x)$; hence we find $x = f'(\alpha)$, and $y = \alpha f'(\alpha)$, and the equation to the circle is

$$\{y - \alpha f'(\alpha)\}^2 + \{x - f'(\alpha)\}^2 = b^2,$$

And the general equation to the curve is

$$\left\{y - \frac{y}{x} f'\left(\frac{y}{x}\right)\right\}^2 + \left\{x - f'\left(\frac{y}{x}\right)\right\}^2 = b^2.$$

314. A perpendicular is drawn from the centre of an hyperbola upon a tangent, find the locus of their intersection.



The equation to the tangent is

$$a^2\, y\, y' - b^2\, x\, x' = -\, a^2\, b^2. \qquad (1)$$

The equation to the perpendicular on it from the centre is

$$y = -\,\frac{a^2}{b^2}\,\frac{y'}{x'}\,x. \qquad (2)$$

In order to get the equation to their intersection, we must eliminate x' and y' from these two equations and that to the hyperbola; from (1) and (2) we find

$$x' = \frac{a^2\, x}{x^2 + y^2},\ y' = \frac{-\,b^2\, y}{x^2 + y^2}.$$

Substituting in the equation $a^2\, y'^2 - b^2\, x'^2 = -\,a^2\, b^2$, we have

$$(x^2 + y^2)^2 + b^2\, y^2 - a^2\, x^2 = 0,$$

which is an equation of the fourth degree.

We shall only investigate the figure in the case when $b = a$, that is, when the hyperbola is equilateral, in which case the equation is $(x^2 + y^2)^2 = a^2 (x^2 - y^2)$,

$$\therefore y^4 + (2x^2 + a^2) y^2 + x^4 - a^2 x^2 = 0,$$

$$\text{and } y = \pm \sqrt{\left\{ -\left(x^2 + \frac{a^2}{2}\right) \pm a \sqrt{2x^2 + \frac{a^2}{4}} \right\}}.$$

If the sign of the interior root be negative, y is impossible; hence we shall only examine the equation

$$y = \pm \sqrt{\left\{ -\left(x^2 + \frac{a^2}{2}\right) + a \sqrt{2x^2 + \frac{a^2}{4}} \right\}};$$

here y is impossible, if $x^2 + \frac{a^2}{2}$ is $> a \sqrt{2x^2 + \frac{a^2}{4}}$,

$$\text{if } x^4 + a^2 x^2 + \frac{a^4}{4} \text{ is } > 2a^2 x^2 + \frac{a^4}{4},$$

$$\text{if } x^4 \text{ is } > a^2 x^2,$$

$$\text{if } x \text{ is } > \pm a;$$

hence we have the following table:

	1	2	3	4
Values of x	0	$\pm a$	$< \pm a$	$> \pm a$
Values of y	0	0	pos.	imp.

From (1) the curve passes through C; from (2) it passes through A and A′; from (3) it has two branches from C to A and from C to A′; from (4) it does not extend beyond A and A′.

We may judge yet more nearly of the form of these ovals, for the tangent at the vertex of the hyperbola being perpendicular to the axis, the oval will cut the axis at A at a right angle; and again at C in an angle of 45°, because the tangent nearly coinciding with the asymptote, the perpendicular on it makes an angle of 45° with the axis ultimately.

This curve was invented by James Bernouilli; it is called the Lemniscata, and forms one of a series of curves corresponding to different values of b.

To find the polar equation to the lemniscata,

$$\text{Let } y = r \sin.\theta, \text{ and } x = r \cos.\theta;$$

hence the equation $(x^2 + y^2)^2 = a^2 (x^2 - y^2)$ becomes $r^2 = a^2 \cos. 2\theta$.

Any curve that is of the form of this figure is called a lemniscata.

315. In the following example the curve may be easily traced by points.

Let a circle be described with centre C and any radius C Q; draw the ordinate Q M, and in Q C take Q P = Q M; the locus of P is a lemniscata.

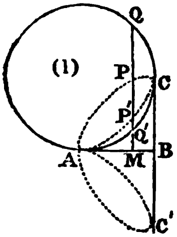
Again, if in M Q we take M R = a third proportional to M Q and C M, the locus of R is another lemniscata whose equation is

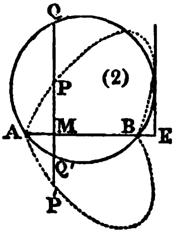
$$x^4 - a^2 x^2 + a^2 y^2 = 0.$$

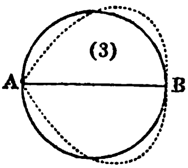
The equation $a^2 (y - a)^2 = (x - a)^2 (2ax - x^2)$ belongs to the same curve referred to a different origin.

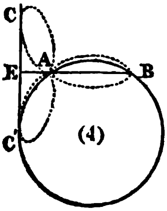
Ex. Trace the locus of the equation $y^2 = x^2 \dfrac{a^2 + x^2}{a^2 - x^2}$.

316. AM, fig. 1, is a tangent to a circle ACQ, MQ an ordinate to the abscissa AM; MP is taken a mean proportional between AM and MQ; required the locus of P.









Let AM $= x$, and MP $= y$, be the rectangular co-ordinates of P, and let the radius of the circle $= b$,

then the square on MP = the rectangle AM, MQ.

To find MQ, we have the equation to the circle

$$(y - y')^2 + (x - x')^2 = r^2,$$

$$\text{or } y^2 - 2by + x^2 = 0, \text{ since } x' = 0, \text{ and } y' = r = b,$$

$$\therefore \text{MQ} = b \pm \sqrt{b^2 - x^2},$$

$$\therefore \text{MP or } y = \pm \sqrt{\{bx \pm x\sqrt{b^2 - x^2}\}}.$$

Since $b^2 - x^2$ is $< b^2$, there are four values of y to each positive value of $x < b$, and no value of y to x negative; hence if AB $= b$, fig. 1, the straight line CBC′ perpendicular to AB is a limit to the curve, and when $x = b$, the ordinate to the curve is equal to the extreme ordinate of the circle, that is, to the tangent BC.

Between $x = 0$, and $x = b$, we have four values of y, which give the two dotted ovals of fig. (1).

To make the question more general we shall suppose the line AB to be a chord of the circle, figs. (2) (3) (4).

Then if b and a are the co-ordinates to the centre of the circle, and A the origin, the equation to the curve will be

$$y = \pm \sqrt{\{bx \pm x\sqrt{b^2 + 2ax - x^2}\}},$$

and we have four cases depending on the values of b and a; hence we have four curves of different forms, yet partaking of the same character and generation.

Case (1). $a = 0$, fig. (1) already discussed.

Case (2). a and b positive, fig. (2). A E $= a + \sqrt{a^2 + b^2}$.

Case (3). $b = 0$, fig. (3).

Case (4). b negative, fig. (4), the equation is

$$y = \pm \sqrt{\{-bx \pm x\sqrt{b^2 + 2ax - x^2}\}}.$$

There are two values of y for x positive, and $< 2a$; but four values for x negative, and $< \sqrt{a^2 + b^2} - a$, that is, $<$ A E.

The gradual transition of one curve to another is apparent, but that the same problem should produce such very different curves as (2) and (4) requires some explanation.

In fig. (1) P and P′ are determined by mean proportionals between A M and M Q, and also between A M and M Q′. Moreover P may be in Q M produced as well as in M Q, thus we have the double oval, fig. (1.) On the left of A the abscissas A M will be negative, and the ordinates M Q positive; hence no possible mean proportional can exist, or no part of the curve can be on the left of A.

In fig. (2) A M and M Q determine the points P and P′; but A M and M Q′ give only an imaginary locus.

Fig. (3) requires no comment.

In fig. (4) the reasoning on fig. (2) will explain the positive side of A; on the left of A the abscissa and both ordinates are negative; therefore two mean proportionals can be found, or four points in the curve for each abscissa.

Such curves may be invented at pleasure, by taking the parabola or other curves for the base instead of the circle.

Ex. To find the locus of the equation $y^4 + 2axy^2 - ax^3 = 0$.

817. To find a point P′, such that drawing straight lines to two given points S and H, we may have the rectangle S P, H P constant.

Join the points S and H, and bisect S H in C; let C be the origin of rectangular axes, S H $= 2a$, C M $= x$, M P $= y$ and let the rectangle S P, H P, $= ab$.

Then since S M $= a + x$, and H M $= a - x$, we have

$$\{y^2 + (a + x)^2\}\{y^2 + (a - x)^2\} = a^2 b^2,$$

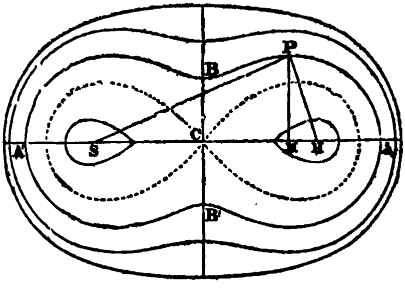
$$\text{or } (y^2 + x^2 + a^2 + 2ax)(y^2 + x^2 + a^2 - 2ax) = a^2 b^2,$$

$$\text{or, } \{y^2 + x^2 + a^2\}^2 - 4a^2x^2 = a^2 b^2;$$

$$\text{hence } \quad y = \pm\sqrt{\{-(a^2 + x^2) + a\sqrt{b^2 + 4x^2}\}}.$$

$$\text{Let } y = 0, \quad \therefore x = \pm\sqrt{a(a \pm b)} \qquad (1).$$

$$\text{Let } x = 0, \quad \therefore y = \pm\sqrt{a(b - a)} \qquad (2).$$



1. Let a be less than b.

Then from (1) we have the points A and A′, and from (2) we have the points B and B′.

Also by comparing the values of y in the original equation and in equation (2) we shall find that M P is greater than C B as long as x is greater than $\sqrt{2a(2a-b)}$; thus the form of the curve must be like that of the figure A P B A′ B′ A.

As b increases, the oval becomes flatter at the top, and takes the form of the outer curves.

2. Let $a = b$, then we have the dotted curve passing through C; also since the equation becomes $(x^2 + y^2)^2 = 2a^2(x^2 - y^2)$ the locus is in this case the lemniscata of Bernouilli.

3. Let a be greater than b.

Then from (1) we have two values of x, and from (2) an impossible value of y; hence the curve must consist of the two small oval figures round S and H.

As b decreases, the little ovals decrease; and when $b = 0$, we have the points S and H themselves for the locus.

These curves are called the ovals of Cassini, that celebrated astronomer having imagined that the path of a planet was a curve like the exterior one in the above figure.

The equation $(y^2 + x^2)^2 = b^2 y^2 + a^2 x^2$, found in art. (123), gives a figure like that in case 1.

318. There are some cases in which it is useful to introduce a third variable; for example, if the equation be $y^4 + x^2 y^2 + 2y^3 + x^3 = 0$, it requires the solution of an equation of three or four dimensions, in order to find corresponding values of x and y; to avoid this difficulty, assume $x = uy$,

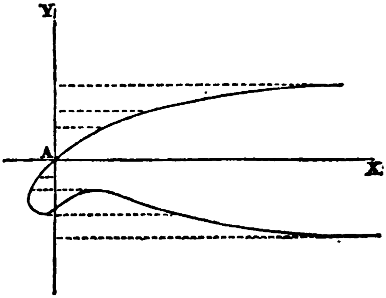
$$\therefore\ y^4 + u^2 y^4 + 2y^3 - u^3 y^3 = 0,$$

$$\text{or,}\quad y + u^2 y + 2 - u^3 = 0,$$

$$\therefore\ y = \frac{u^3 - 2}{u^2 + 1}, \text{ and } x = u.\frac{u^3 - 2}{u^2 + 1};$$

from these equations we can find a series of corresponding values for x and y.

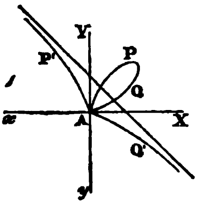
$$\begin{array}{llllll}
u = -3 & , & y = -2\tfrac{9}{10} & , & x = & 8\tfrac{7}{10} \\
u = -2 & & = -2 & & = & 4 \\
 = -1 & & = -1\tfrac{1}{2} & & = & 1\tfrac{1}{2} \\
 = -\tfrac{1}{2} & & = -1\tfrac{7}{10} & & = & \tfrac{17}{20} \\
 = 0 & & = -2 & & = & 0 \\
 = \tfrac{1}{2} & & = -1\tfrac{1}{2} & & = & -\tfrac{3}{4} \\
 = 1 & & = -\tfrac{1}{2} & & = & -\tfrac{1}{2} \\
 = 1\tfrac{1}{2} & & = \tfrac{11}{26} & & = & \tfrac{33}{52} \\
 = 2 & & = 1\tfrac{1}{5} & & = & 2\tfrac{2}{5} \\
 = 3 & & = 2\tfrac{1}{2} & & = & 7\tfrac{1}{2} \\
 = 4 & & = 3\tfrac{11}{17} & & = & 14\tfrac{10}{17} \\
\&c. & & \&c. & & \&c. &
\end{array}$$

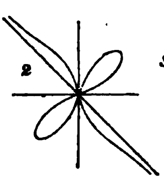


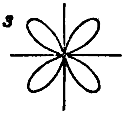
Also when $y = 0$, $x = 0$, hence the curve passes through A. Let A X, A Y be the axes; along the axis of y take values equal to those in the table for y; and from the points thus determined draw lines equal to the corresponding values in the table for x (these are the dotted lines in the figure); by this method we obtain a number of points in the curve sufficient to determine its course.

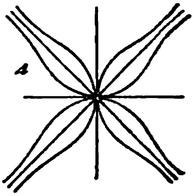
This example is taken from the "Analyse des Lignes Courbes, by G. Cramer. Geneva. 1750," a work which will be found extremely useful in the study of algebraical curves.

319. To trace the curve whose equation is $y^5 - 5ax^2y^2 + x^5 = 0$.









Let x be very small $\therefore$ x^5 being exceedingly small may be omitted, and the equation becomes $y^5 = 5\,a\,x^2\,y^2$, or $y^3 = 5\,a\,x^2$, which is the equation to a semi-cubical parabola P A P′ fig. (1.); and if y be very small, we have $x^3 = 5\,a\,y^2$, which gives the parabola Q A Q′; hence near the origin the curve assumes the forms of the two parabolic branches. Again when x is infinitely great, x^2 may be neglected in comparison with x^5 and the equation becomes $y^5 = -x^5$, $\therefore y = -x$; hence for x positive, we have an infinite branch in the angle X A y, and for x negative an infinite branch in the angle Y A x.

To find the asymptote:

$$y^5 = -x^5 + 5\,a\,x^2\,y^2,$$

$$= -x^5\left(1 - 5\frac{a\,y^2}{x^3}\right),$$

$$\therefore y = -x\left(1 - 5\,a\frac{y^2}{x^3}\right)^{\frac{1}{5}},$$

$$= -x\left\{1 - a\frac{y^2}{x^3} - 2\,a^2\frac{y^4}{x^6} -, \&c.\right\}$$

$$= -x + a\left(\frac{y}{x}\right)^2 + 2a^2\frac{y^3}{x^5} +, \&c.$$

$$= -x + a + 2\frac{a^2}{x^2} +, \&c. \text{ when } y = -x:$$

Therefore the equation to the asymptote is $y + x = a$; this being drawn and the branches A P′, A Q′ produced towards it, we have nearly a correct idea of the curve.

If the equation be $y^5 - 5a^2x^2y + x^5 = 0$, the curve will be traced in the same manner, fig. (2).

If the equation be $y^6 - a^2x^2y^2 + x^6 = 0$, we have fig. (3);

And the equation $y^6 - a^2x^2y^2 - x^6 = 0$ will give fig. (4).

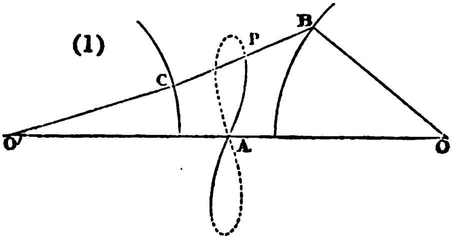
Ex. Find the locus of the equation $y^4 - 4a^2xy - x^4 = 0$.

For the above method of tracing curves of this species, see a treatise on the Differential Calculus, by Professor Miller. Cambridge, 1832.

320. B C is a straight line of given length ($2b$), having its extremities always in the circumferences of two equal circles, to find the locus of the middle point P of the line B C.

Let the line joining the centres O, O′ of the circles be the axis of x, and let the origin of rectangular axes be at A, the bisecting point of O O′.

Let $x\,y$ be the co-ordinates of B.
$x'y'$ . . . . . C.
X Y . . . . . P.



$$AO = AO' = a,$$
$$OB = O'C = c,$$

$$\left.\begin{array}{rr} \text{the equation to B is } y^2 + (x-a)^2 = c^2 & (1) \\ \text{to C is } y'^2 + (x'+a^2)^2 = c^2 & (2) \\ \text{also } (y-y')^2 + (x-x')^2 = 4b^2 & (3) \\ 2Y = y + y' & (4) \\ 2X = x + x' & (5) \end{array}\right\}$$

From these five equations we must eliminate the four quantities y, x, y' and x'; from (1) and (2)

$$y^2 - y'^2 + x^2 - x'^2 - 2a(x + x') = 0,$$

$$\text{or } (y - y')Y + (x - x')X - 2aX = 0 \qquad (6),$$

from (4) and (5) $y^2 + y'^2 + x^2 + x'^2 + 2yy' + 2xx' = 4Y^2 + 4X^2$,

from (3) $y^2 + y'^2 + x^2 + x'^2 - 2yy' - 2xx' = 4b^2$,

from (1) and (2) $2y^2 + 2y'^2 + 2x^2 + 2x'^2 - 4a(x - x') = 4c^2 - 4a^2$,

$\therefore$ by substitution $4a(x - x') = 4(Y^2 + X^2 + b^2 + a^2 - c^2)$,

$$\text{or } (x - x') = \frac{Y^2 + X^2 + m^2}{a}, \text{ if } m^2 = a^2 + b^2 - c^2,$$

and from (6)

$$y - y' = \{2a - (x - x')\} \frac{X}{Y} = \left(2a - \frac{Y^2 + X^2 + m^2}{a}\right) \frac{X}{Y}.$$

Substituting these values of $x - x'$ and $y - y'$ in (3), we have

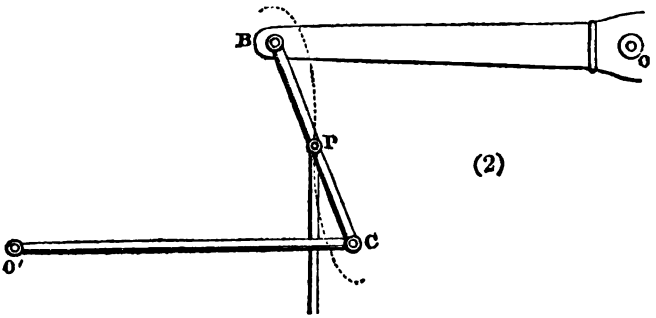
$$\left\{2a - \frac{Y^2 + X^2 + m^2}{a}\right\}^2 \frac{X^2}{Y^2} + \left\{\frac{Y^2 + X^2 + m^2}{a}\right\}^2 = 4 b^2,$$

$$\text{or } 4 a^2 X^2 - 4 (Y^2 + X^2 + m^2) X^2 + \left(\frac{Y^2 + X^2 + m^2}{a}\right)^2 (X^2 + Y^2) = 4 b^2 Y^2$$

$$\therefore \; a^2 X - b^2 Y^2 - X^2 (X^2 + Y^2 + m^2) + (X^2 + Y^2) \left(\frac{X^2 + Y^2 + m^2}{2a}\right)^2 = 0.$$

This equation, being of the sixth dimension, and the highest terms being both positive, the curve must be limited in every direction: when X is very small, there are four values of Y; also when X = 0, we have Y = 0; hence the curve is a species of double oval, or lemniscata.

If the circles be unequal, and P be any point in the line B C, the curve will be of the same nature, but the investigation is much longer.



(2)

The very beautiful contrivance of Watt to reduce a circular to a rectilinear motion is well known to every one. Suppose the point B to be the extremity of an engine-beam, moveable about its centre O, this beam is required to moved a piston-rod always in the same vertical position; it is plain that this motion cannot be obtained by fixing the piston-rod to B, or to any point in O B. Suppose now, a beam O′ C, called the radius-rod, to move about a centre O′, and join the extremities B, C, by a bar B C: the extremity of the piston-rod is fixed to the middle of the beam B C, and its motion, according to the above demonstration, is in a portion of the curve, such as the dark part of the lemniscate in the first figure, and consequently the rod itself continues much more in the same vertical line than if attached to B. The comparative lengths of the rods necessary to render the motion as nearly vertical as possible are stated in most works on the steam-engine, and in the Mechanics' Magazine. For a more complete but very different method of finding the equation to the above curve, see "Prony, Hydraulique."

321. We have no space for the discussion of any higher algebraic curves, if it were necessary; but in fact we have not the means: it must

have been already seen that many of the preceding curves have not been drawn with mathematical exactness; for unless we took the trouble of tracing them by points, we could not easily determine their curvature; we shall therefore pass to the consideration of the general equation of the nth dimension, and then proceed to the intersection of algebraic curves*.

322. The general equation of the nth degree, with all its terms complete, is

$$y^n + (a x + b)\, y^{n-1} + (c x^2 + d x + e)\, y^{n-2} + \dots + f x^n + g x^{n-1} + h x^{n-2} + \dots + k x + l = 0;$$

it contains all the possible combinations of x and y, so that the sum of the exponents in no one term exceeds n.

The number of terms is $1 + 2 + 3 + \dots + (n + 1)$, or is the sum of an arithmetic progression, whose first term and common difference is unity, and the number of terms is $n + 1$; therefore the sum of this series is $\dfrac{(n + 2)\ \ (n + 1)}{2}$.

The number of independent constants is (dividing by the co-efficient of y^n if necessary) one less than the number of terms in the equation, that is,

$$= \frac{(n + 2)\ \ (n + 1)}{2} - 1 = \frac{n\ \ (n + 3)}{2}.$$

323. An algebraic curve of the nth degree may pass through as many given points as it has arbitrary constants, that is, through $\dfrac{n\ (n+3)}{2}$ points, for giving to x and y their values at each one of the given points, we have $\dfrac{n\ (n + 3)}{2}$ different equations, by which the values of the constants may be determined. For example,

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* We must refer our readers to our treatise on the Differential Calculus for information on the curvature of lines. It must not, however, be imagined that algebraic geometry is incapable of exhibiting the form of curves; the following method of determining the curvature is an instance to the contrary.

Let y_1, y_2, and y_3 be three consecutive ordinates, at equal distances from each other; then drawing a corresponding figure, it will be seen that the curve is concave or convex to the axis, according as y_2 is $>$ or $< \dfrac{y_1 + y_3}{2}$; as an example, take the cubical parabola, whose equation is $a^2\, y = x^3$, then the curve is convex, if $2\, x^3$ is $< (x - 1)^3 + (x + 1)^3$ is $> 2 x^3 + 6 x$, which it is, and therefore the curve is convex. The distances at which the ordinates are drawn from each other must depend on the constants in the equation.

Again, to determine the angle at which a curve cuts the axis of x, transfer the origin to that point; then the tangent to the curve at that point and the curve itself make the same angle with the axis; but the value of the tangent of the angle which the tangent to the curve makes with the axis is then $\dfrac{y}{x} = \dfrac{0}{0}$, which may be any value whatever: for example, let $a^2\, y = x$ $\therefore \dfrac{y}{x} = \dfrac{x^2}{a^2} = 0$ when $x = 0$, therefore the curve coincides with the axis at the origin. Again, take the example in art. 307, $y^3 = x^2\, (2 a - x)$; at A we have $\dfrac{y^3}{x^3} = \dfrac{2 a - x}{x} = \dfrac{1}{0}$, and at B we have $\dfrac{y}{2 a - x} = \dfrac{x^2}{y^2} = \dfrac{4\, a^2}{0}$; hence the curve cuts the axis of x in both cases at an angle of 90°.

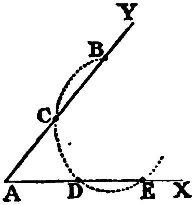
The general equation to the conic-sections, dividing by the co-efficient of y^2, is

$$y^2 + bxy + cx^2 + dy + ex + f = 0,$$

in which there are five co-efficients, and therefore a conic section may pass through five given points; substituting the co-ordinates of the given points separately for x and y we obtain five equations from which the constants can be determined, and thence we have the particular curve required; it will be an ellipse, hyperbola, or parabola, according as $b^2 - 4c$ is negative, positive, or nothing. (79.)

324. The elimination is long, but the trouble may be much lessened by assuming one of the given points for the origin, and two lines drawn from the origin to other two given points for the axes.

For example, if it be required to pass a conic section through four given points B C D E, join B C and D E, and let them meet in A; let A B be the axis of y and A D the axis of x,



Let $AC = y_1$, $AB = y_2$,

$AD = x_1$, $AE = x_2$;

Assume the equation to be

$$y^2 + bxy + cx^2 + dy + ex + f = 0;$$

we have for $x = 0$, $y_1^2 + dy_1 + f = 0$

and $y_2^2 + dy_2 + f = 0$;

$$\therefore \quad d = -(y_1 + y_2), \text{ and } f = y_1 y_2.$$

Similarly for $y = 0$, $e = -c(x_1 + x_2)$, and $f = cx_1 x_2$;

equating the values of f, we have $c = \dfrac{y_1 y_2}{x_1 x_2}$.

Substituting and dividing by $y_1 y_2$, we have

$$\frac{y^2}{y_1 y_2} + \frac{b}{y_1 y^2} x y + \frac{x^2}{x_1 x_2} - \frac{y_1 + y_2}{y_1 y_2} y + \frac{x_1 + x_2}{x_1 x_2} x + 1 = 0,$$

an equation involving only one unknown co-efficient b.

There are some restrictions depending on the situation of the given points; thus no more than two can be in the same straight line, or else the conic section degenerates into two straight lines.

The five given points are the same as five conditions expressed analytically; four are sufficient if the curve is to be a parabola; for $b^2 - 4c = 0$, is equivalent to one. If the curve has a centre, whose position is given, three other conditions suffice, because we may assume the equation to be $y^2 + bxy + cx^2 + f = 0$. If the position of two conjugate diameters be given, only two more conditions are requisite.

Newton, in his Universal Arithmetic, gives excellent methods for describing, by continued motion, a conic section passing through five given points.

325. If it be required to pass a curve, whose species is not given, through a number of given points, we may with advantage assume the equation to be of the form

$$y = a + bx + cx^2 + dx^3 +, \text{ \&c.}$$

The elimination of the constants is more regular, and therefore easier in this equation than in any other: such curves are called parabolic (the three first terms giving the common parabola) and consist of a series of sinuosities, such as in (309), which are easily traced. For the elimination of the constants, see Lagrange, or Lardner's Algebraic Geometry, art. 617.

326. We saw in article (79) that the general equation of the second order sometimes gave straight lines for the loci; such will be the case whenever any equation is reducible into rational factors of the first degree; so that we must not always conclude that an equation of the nth order has a curve of the nth order for its locus. If the equation be reducible into factors of lower degrees, there will be a series of lines corresponding to those factors; thus if an equation of the 4th degree be composed of one factor of two degrees, and two factors of the 1st degree, the loci are a conic section and two straight lines; and hence a general equation of any order embraces under it all curves of inferior orders: if any of the factors be impossible, their loci are either points, or imaginary.

If the sum of the indices of x and y be the same in every term, the loci are either straight lines or points; for an equation of this species will have the form

$$y^n + ay^{n-1}x + by^{n-2}x^2 \dots\dots + lx^n = 0,$$

$$\text{or} \quad \left(\frac{y}{x}\right)^n + a\left(\frac{y}{x}\right)^{n-1} + b\left(\frac{y}{x}\right)^{n-2} \dots\dots + l = 0,$$

let the roots of this equation be α, β, γ, &c., then the equation will be

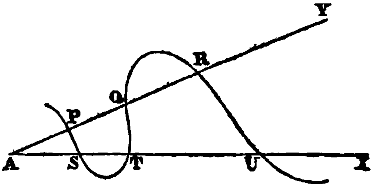
$$\left(\frac{y}{x} - \alpha\right)\left(\frac{y}{x} - \beta\right)\left(\frac{y}{x} - \gamma\right) \dots = 0,$$

each factor of which being $= 0$, its corresponding locus is evidently a straight line; if the roots of the equation be impossible, the corresponding loci are points.

Ex. $y^2 - 2xy \sec.\alpha + x^2 = 0$. The locus consists of two straight lines whose equations are $y = x\,\dfrac{1 \pm \sin.\alpha}{\cos.\alpha} = x \tan.\left(45° \pm \dfrac{\alpha}{2}\right)$ and therefore the lines pass through the origin, and are inclined to the axis of x at angles of $45° \pm \dfrac{\alpha}{2}$.

327. Since the general equation includes all equations below it, the properties of the curve of n dimensions will generally be true for the lower orders, and also for certain combinations of the lower orders; thus, a property of a line of the third degree will be generally true for a conic section, or for a figure consisting of a conic section and a straight line, or for three straight lines. Moreover the lower orders of curves have generally some analogy to the higher curves, and hence the properties of inferior orders often lead to the discovery of those of the superior.

328. From the application of the theory of equations to curves, an immense number of curious theorems arise, which may be seen in the works of Waring and Maclaurin: we have only room for two or three of the most important.



If two straight lines, A X, A Y cut a curve of n dimensions, in the points P Q R, &c., S T U, &c., so that A P, A Q, A R, &c. $= y_1, y_2, y_3$, &c. respectively, and A S, A T, A U, &c. $= x_1, x_2, x_3$, &c. respectively, then if A X and A Y move parallel to themselves, we shall always have $y_1 \,.\, y_2 \,.\, y_3$. &c. : $x_1 \,.\, x_2 \,.\, x_3$. &c., in a constant ratio.

Let the equation to the curve be referred to the origin A, and to axes A X, A Y, by means of the transformation of co-ordinates, and suppose the equation to be

$$y^n + (ax+b)\,y^{n-1} + \ldots cx^n + dx^{n-1} + \ldots kx + l = 0.$$

$$\text{Let } y = 0 \quad \therefore\ cx^n + dx^{n-1} + \ldots\ldots kx + l = 0. \qquad (1)$$

$$x = 0 \quad \therefore\ y^n + by^{n-1} + \ldots\ldots ky + l = 0. \qquad (2)$$

The roots of (1) are A S, A T, A U, &c.; $\therefore\ x_1 \,.\, x_2 \,.\, x_3 .$ &c. $= \frac{l}{c}$.

The roots of (2) are A P, A Q, A R, &c.; $\therefore\ y_1 \,.\, y_2 \,.\, y_3 .$ &c. $= l$.

$$\therefore\ y \,.\, y_2 \,.\, y_3, \text{&c.} : x_1 \,.\, x_2 \,.\, x_3, \text{&c.} :: c : 1.$$

Now the transformation of the axes, parallel to themselves, never alters the co-efficients of y^n and x^n; hence the above ratio is constant for any parallel position of A X and A Y.

Article 293 is an example of this theorem.

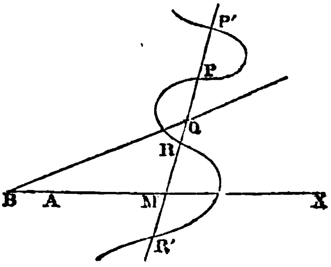
329. A diameter was defined in (76) to be a straight line, bisecting a system of parallel chords; more generally it is a line, such that if any one of its parallel chords be drawn, meeting the curve in various points, the sum of the ordinates on one side shall equal the sum on the other; thus, in the figure, if P Q + P′ Q + &c. = R Q + R′ Q = &c., and the same be true for all lines parallel to P R, then B Q is a diameter.

To find the equation to the diameter B Q let the equation to the curve, referred to A X and a parallel to P Q, &c.

$$y^n + (ax+b)\,y^{n-1} + (cx^2 + dx + e)\,y^{n-2} +, \text{&c.} = 0.$$

Let M Q $= u$, and P Q $= y'$, $\therefore\ y = y' + u$, by substitution we have

$$y'^n + (ax + b + nu)\,y'^{n-1} + \{cx^2 + dx + e + \overline{n-1}\,u \,.\, \overline{ax+b}$$
$$+ n \,.\, \frac{n-1}{2}\,u^2\}\,y'^{n-2} +, \text{&c.} = 0,$$



By the definition the sum of the values of y' must equal nothing, and that sum is the co-efficient of the second term in the last equation with its sign changed,

$$\therefore ax + b + nu = 0,$$

$$\text{or} \quad u = -\frac{ax + b}{n},$$

and this is the equation to the diameter B Q.

Again, by the same reasoning, the equation

$$cx^2 + dx + e + \overline{n-1}u \, . \, \overline{ax + b} + n \, . \, \frac{n-1}{2} u^2 = 0$$

is that to a conic section drawn so that the sum of the products of the values of y, taken two and two together, shall equal nothing.

We might proceed on with the co-efficient of the fourth term.

These curves are sometimes called curvilinear diameters.

330. The method of finding the centre, if any, of a curve, is given in (81); the operation is too long to apply it to a general equation of high dimensions, and therefore we shall take an example among the lines of the third order as fully illustrating the subject.

Let the equation be $xy^2 + ey = ax^3 + bx^2 + cx + d$, under which form are comprehended most of the curves of the third order.

Let $x = x + m$, $y = y + n$; the transformed equation is

$$\begin{gathered} xy^2 + 2nxy + my^2 + (2nm + e)y - ax^3 - (3am + b)x^2 \\ + (n^2 - 3am^2 - 2bm - c)x + mn^2 + en - am^3 - bm^2 \\ - cm - d = 0; \end{gathered}$$

in order that the curve may have a centre, the 2nd, 3rd, 6th, and last or constant term must each $= 0$; $\therefore n = 0$, $m = 0$, $b = 0$, $d = 0$, so that the corresponding curve has a centre, which is the origin, only when the co-efficients b and d are wanting.

CHAPTER XIII.

ON THE INTERSECTION OF ALGEBRAIC CURVES.

331. The intersection of a straight line with a line of the nth order is found by eliminating y from the two equations; hence the resulting equation in terms of x will be of the nth order, and therefore may have n real roots; thus there may be n intersections: there may be less, since some of the roots of the resulting equation may be equal to one another, or some impossible.

Generally speaking, a curve of n dimensions may be cut by a straight line drawn in some direction in n points; but the curve, in its most general form, must be taken; otherwise certain points as conjugate and multiple points, must be considered as evanescent ovals or evanescent branches of the curve, and thus a line passing through such points is equivalent to two or more intersections.

332. The intersections of two lines of the mth and nth orders are found also by eliminating y from both; hence the resulting equation may be of the mnth order, or there may be $m\,n$ intersections; there are often less, for not all the real roots of the equation $X = 0$ will give points of intersection: for example, if we eliminate y from the equations

$$y^2 = 2\,a\,x - x^2 \text{ and } y^2 = 2\,a\,(x - b) \text{ we find } x = \sqrt{2\,a\,b};$$

hence, apparently, there is always an intersection corresponding to the abscissa $\sqrt{2\,a\,b}$; but this is not the case; for then $y^2 = 2\,a\,(\sqrt{2ab} - b)$, and therefore y is impossible, if b is $> 2\,a$, which is evident on drawing the two curves; hence after the abscissa is found, we must examine the corresponding ordinates in each curve; if they are not real, there can be no intersection corresponding to such abscissa.

If we have the two equations $y^2 + 2\,x = 0$, $y^2 + 4\,x^2 - 10\,x - 16 = 0$, the elimination y gives the abscissas of intersection $x = 4$ and $x = -1$, the second of which alone determines a point of intersection.

333. In finding the intersections of lines, we often fall upon a final equation of an order higher than the second, or arrive at an equation whose roots are of a form not readily constructed; to avoid this difficulty a method is often used which consists in drawing a line which shall pass through all the required points of intersection, and thus determine their situation.

Let $y = f\,(x)$* (1), and $y = \phi\,(x)$ (2), be the equations to two lines, then at the point of intersection they have the same ordinates and abscissas; or calling X and Y the co-ordinates of the point of intersection, we have simultaneously $Y = f\,(X)$ and $Y = \phi\,(X)$; hence $f\,(X) = \phi\,(X)$, from which equation X and Y might be obtained, and their values constructed.

But since $Y = f\,(X)\ \ldots\ldots \quad (3)$

and $Y = \phi\,(X)\ \ldots\ldots \quad (4)$

we have by addition $2\,Y = f\,(X) + \phi\,(X)\ \ldots\ldots \quad (5)$

or by multiplication $Y^2 = f\,(X)\,.\,\phi\,(X)\ \ldots\ldots \quad (6)$

or generally $Y = F\,\{f(X),\, \phi(X)\}\ \ldots\ldots (7)$

F implying any function arising from the addition, subtraction, multiplication, &c. of (3) and (4).

Now any one of these equations gives a true relation between the co-ordinates X and Y of the point of intersection of (1) and (2); but by supposing X and Y to vary, it will give a relation between a series of points, of which the required point of intersection is certainly one; that is, drawing the locus of (5) or (6) or (7), it must pass through the required point of intersection of (1) and (2).

It is manifest that if one of the equations (5), (6), or (7), be a circle or straight line, it will be much easier to draw this circle or straight line than to find the intersection by means of elimination.

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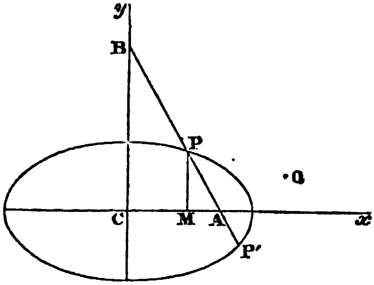
* The symbols $F\,(x)$, $f\,(x)$, $\phi\,(x)$, serve to denote different functions of x, that is, indicate expressions into which the same quantity x enters, but combined in different ways with given quantities. But $f\,(x)$, $f\,(y)$, indicate similar formulæ for both x and y; thus, if $f\,(x) = 2\,a\,x - x^2$, then $f\,(y) = 2\,a\,y - y^2$, or $2\,b\,y - y^2$.

Also we may often find the intersection of (1) and (2), when one of them is a given curve, by drawing the locus of the other, and this method is the simplest when that other is a straight line.

We shall give a few examples to illustrate the subject.

334. From a given point Q without an ellipse, to draw a tangent to it.



Let the co-ordinates of Q be m and n, and let X and Y be the co-ordinates of the point P, where the required tangent meets the curve.

Then by (111) the equation to the tangent through P is

$$a^2 y \mathrm{Y} + b^2 x \mathrm{X} = a^2 b^2,$$

and since this passes through Q we have

$$a^2 n \mathrm{Y} + b^2 m \mathrm{X} = a^2 b^2 \qquad (1)$$

$$\text{and} \quad a^2 \mathrm{Y}^2 + b^2 \mathrm{X}^2 = a^2 b^2. \qquad (2)$$

From (1) and (2) we might, by elimination, find X and Y, and their constructed values would be the co-ordinates C M, M P of the required point.

Now (1) is not the equation to any straight line, but only gives the relation between C M and M P; but if we suppose X and Y to vary, it will give the relation between a series of points, of which P is certainly one; and therefore, if the line whose equation is (1) be drawn, it must pass through P, and consequently, with the ellipse (2), will completely fix the situation of P.

To draw the line (1),

Let $\mathrm{X} = 0$; $\therefore \mathrm{Y} = \dfrac{b^2}{n}$; Let $\mathrm{Y} = 0$; $\therefore \mathrm{X} = \dfrac{a^2}{m}$;

in C y take C B $= \dfrac{b^2}{n}$, and in C x take C A $= \dfrac{a^2}{m}$; join B A; B A produced is the locus of (1), and it cuts the ellipse in two points P and P′; hence if Q P and Q P′ be joined, they are the tangents required.

The same method may be employed in drawing tangents to the parabola and hyperbola.

To take the more general case, let $a y^2 + c x^2 + d y + e x = 0$ (1) be the equation to the curves of the second order referred to axes parallel to conjugate diameters.

Then the equation to a tangent at a point $x' y'$ is

$$a y y' + c x x' + \frac{d}{2}(y + y') + \frac{e}{2}(x + x') = 0,$$

or $(2ay'+d)y+(2cx'+e)x+dy'+ex'=0$ (2).

Let this tangent pass through a point $m\,n$, then (2) becomes

$$(2ay'+d)n+(2cx'+e)m+dy'+ex'=0 \qquad (3),$$

or, $$(2an+d)y'+(2cm+e)x'+dn+em=0 \qquad (4).$$

Now let x' and y' in (4) be considered variable, and construct the straight line, which is the locus of (4); this with the curve itself, determines the position of the secant line which joins the two points on the curve, whence tangents are drawn to the point $m\,n$.

335. Again, suppose the secant line (4) to pass through a given point $m'\,n'$; Then the equation (4) becomes

$$(2an+d)n'+(2cm+e)m'+dn+em=0 \qquad (5),$$

and of course the point $m\ n$, whence tangents were originally drawn, must have a particular position corresponding to each secant line passing through $m'\,n'$; if therefore we make m and n variable in (5) we shall have the equation to the locus of the point $m\,n$

$$(2an'+d)n+(2cm'+e)m+dn'+em'=0$$

where m and n are the variable co-ordinates.

Hence we have the following theorem: if from any point secants be drawn to a line of the second order, and from the two points where each of these secants intersect the curve, tangents be drawn meeting each other, the locus of all such points of concourse is a straight line.

336. To draw a normal to a parabola from a point Q $(a, b,)$ not on the curve.

Let $y^2=4mx$, be the equation to the curve, and let X and Y be the co-ordinates of the required point, then the equation to the tangent at the point X Y, is by (232)

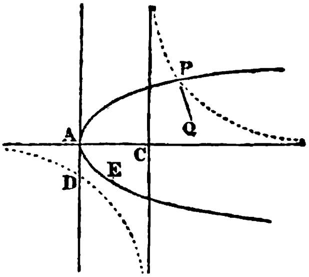
$$Yy=2m(X+x),$$

and therefore that to the normal at X Y is

$$y-Y=-\frac{Y}{2m}(x-X),$$

and since it passes through $(a\,b)$ we have

$$b-Y=-\frac{Y}{2m}(a-X),$$



or, $$XY-(a-2m)Y-2mb=0, \qquad (1)$$

also, $$Y^2=4mX \ldots\ldots \qquad (2)$$

The elimination of X gives $Y^3-4m(a-2m)Y-8m^2b=0$ (3), an equation whose roots would give the three required ordinates.

To avoid this equation we shall construct the locus of (1), which is the equation to an equilateral hyperbola. The axis of x is one asymptote (198), and the other is parallel to the axis of y, and at a distance $AC=a-2m$ from A; the equation to the hyperbola referred to its centre C and asymptotes is $XY=2mb$; moreover the hyperbola cuts the axis of y in the point D, where $AD=\frac{2mb}{2m-a}$; hence this hyperbola (the dotted curve in the figure) may be constructed.

We have drawn the figure, so that there shall be only one intersection of the curves, and hence only one normal is drawn from Q. If the curves touched, as at E, there would be two normals; and if the hyperbola cut the parabola in the lower branch, there would be three normals drawn from Q. These cases correspond respectively to the equation (3), having one real root; three real roots of which two are equal; and, lastly, three real and unequal roots.

337. We must particularly observe that, in the construction of loci, those are to be selected which admit of the easiest description, and of all curves the circle is to be preferred; hence, in the present case, we must look carefully to see if it is possible, by any combination of (1) and (2), to obtain the equation to the circle; for by 333 this will pass through the required normal points.

Multiply (1) by Y, then

$$XY^2 - (a - 2m)\,Y^2 - 2mbY = 0,$$

$$\text{or,}\quad X\,.\,4mX - (a - 2m)\,4mX - 2mbY = 0;$$

$$\therefore X^2 - (a - 2m)X - \frac{b}{2}Y = 0,$$

$$\text{and } Y^2 - 4mX = 0, \text{ from (2)}$$

$$\therefore \text{by addition}\quad Y^2 + X^2 - (a + 2m)X - \frac{b}{2}Y = 0,$$

which is the equation to a circle, the co-ordinates of whose centre are $\frac{a}{2} + m$ and $\frac{b}{4}$, and whose radius is $\sqrt{\left\{\left(\frac{a}{2} + m\right)^2 + \frac{b^2}{16}\right\}}$. Although this circle passes through the vertex of the parabola, yet that point is not one of the required intersections, but merely arises from the multiplication of (1) by Y.

If the parabola and circle be drawn, the latter in various situations according to the position of Q, we shall see, as before, that there will be one, two or three intersections: such practice will be found very useful.

The problem of drawing a normal to an ellipse is of the same nature, only in this case there may be four intersections.

338. The intersection of curves has been employed in the last articles to avoid the resolution of equations resulting from elimination, but the principle may be extended, so as to render curves generally subservient to the solution of equations; for as two equations combined produce one whose roots give the intersection of their loci, so that one may in its turn be separated into two, whose loci can be drawn, and their intersection will determine the roots of the one.

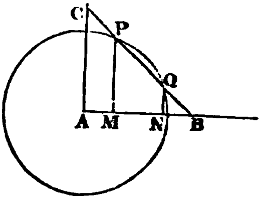
This method, known by the name of "the construction of equations," was much used by mathematicians before the present methods of approximation were invented; it is even now useful to a certain extent, and therefore we proceed to explain it.

Let there be two equations: $y + x = a$ (1), $y^2 + x^2 = b^2$ (2), by elimination we find

$$x^2 - ax + \frac{a^2 - b^2}{2} = 0 \quad (3).$$

We already know that the roots of (3) are the abscissas to the points of intersection of the loci (1) (2); but, conversely, it is manifest that the roots of (3) can be determined by drawing the loci of (1) and (2), and measuring the abscissas of intersection.

Hence if it be required to exhibit geometrically the roots of (3), let it be decomposed into the two equations (1) and (2), and let C P Q B be the locus of (1), and the circle E P Q of (2), having the same origin and axes: draw the ordinates M P, N Q, then A M and A N are the roots of (3).



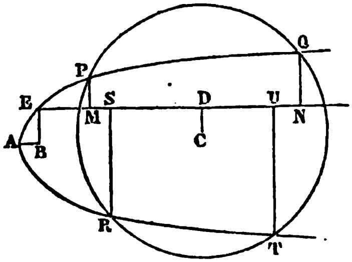
The method consists in parting any given equation into two others, and then drawing the loci of those two; and as it is obvious that there are a great many equations which, when combined together, may produce the given equation, so we may construct a great many loci, whose intersections will give the required roots: thus, in the above case, the equation (3) may be resolved into the two $x^2 = a\,y$, and $a\,y - a\,x + \frac{a^2 - b^2}{2} = 0$, and the corresponding parabola and straight line being drawn, their intersections will give the roots of (3).

In general the roots of an equation can be found by the intersection of any two species of curves whose indices, multiplied together, are equal to the index of the equation: thus, a straight line and a curve of the third order will give the solution of an equation of the third order; and any two conic sections, except two circles, will give the roots of an equation of the fourth order.

339. As equations of the third and fourth order are of frequent recurrence in mathematical researches, we proceed to the solution of the complete equation of the fourth order,

$$y^4 + p y^3 + q y^2 + r y + s = 0.$$

Here the circle and parabola, as curves of easy description, ought to be chosen, and assuming the equation to the parabola a slight artifice will give us that to the circle.



$$\text{Let } y^2 + \frac{p}{2} y = x \quad (1);$$

$$\therefore\ y^4 + p\,y^3 + \frac{p^2}{4}\,y^2 = x^2$$

$$\text{but}\quad y^4 + p y^3 + q y^2 + r y + s = 0,$$

$$\therefore \text{ by subtraction } \left(q - \frac{p^2}{4}\right) y^2 + x^2 + r y + s = 0,$$

$$\text{or from (1), } \left(q - \frac{p^2}{4}\right)\left(x - \frac{p}{2}\,y\right) + x^2 + r y + s = 0;$$

$$\text{or } x^2 + \left(q - \frac{p^2}{4}\right)x + \left(r - \frac{p\,q}{2} + \frac{p^3}{8}\right)y + s = 0,$$

$$\text{and from (1), } y^2 + \frac{p}{2}\,y - x = 0,$$

$$\therefore\ y^2 + x^2 + \left(r + \frac{p^3}{8} + \frac{p}{2} - \frac{p\,q}{2}\right)y + \left(q - 1 - \frac{p^2}{4}\right)x + s = 0 \quad (2).$$

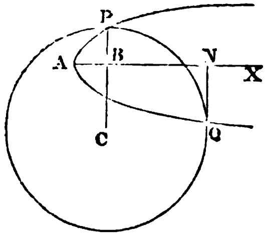
The locus of (1) is the parabola A E Q, the origin being at E $\left(\text{BE} = \frac{p}{4}\right)$, and the co-ordinates rectangular. The locus of (2) is the circle Q P R; the co-ordinates E D, D C of the centre, and the radius are readily determined from (2). The roots of the equation are drawn as if two, P M, Q N were positive, and other two R S, T U were negative. If the circle touch the parabola, two roots are equal; the cases of three or four equal roots can only be discussed by the principles of osculation, but as two roots are sufficient to depress the equation to one of the second order, we need not here consider those cases. If there be only two intersections, two roots are impossible; and if there be no intersection, all four roots are impossible.

340. In practice the operation is shortened by first taking away the second term of the equation; for example, to construct the roots of the equation

$$x^4 + 8\,x^3 + 23\,x^2 + 32\,x + 16 = 0. \qquad (1).$$

Let $x = y - 2$, and the reduced equation is

$$y^4 - y^2 + 4\,y - 4 = 0. \qquad (2).$$



Let $y^2 = x$ (3);

$$\therefore \text{ by substitution } x^2 - x + 4\,y - 4 = 0,$$

$$\therefore \text{ by addition } y^2 + x^2 + 4\,y - 2\,x - 4 = 0,$$

$$\text{or, } (y + 2)^2 + (x - 1)^2 = 9 \quad (4).$$

Let P A Q be the parabola (3), whose parameter is unity, the co-ordinates of the centre C of the circle (4) are A B = 1, and B C = − 2, the radius = 3. Describing this circle, the ordinates B P and Q N are the possible roots of (2); measuring these values we shall find P B = 1, and Q N = − 2; hence the possible roots of (2) are 1 and − 2, and therefore those of (1) are − 1 and − 4.

341. The construction of equations of the third order is involved in that

of the fourth order. Take away the second term, if necessary, multiply the resulting equation $Y = 0$ by y, and then proceed precisely as in the last article. The circle will always pass through the vertex of the parabola, but this intersection gives the root $y = 0$, introduced by multiplication, and has therefore nothing to do with the roots of the given equation. This circumstance of the circle passing through the vertex of the parabola, is singularly convenient, as it entirely saves the trouble of calculating the radius to decimal places, which is often necessary in the preceding cases.

Ex. 1. $x^3 - 6x^2 - x + 6 = 0$. Let $x = y + 2$;

$$\therefore y^3 - 13y - 12 = 0,$$

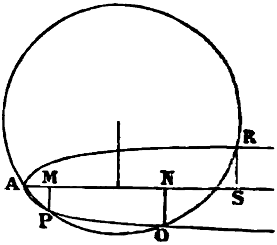
$$\text{or, } y^4 - 13y^2 - 12y = 0.$$

$$\text{Let } \quad y^2 - x \qquad = 0 - (1)$$

$$\therefore x^2 - 13x - 12y = 0,$$

$$\therefore y^2 - 12y + x^2 - 12x = 0,$$

$$\text{or, } (y - 6)^2 + (x - 6)^2 = 72 \quad (2).$$



The three roots of y, as given by the figure, are 4, − 1 and − 3; hence the values of x are 6, − 1 and − 1.

Ex. 2. $4y^3 + 6y - 5 = 0$. There is one possible root nearly $= \frac{1}{\sqrt{2}}$.

Ex. 3. $4y^3 - 3y + 1 = 0$.

There never can be any difficulty in constructing the loci of these equations; having once drawn a parabola, whose parameter is unity, with tolerable exactness, it will serve for the construction of any number of such equations.

As another example, we take the following question.

342. To find two mean proportionals between two given lines a and b.

Let y and x be the required lines;

$$\text{then } \quad a : y :: y : x, \quad \therefore y^2 = ax \qquad (1),$$

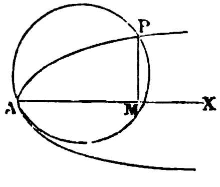
$$y : x :: x : b, \quad \therefore x^2 = by \qquad (2),$$

$$\therefore y^4 = a^2x^2 = a^2by, \text{ or } y^3 - a^2b = 0;$$

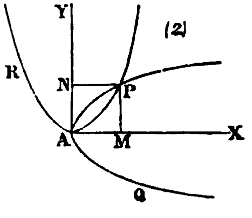
but by addition of (1) and (2), $\quad y^2 - by + x^2 - ax = 0,$

$$\text{or, } \left(y - \frac{b}{2}\right)^2 + \left(x - \frac{a}{2}\right)^2 = \frac{a^2 + b^2}{4} \quad (3).$$

(1.)



(2)



Let P A Q be the parabola (1), then the intersection of the circle (3) will give M P and A M, the two mean proportionals required.

The other roots of the equation $y^3 - a^2b = 0$ are impossible.

This problem was one of those so much celebrated by the ancient mathematicians. Menechme, of the school of Plato, was the first who gave a solution of it: his method being particularly ingenious, as well as being the first instance known of the application of geometrical loci to plain problems, is well worth insertion.

With a parameter a, draw the parabola P A Q (fig. 2), and on A Y perpendicular to A X describe the parabola P A R with parameter b.

Then the rectangle a, A M or a, N P is equal to the square on M P;

$$\therefore\ a,\ \mathrm{MP} \text{ and } \mathrm{NP} \text{ are in continued proportion.}$$

Again, the rectangle b, A N or b, M P is equal to the square on N P;

$$\therefore\ \mathrm{MP},\ \mathrm{NP}, \text{ and } b, \text{ are in continued proportion;}$$

hence we have at the same time the two proportions

$$a : \mathrm{MP} :: \mathrm{MP} : \mathrm{NP} \text{ and } \mathrm{MP} : \mathrm{NP} :: \mathrm{NP} : b;$$

$$\therefore\ a,\ \mathrm{MP},\ \mathrm{NP}, \text{ and } b, \text{ are in continued proportion.}$$

Menechme also gave a second solution depending on the intersection of a parabola and hyperbola.

343. To find a cube which shall be double of a given cube.

Let a be a side of the given cube, then the equation to be solved is

$$y^3 = 2a^3, \text{ or } y^4 - 2a^3 y = 0,$$

$$\text{Let } y^2 = ax \ (1), \quad \therefore\ a^2x^2 - 2a^3y = 0, \quad \text{or, } x^2 - 2ay = 0;$$

$$\therefore \text{ by addition, } \quad y^2 - 2ay + x^2 - ax = 0 \quad (2);$$

The loci of (1) and (2) being drawn, the ordinate P M of their intersection is the side of the required cube.

This problem, like the former, occupied the attention of the early geometricians; they soon discovered that its solution is involved in the preceding one; for if $b = 2a$, the resulting equations are the same.

In this manner a cube may be found which shall be m times greater than a given cube.

344. We may thus find any number of mean proportionals between two given quantities a and b.

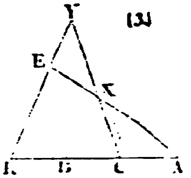
For if y be the first of the mean proportionals, they will form the following progression:

$$a,\ y,\ \frac{y^2}{a},\ \frac{y^3}{a^2},\ \frac{y^4}{a^3}, \ \&\text{c}.$$

Let there be four mean proportionals, then the sixth term of the progression being b we have $\frac{y^5}{a^4} = b$, or $y^5 - a^4 b = 0$.

Describe the parabola whose equation is $y^2 = ax$, and then draw the locus of the equation $y x^2 - a^2 b = 0$. The last curve consists of an hyperbolic branch in each of the angles Y A X, Y A x, and therefore the ordinate corresponding to the real root is readily found.

345. Newton constructed equations by means of the conchoid of Nicomedes: he justly observes that those curves are to be preferred whose mechanical description is the easiest; and he adds, that of all curves, the conchoid next to the circle is, in this respect, the most simple. See the instrument in (312). The following is one of the many examples given in the Universal Arithmetic.

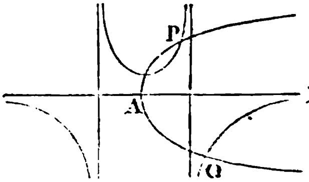


Let the equation be $x^3 + qx + r = 0$, draw a straight line K A, of any length n. In K A take K B $= \frac{q}{n}$, and bisect B A in C; with centre k and radius K C describe a circle, in which inscribe the straight line C X $= \frac{r}{n^2}$; join A X, and between the lines C X and A X produced, inscribe E Y equal to C A, so that, when produced, it passes through the point K.

A geometrical proof follows to show that, from this construction, the equation for the length of X Y is $x^3 + qx + r = 0$, so that X Y is a root of the equation.

The conchoid is employed to insert the line E Y between C X and C A. Let K be the pole, A X E the base, and C A the modulus; then the common description of the curve determines the point Y on the line C X Y, such that E Y = C A.

346. With regard to the higher equations, there is not much advantage in constructions, since it is extremely difficult to draw the curves with sufficient exactness. The method, however, is so far useful as enabling us to detect the number of impossible roots in any equation, as we can generally trace the curves with sufficient accuracy to determine the number of intersections, though not the exact points of intersection.



Ex. $y^3 - 3y + 1 = 0$.

Let $y^2 = x$ . . . . . (1),

$\therefore yx^2 - 3y + 1 = 0$, (2)

or $y = \frac{1}{3 - x^2}$;

the locus of (1) is a parabola P A Q, that of (2) is a curve of the third order, and there are three intersections; and, therefore, three possible roots, two positive, and one negative.

347. There is some uncertainty in the employment of curves in finding roots; we stated in (332), that real roots may correspond to imaginary intersections; so, on the contrary, imaginary intersections, or what is the same, the absence of intersections, does not always prove the absence of real roots; for example, if to prove the equation $x^4 + 15x + 14 = 0$ we assume $y^2 = x^3$ (1), and therefore $xy^2 + 15x + 14 = 0$ (2), the loci of (1) and (2) will not intersect, but yet two roots are possible. The error was in choosing a curve (1), which proceeds only in the positive direction, when from the form of the equation it is apparent that there are negative roots. Taking the circle and common parabola for the loci, as in (340), we shall find the roots to be -1 and -2. Hence, in general, to ascertain real roots it will be advisable to try more than two curves.

CHAPTER XIV.

TRANSCENDENTAL CURVES.

348. It was stated in art. (23), that those equations which cannot be put into a finite and rational algebraical form with respect to the variables, are called Transcendental; of this nature are the equations $y = \sin. x$ and $y = a^x$. In Chapter XII. we have obtained the equations to curves, generally from some distinct Geometrical property of those curves; but there are many curves whose equations thus obtained cannot be expressed in the ordinary language of algebra; that is, the equation resulting from the description or generation of the curve is dependent upon Trigonometrical or Logarithmical quantities; these curves, from the nature of their equations, are called Transcendental.

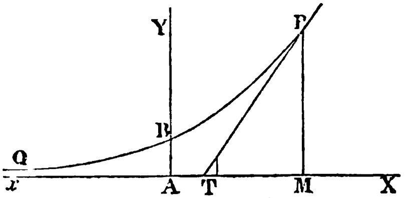
We shall here investigate the equations and the forms of the most celebrated of these curves, and mention a few of the remarkable properties belonging to them, although they can be only fully investigated by the higher calculus.

349. In this class will be found some curves, as the Cardioide, whose equations may be expressed in finite algebraic terms; but these examples are only particular cases of a species of curves decidedly Transcendental, and which cannot be separated from the rest without injury to the general arrangement.

Some of the Transcendental class have been called Mechanical curves, because they can be described by the continued motion of a point; but this name as a distinction is erroneous, for it is very probable that all curves may be thus described by a proper adjustment of machinery.

THE LOGARITHMIC CURVE.

350. The curve Q B P, of which the abscissa A M is the logarithm of the corresponding ordinate M P, is called the Logarithmic curve.



Let A M $= x$, M P $= y$, then $x = \log. y$, that is, if a be the base of the system of logarithms, $y = a^x$.

To examine the course of the curve we find when $x = 0$, $y = a^0 = 1$; as x increases from 0 to ∞, y increases from 1 to ∞; as $-x$ increases to ∞, y decreases from 1 to 0. In A Y take A B = the linear unit, then the curve proceeding from B to the right of A B, recedes from the axis of x, and on the left continually approaches that axis, which is therefore an asymptote.

This curve was invented by James Gregory; Huyghens discovered that if P T be a tangent meeting A X in T, M T is constant and equal to the modulus $\left(\frac{1}{\log. a}\right)$ of the system of logarithms. Also that the whole area M P Q x extending infinitely towards x is finite, and equal to twice the triangle P M T, and that the solid described by the revolution of the same area about X x is equal to 1½ times the cone, by the revolution of P T M about X x.

That such areas and solids are finite is curious, but not unintelligible to those who are accustomed to the summation of decreasing infinite series.

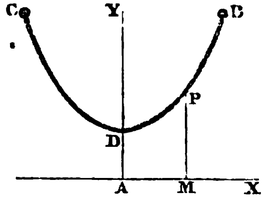
If the equation be $y = a^{-x}$, the curve is the same, but placed in the opposite direction with regard to the axis of y.

351. The equation to the curve called the Catenary, formed by suspending a chain, or string, between two points B and C, is

$$y = \tfrac{1}{2}\,(e^{x} + e^{-x})$$

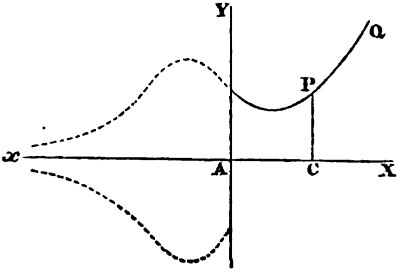
where $\text{A M} = x$, $\text{M P} = y$,

and $\text{A D} = 1$.



This equation cannot be obtained by the ordinary algebraical analysis; but it is evident that the curve may be traced from this equation, by adding together the ordinates of two logarithmic curves corresponding to the equations $y = e^{x}$ and $y = e^{-x}$.

352. Trace the locus of the equation $y = a^{\frac{1}{x}}$. (Fig. 1.)



353. To trace the curve whose equation is $y = x^{x}$. Let $x = 0$ $\therefore$ $y = 1$; let $x = 1$ $\therefore$ $y = 1$; and between $x = 0$ and $x = 1$, we have y less than 1; also x increases from 1 to ∞, y increases to infinity:

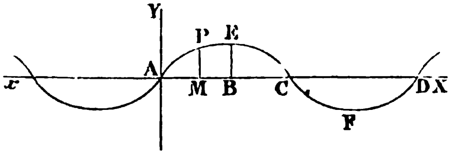
hence if A B = 1 (fig. 2,) and A C = 1, we have the branch B P Q corresponding to positive values of x.

Let x be negative $\therefore\ y = (-x)^{-x} = \frac{1}{(-x)^x}$; now if we take for x three consecutive values, as 2, $2\frac{1}{2}$, 3, it is evident that y will be positive, impossible, or negative; hence the curve must consist of a series of isolated points above and below the axis A x.

For further information on this subject see a very interesting memoir by M. Vincent, in the fifteenth volume of the " Annales des Math." M. Vincent calls such discontinuous branches by the name " Branches Ponctuées ;" and he also shows, that in the common logarithmic curve there must be a similar branch below the axis of x, corresponding to fractional values of x with even denominators.

THE CURVE OF SINES.

354. The curve A P E C, of which the ordinates M P, B E are the sines of the corresponding abscissas A M, A B, is called the Curve of Sines.



Let A M = x, M P = y, then the equation is $y = \sin. x$,

$$y = r \sin. \frac{x}{r}.$$

	1	2	3	4	5
Values of x	0	$\frac{\pi r}{2}$	πr	$\frac{3\pi r}{2}$	$2\pi r$
Values of y	0	r	0	$-r$	0

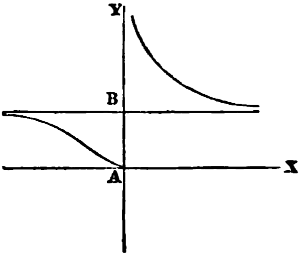
Take A B = $\frac{\pi r}{2}$, A C = πr, A D = $2\pi r$; then from (1) the curve cuts the axis at A; from (2), if B E = r, the curve passes through E, and this is the highest point of its course, because between (1) and (2) y increases, and between (2) and (3) y decreases; the curve cuts the axis again in C; from C, y increases negatively until it equals $-r$, and then decreases to 0, so that we have a second branch C F D equal and similar to the first. Beyond D the values of y recur, and the curve continues the same course *ad infinitum;* also since $\sin. (-x) = -\sin. x$ there is a similar branch to the left of A.

This curve may be supposed to arise from the development of circular arcs into a straight line X x, ordinates being drawn corresponding to the sines of these circular arcs.

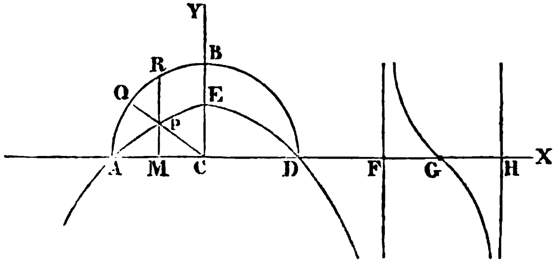
In a similar manner the curve of cosines, of versed sines, of tangents, &c., may readily be investigated.

If the ordinates of the curve of sines be increased or diminished in a given ratio, the resulting curve ($y = m$ sin. x) is the curve formed by the simple vibration of a musical chord: hence this curve is called the Harmonic Curve.

355. The accompanying figure belongs to the curve whose equation is $y = x$ tan. x. Such curves are useful in finding the roots of an equation as x tan. $x = a$; for, supposing the curve to be described, in A Y take A B $= a$, and from B draw a line parallel to A X; then the ordinates corresponding to the points of intersection of this straight line with the curve are the values of y, that is, of x tan. x.



THE QUADRATRIX.



356. Let C be the centre of a circle A Q B D; let the ordinate M R move uniformly from A to B C, and in the same time let the radius C Q, turning round C, move from C A to C B; then the intersection P of C Q and R M traces out a curve called the Quadratrix.

Let A be the origin, A M $= x$, M P $= y$, A C $= r$, angle A C Q $= \theta$,

$$\text{Then } AM : AC :: AQ : AB,$$

$$x : r :: r\theta : \frac{\pi r}{2} \therefore \theta = \frac{\pi x}{2r}.$$

$$\text{But } MP = MC \tan. \theta,$$

$$\therefore y = (r - x) \tan. \frac{\pi x}{2r}, \text{ which is the equation to the curve.}$$

When $x = 0$, $y = 0$; $\therefore$ the curve passes through A; as x increases from 0 to r, y increases, because the tangent increases faster than the angle; when $x = r = AC$, $y = \frac{0}{0}$, the real value of which found by the Differential Calculus is $\frac{2r}{\pi}$; hence if $CE = \frac{2r}{\pi}$, the curve passes through E; as x increases beyond r the tangent diminishes but is negative, and so is $r - x$; $\therefore y$ is positive and diminishes until it finally becomes 0, when $x = 2r = AD$; when x is greater than $2r$ the tangent is positive, therefore y is negative and increases; when $x = 3r$, the tangent $= \infty$; $\therefore y = -\infty$; this gives an asymptote through F. As x increases beyond $3r$ the tangent decreases but is negative; hence y is positive; when $x = 4r$, $y = 0$, when $x = 5r$, $y = -\infty$, and between $x = 4r$ and $5r$, y is negative: therefore we have the branch between the asymptotes at F and H, and proceeding onwards we should find a series of branches like the last. The part of the curve to the left of A is the same as that to the right of D.

This curve was invented most probably by a Greek mathematician of the name of Hippias, a cotemporary of Socrates; his object was to trisect an angle, or generally to divide an angle into any number of equal parts, and this would be done if the curve could be accurately drawn; thus to trisect an angle A C Q, draw the quadratrix and the ordinate M P, trisect the line A M in the points N and O, draw the ordinates N S, O T to the quadratrix. Then from the equation $\theta = \frac{\pi x}{2r}$, we shall see that C S and C T trisect the angle A C Q.

This curve was afterwards employed by Dinostratus to find the area or quadrature of the circle, and hence its name: supposing the point E to be determined by mechanical description we have the value of π given by the equation $CE = \frac{2r}{\pi}$, and therefore the ratio of the circumference to the diameter of the circle would be known.

There is another quadratrix, that of Tschirnhausen, which is generated by drawing two lines through Q and M parallel respectively to A C and B C, and finding the locus of their intersection; its equation will be

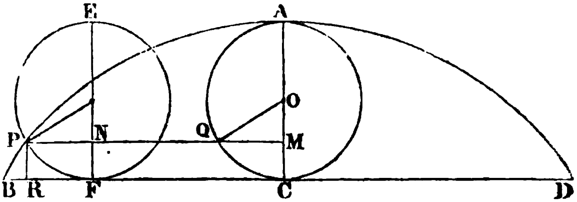
$$y = r \cos.\left(\frac{\pi}{2} - \theta\right) = r \sin.\theta = r \sin.\frac{\pi x}{2r}.$$

THE CYCLOID.

357. If a circle E P F be made to roll in a given plane upon a straight line B C D, the point in the circumference which was in contact with B at the commencement of the motion, will, in a revolution of the circle, describe a curve B P A D, which is called the cycloid.

This is the curve which a nail in the rim of a carriage-wheel describes in the air during the motion of the carriage on a level road; hence the generating circle E P F is called the wheel. The curve derives its name from two Greek words signifying "circle formed."

The line B D which the circle passes over in one revolution is called the base of the cycloid; if A Q C be the position of the generating circle in the middle of its course, A is called the vertex and A C the axis of the curve. The description of the curve shows that the line B D is equal to the circumference of the circle, and that B C is equal to half that circumference. Hence also if E P F be the position of the generating circle, and P the generating point, then every point in the circular arc P F having coincided with B F, we have the line B F = the arc P F, and F C = the arc E P or A Q;



Draw P N Q M parallel to the base B D.

Let A be the origin of rectangular axes,

A C the axis of x, and O the centre of the circle A Q C.

Let A M $= x$, A O $= a$,

M P $= y$, angle A O Q $= \theta$;

then by the similarity of position of the two circles, we have

$$PN = QM, \text{ and } PQ = NM;$$

$$\therefore MP = PQ + QM = NM + QM = FC + QM = \text{arc } AQ + QM$$

$$\text{that is, } y = a\theta + a \sin.\theta = a(\theta + \sin.\theta) \quad (1)$$

$$x = a - a\cos.\theta = a \text{ vers.}\theta \quad (2)$$

The equation between y and x is found by eliminating θ between (1) and (2)

$$\cos.\theta = \frac{a-x}{a} \therefore \sin.\theta = \frac{\sqrt{2ax - x^2}}{a}$$

$$\text{and } y = a\theta + a\sin.\theta$$

$$= a\cos.^{-1}\left(\frac{a-x}{a}\right) + \sqrt{2ax - x^2}$$

But we can obtain an equation between x and y from (1) alone; that is, from the equation, M P = arc A Q + Q M.

For arc A Q = a circular arc whose radius is a and versed sine x

$$= a\left\{\text{a circular arc whose radius is unity and vers. sin. } \frac{x}{a}\right\}$$

$$= a \text{ vers.}^{-1}\frac{x}{a}$$

$$\therefore y = a \text{ vers.}^{-1}\frac{x}{a} + \sqrt{2ax - x^2}.$$

If the origin is at B, $BR = x$ and $RP = y$, the equations are

$$x = a\theta - a \sin.\theta$$

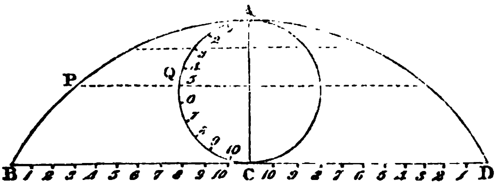
$$y = a - a \cos.\theta.$$

We shall not stop to discuss these equations, as the mechanical description of the curve sufficiently indicates its form.

The cycloid, if not first imagined by Galileo, was first examined by him; and it is remarkable for having occupied the attention of the most eminent mathematicians of the seventeenth century.

Of the many properties of this curve the most curious are that the whole area is three times that of the generating circle, that the arc A P is double of the chord of A Q, and that the tangent at P is parallel to the same chord. Also that if the figure be inverted, a body will fall from any point P on the curve to the lowest point A in the same time; and if a body falls from one point to another point, not in the same vertical line, its path of quickest descent is not the straight line joining the two points, but the arc of a cycloid, the concavity or hollow side being placed upwards.

358. Given the base of a cycloid to trace the curve.

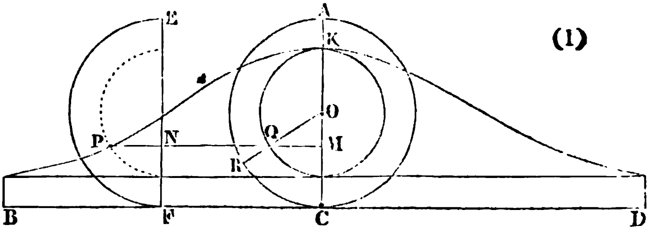


Let the base B D be divided into twenty-two equal parts, and let them be numbered from B and D towards the middle point C; from C draw the perpendicular line C A equal to 7 of these parts; and on A C describe a circle A Q C. Along the circumference mark off the same number of equal parts, either by measurement or by applying the line B C to the circle C A. In the figure the point Q is supposed to coincide with the end of the fifth division from the top.

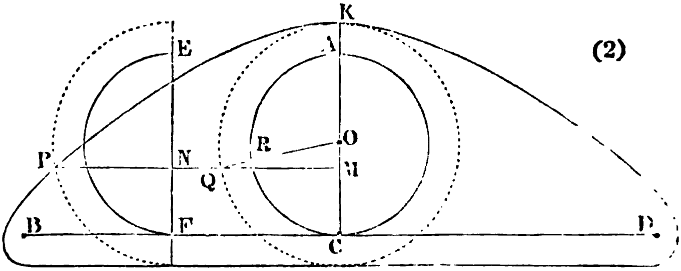
Then the arc C Q being equal to the length C 5 measured on the base, if P Q be drawn parallel to the base, and equal to the remainder of the base, that is, to B 5 or A Q, it is evident that P is a point in the cycloid, and thus any number of points may be found.

The ratio of the circumference to the diameter of a circle is generally taken as in this case to be as 22 to 7.

359. Instead of the point P being on the circumference of the circle, it may be anywhere in the plane of that circle, either within or without the circumference. In the former case the curve is called the prolate cycloid or trochoid, (fig. 1,) in the latter case the curtate or shortened cycloid, (fig. 2.)



(1)



(2)

B D is the base on which the generating circle A R C rolls, O the centre of the generating circle, P the describing point when that circle is at F. Draw P N Q M parallel to the base.

Let A be the origin of rectangular axes,

$$A M = x,\ M P = y,\ A O = a,\ K O = m a,\ \text{angle } A O R = \theta, \text{ then}$$

$$M P = M N + P N = M N + Q M = F C + Q M = \text{arc } A R + Q M$$

$$\text{and } A M = A O + O M;$$

$$\therefore\ y = a\theta + m a \sin.\theta$$

$$\text{and}\quad x = a \text{ vers. } \theta.$$

These are the equations to the prolate, curtate, or common cycloid, according as m is less than, greater than, or equal to, unity.

If the vertex K of the curve be the origin of co-ordinates in figs. (1) and (2,) we have $K O = a$, and $A O = m a$: also $M P = F C + Q M = \text{arc } A R + Q M$

$$\therefore\ y = m a \theta + a \sin.\theta$$

$$= m \text{ vers.}^{-1}\frac{x}{a} + \sqrt{2 a x - x^2}$$

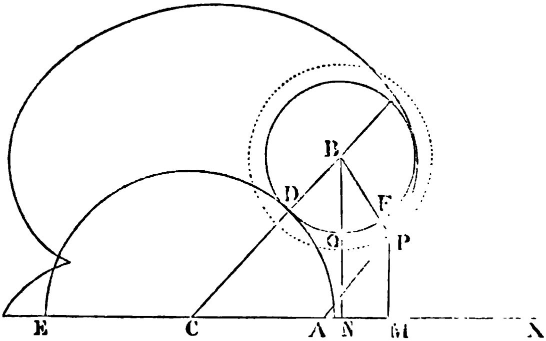
The curve whose equations are $y = a\,\theta$, and $x = a$ vers. θ is called the companion to the cycloid.

360. The class of cycloids may be much extended by supposing the base on which the circle rolls to be no longer a straight line, but itself a curve: thus let the base be a circle, and let another circle roll on the circumference of the former; then a point either within or without the circumference of the rolling circle will describe a curve called the epitrochoid; but if the describing point is on the circumference, it is called the epicycloid.

If the revolving circle roll on the inner or concave side of the base, the curve described by a point within or without the revolving circle is called the hypotrochoid; and when the point is on the circumference it is called the hypocycloid.

To find the equation to the epitrochoid.

Let C be the centre of its base E D, and B the centre of the revolving circle D F in one of its positions: C A M the straight line passing through the centres of both circles at the commencement of the motion; that is, when the generating point P is nearest to C or at A.



Let C A be the axis of x,

$CM = x$, $MP = y$,

$CD = a$, $DB = b$,

$BP = mb$, angle $ACB = \theta$

Draw B N parallel to M P, and P Q parallel to E M. Then, since every point in D F has coincided with the base A D, we have $DF = a\theta$, and angle $DBF = \frac{a\theta}{b}$; also angle FBQ = angle FBD − angle QBD

$$= \frac{a\theta}{b} - \left(\frac{\pi}{2} - \theta\right) = \frac{a+b}{b}\theta - \frac{\pi}{2}$$

Now $CM = CN + NM = CB \cos. BCN + PB \sin. PBQ$

$$= (a+b)\cos.\theta + mb\sin.\left(\frac{a+b}{b}\theta - \frac{\pi}{2}\right).$$

And $MP = BN - BQ = (a+b)\sin.\theta - mb\cos.\left(\frac{a+b}{b}\theta - \frac{\pi}{2}\right)$

$$\left.\begin{array}{l} \text{or } x = (a+b)\cos.\theta - mb\cos.\frac{a+b}{b} \\ \text{and } y = (a+b)\sin.\theta - mb\sin.\frac{a+b}{b}\theta \end{array}\right\} (1)$$

The equations to the epicycloid are found by putting b for $m\,b$ in (1.)

$$\left.\begin{aligned} \therefore\ x &= (a + b)\cos.\theta - b\cos.\frac{a+b}{b}\theta \\ \text{and } y &= (a + b)\sin.\theta - b\sin.\frac{a+b}{b}\theta \end{aligned}\right\}\ (2)$$

The equations to the hypotrochoid may be obtained in the same manner as the system (1), or more simply by putting $-b$ for b in the equations (1.)

$$\left.\begin{aligned} \therefore\ x &= (a - b)\cos.\theta + m\,b\cos.\frac{a-b}{b}\theta \\ \text{and } y &= (a - b)\sin.\theta - m\,b\sin.\frac{a-b}{b}\theta \end{aligned}\right\}\ (3)$$

The equations to the hypocycloid are found by putting $-\ b$ for both b and $m\,b$ in system (1.)

$$\left.\begin{aligned} \therefore\ x &= (a - b)\cos.\theta + b\cos.\frac{a-b}{b}\theta \\ \text{and } y &= (a - b)\sin.\theta - b\sin.\frac{a-b}{b}\theta \end{aligned}\right\}\ (4)$$

We have comprehended all the systems in (1), but each of them might be obtained from their respective figures.

361. The elimination of the trigonometrical quantities is possible, and gives finite algebraic equations whenever a and b are in the proportion of two integral numbers. For then $\cos.\theta$, $\cos.\frac{a+b}{b}\theta$, $\sin.\theta$, &c., can be expressed by trigonometrical formulas, in terms of $\cos.\phi$ and $\sin.\phi$, where ϕ is a common submultiple of θ and $\frac{a+b}{b}\theta$; and then $\cos.\phi$ and $\sin.\phi$ may be expressed in terms of x and y. Also since the resulting equation in $x\,y$ is finite, the curve does not make an infinite series of convolutions, but the wheel or revolving circle, after a certain number of revolutions, is found, having the generating point exactly in the same position as at first, and thence describing the same curve line over again.

For example, let $a = b$, the equations to the epicycloid become

$$x = a\,(2\cos.\theta - \cos.2\,\theta)$$
$$y = a\,(2\sin.\theta - \sin.2\,\theta)$$
$$\therefore\ x = a\,(2\cos.\theta - 2\,(\cos.\theta)^2 + 1)$$
$$\text{and } y = 2\,a\sin.\theta\,(1 - \cos.\theta).$$

From the first of these equations we find $\cos.\theta$, and then from the second we have $\sin.\theta$, adding the values of $(\cos.\theta)^2$ and $(\sin.\theta)^2$ together and reducing, we have

$$(y^2 + x^2 - 3\,a)^2 = 4\,a^4\left(3 - \frac{2\,x}{a}\right)$$
$$\text{or } \{x^2 + y^2 - a^2\}^2 - 4\,a^2\,\{(x - a)^2 + y^2\} = 0.$$

This curve, from its heart-like shape, is called the cardioide.

Let A be the origin; that is, for x put $x + a$ in the last algebraical equation, and then by transformation into polar co-ordinates, the equation to the cardioide becomes

$$r = 2a\,(1 - \cos.\,\phi).$$

362. If $b = \frac{a}{2}$ the equations (4) to the hypocycloid become

$$x = a\cos.\,\theta$$
$$\text{and } y = 0\,;$$

and the hypocycloid has degenerated into the diameter of the circle A C E.

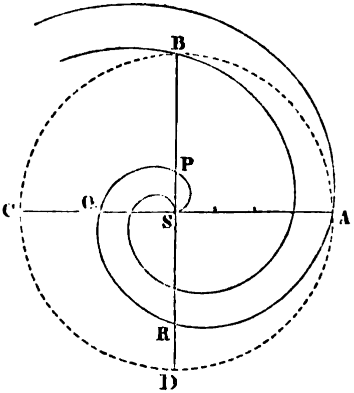
In the same case the equations to the hypotrochoid become

$$x = \frac{a}{2}\,(1 + m)\cos.\,\theta$$

$$y = \frac{a}{2}\,(1 - m)\sin.\,\theta\,;$$

which by the elimination of θ give the equation to an ellipse, whose axes are $a\,(1 + m)$ and $a\,(1 - m)$.

363. If a thread coinciding with a circular axis be unwound from the circle, the extremity of the thread will trace out a curve called the involute of the circle.



Thus suppose a thread fixed round the circle A B C D; then if it be unwound from A, the extremity in the hand will trace out the curve A P Q R S; the lines B P, D Q, C R, A S, which are particular positions of the thread, are also tangents to the circle, and each of them is equal to the length of the corresponding circular arc measured from A.

The curve makes an infinite number of revolutions, the successive branches being separated by a distance equal to the circumference of the circle.

To find the equation to the involute.

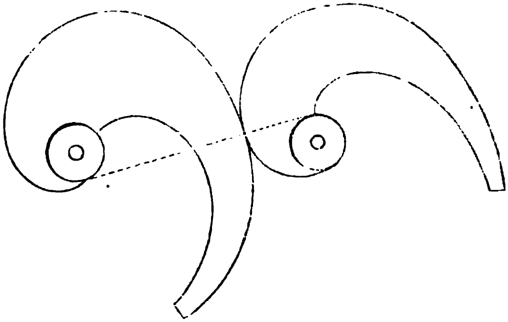
Let C A $= a$, C P $= r$, and angle A C P $= \theta$; then from the triangle B C P, we have B C $=$ P C cos. P C B, or angle P C B $= \cos.^{-1}\frac{a}{r}$;

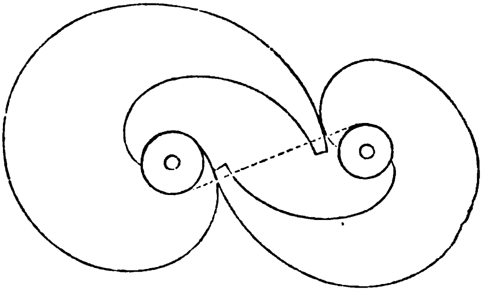
$$\therefore\ \text{B P} = \text{B A} = a\left(\cos.^{-1}\frac{a}{r} + \theta\right)$$

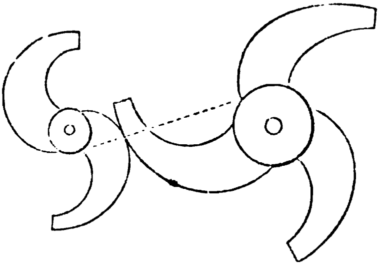
$$\text{or } \surd(r^2 - a^2) = a\left(\cos.^{-1}\frac{a}{r} + \theta\right)$$

$$\therefore\ \theta = \frac{\surd\{r^2 - a^2\}}{a} - \cos.^{-1}\frac{a}{r}.$$

The involute of the circle is usefully employed in toothed wheels; for there is less waste of power in passing from one tooth to another when they are of this form than in any other case.



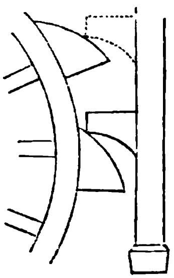




In the figures (2) and (3) we have examples of two equal wheels which have each two teeth; and by turning one wheel the other wheel will be kept in motion by means of the continual contact of the teeth. The dotted line of contact is, by the property of the involute, a common tangent to the two wheels; this dotted line is the constant line of contact, and the force is the same in every part of a revolution.

Fig. (3) is another example; and by making the teeth smaller and more numerous we shall have toothed wheels always in contact, and therefore giving no jar or shake to the machinery.

Again, in raising a piston or hammer, the involute of the circle is the best form for the teeth of the turning-wheel, as the force acts on the piston entirely in a vertical direction.

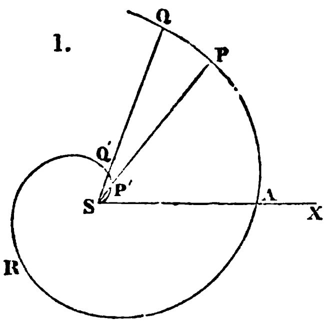


ON SPIRALS.

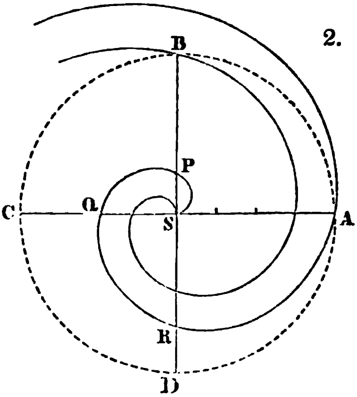
364. There is one class of transcendental curves which are called spirals, from their peculiar twisting form. They were invented by the ancient geometricians, and were much used in architectural ornaments. Of these curves, the most important as well as the most simple, is the spiral invented by the celebrated Archimedes.

This spiral is thus generated: Let a straight line S P of indefinite length move uniformly round a fixed point S, and from a fixed line S X, and let a point P move uniformly also along the line S P, starting from S, at the same time that the line S P commences its motion from S X, then the point will evidently trace out a curve line S P Q R A, commencing at S, and gradually extending further from S. When the line S P has made one revolution, P will have got to a certain point A, and S P still continuing to turn as before, we shall have the curve proceeding on regularly through a series of turnings, and extending further from S.

1.



2.



To examine the form and properties of this curve, we must express this method of generation by means of an equation between polar co-ordinates.

$$\text{Let } SP = r,\ SA = b,\ ASP = \theta;$$

then since the increase of r and θ is uniform, we have

$$SP : SA :: \text{angle } ASP : \text{four right angles} :: \theta : 2\pi$$

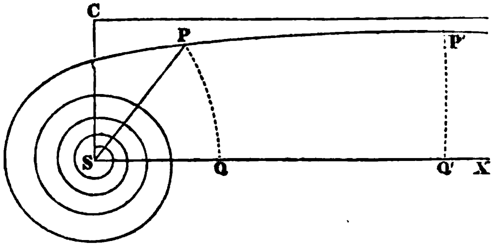
$$\therefore r = \frac{b\,\theta}{2\pi} = a\theta, \text{ if } a = \frac{b}{2\pi}.$$

From this equation it appears that when S P has made two revolutions or $\theta = 4\pi$, we have $r = 2b$, or the curve cuts the axis S X again at a distance 2 S A; and similarly after 3, 4, n revolutions it meets the axis S X at distances 3; 4, n times S A. Archimedes discovered that the area S P Q R A is equal to one-third of the area of the circle described with centre S and radius S A.

365. The spiral of Archimedes is sometimes used for the volutes of the capitals of columns, and in that case the following description by points is useful.

Let a circle A B C D, fig. (2), be described on the diameter C S A, and draw the diameter B D at right angles to C A; divide the radius S A into four equal parts, and in S B take $SP = \frac{1}{4}SA$, in S C take $SQ = \frac{1}{2}SA$, and in S D take $SR = \frac{3}{4}SA$; then from the equation to the curve these points belong to the spiral; by subdividing the radius S A and the angles in each quadrant we may obtain other points as in the figure. In order to complete the raised part in the volute, another spiral commences from S B.

366. The spiral of Archimedes is one of a class of spirals comprehended in the general equation $r = a\theta^n$. Of this class we shall consider the cases where $n = -1$, and $n = -\frac{1}{2}$.



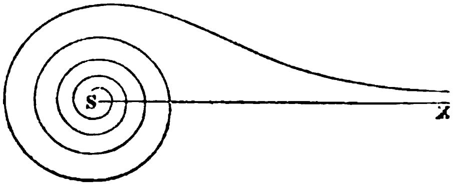
Let $n = -1 \quad \therefore r = a\theta^{-1}$

Let S be the pole, S X the axis from which the angle θ is measured, $SP = r$.

When $\theta = 0$, $r = \infty$; as θ increases, r decreases very rapidly at first and more uniformly afterwards; as θ may go on increasing ad infinitum r also may go on diminishing ad infinitum without ever actually becoming nothing: hence we have an infinite series of convolutions round S: Describe a circular arc P Q with centre S and radius S P, then $PQ = r\theta = a$; and since this value of a is the same for all positions of P, we must have $PQ = P'Q' =$ the straight line S C at an infinite distance, and therefore the curve must approach to an asymptote drawn through C parallel to S X.

This curve is called the reciprocal spiral from the form of its equation, since the variables are inversely as each other, or the hyperbolic spiral, from the similarity of the equation to that of the hyperbola referred to its asymptotes ($xy = k^2$).

367. Let $n = -\frac{1}{2}$; $\therefore r = a\theta^{-\frac{1}{2}}$ or $r^2\theta = a^2$. This curve, called the lituus or trumpet, is described as in the figure; proceeding from the asymptote S X, it makes an infinite series of convolutions round S.



368. If in the equation $r\,\theta = a$, we always deduct the constant quantity, b, we have the equation $(r - b)\,\theta = a$; this curve commences its course like the reciprocal spiral; but as θ increases we have $r - b$ approximating to nothing, or r approximating to b; hence the spiral, after an infinite number of convolutions, approaches to an asymptotic circle, whose centre is S, and radius b.

369. Trace the spiral whose equation is $\theta\,\sqrt{a\,r - r^2} = b$; this curve has an infinite number of small revolutions round the pole, and gradually extends outwards to meet an asymptotic circle whose radius is a.

370. The spiral whose equation is $(r - a)^2 = b^2\,\theta$ commences its course from a point in the circumference of the circle whose radius is a, and extends outwards round S in an infinite series of convolutions. This curve is formed by twisting the axis of the common parabola round the circumference of a circle, the curve line of the parabola forming the spiral.

371. The curve whose equation is $r = a^\theta$ is called the logarithmic spiral, for the logarithm of the radius vector is proportional to the angle θ. Examining all the values of θ from 0 to $\pm\,\infty$ we find that there are an infinite series of convolutions round the pole S. This curve is also called the equiangular spiral, for it is found by the principles of the higher analysis that this curve cuts the radius vector in a constant angle.

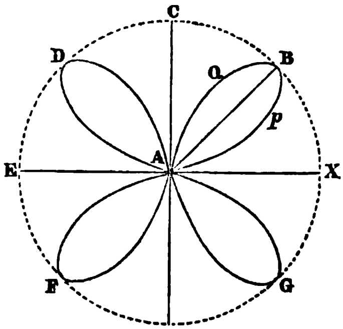
Descartes, who first imagined this curve, found also that the whole length of the curve from any point P to the pole was proportional to the radius vector at P.

372. It will often happen that the algebraical equation of a curve is much more complicated than the polar equation; the conchoid art. 312 is an example. In these cases it is advisable to transform the equation from algebraical to polar co-ordinates, and then trace the curve from the polar equation.

For example, if the equation be $(x^2 + y^2)^{\frac{3}{2}} = 2\,a\,x\,y$, there would be much difficulty in ascertaining the form of the curve from this equation; but let $x = r$ cos. θ and $y = r$ sin. θ (61)

$$\therefore\ r^3 = 2\,a\,r^2 \cos.\,\theta \sin.\,\theta,$$

$$\text{or } r = a \sin.\,2\,\theta.$$



Let A be the origin of polar co-ordinates; A X the axis whence θ is measured; with centre A and radius a describe a circle B C D. Then for $\theta = 0$ we have $r = 0$, as θ increases from 0 to 45°, r increases from

0 to a; hence the branch A P B. Again, as θ increases from 45° to 90°, sin. 2θ diminishes from 1 to 0; ∴ r diminishes, and we trace the branch B Q A. As θ increases from 90° to 180°, sin. 2θ increases and decreases as before; hence the similar oval in the second quadrant. By following θ from 180° to 360°, we shall have the ovals in the third and fourth quadrant: and since the sine of an arc advances similarly in each quadrant of the circle, we have the four ovals similar and equal.

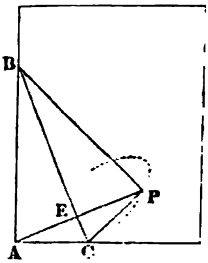
In this case we have paid no regard to the algebraical sign of r; we have considered θ to vary from 0 to 360°, which method we prefer to that of giving θ all values from 0 to 180°, and then making the sign of r to vary.

If the equation had been $(x^2 + y^2)^2 = 2a^2 x y$, we should have found two equal and similar ovals in the first and third quadrant.

The locus of the equation $r = a\,(\cos.\theta - \sin.\theta)$ is the same kind of figure differently situated with respect to the lines A X and A C.

The equation to the lemniscata $r^2 = a^2 \cos.2\theta$ art. (314), may be similarly traced.

373. In many indeterminate problems we shall find that polar co-ordinates may be very usefully employed. For example,



Let the corner of the page of a book be turned over into the position B C P, and in such a manner that the triangle B C P be constant, to find the locus of P.

Let A P $= r$, angle P A C $= \theta$, and let the area A B C $= a^2$; then since the triangles A B E, P B E are equal, we have A E $= \frac{r}{2}$, and the angle A E B a right angle ∴ A E $=$ A C $\cos.\theta$, and A E $=$ A B $\cos.\left(\frac{\pi}{2} - \theta\right) =$ A B $\sin.\theta$ ∴ $\frac{r^2}{4} = \frac{a^2}{2}\sin.\theta\cos.\theta$, or $r^2 = a^2 \sin.2\theta$. Hence the locus is an oval A P B Q as in the last figure.

If a point be taken in the radius vector S P of a parabola so that its distance from the focus is equal to the perpendicular from the focus on the tangent, the locus of the point is the curve whose equation is $r = a\sec.\frac{\theta}{2}$.

PART II.

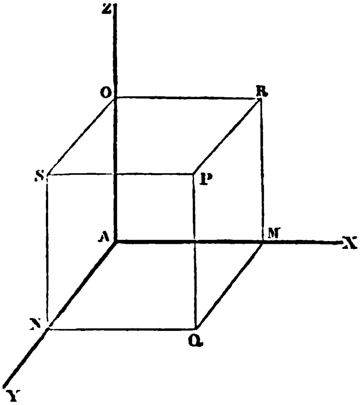
APPLICATION OF ALGEBRA TO SOLID GEOMETRY.

CHAPTER I.

INTRODUCTION.

374. In the preceding part of this Treatise lines and points have always been considered as situated in one plane, and have been referred to two lines called axes situated in that plane. Now we may readily imagine a curve line, the parts of which are not situated in one plane; also, if we consider a surface, as that of a sphere, for example, we observe immediately that all the points in such a surface cannot be in the same plane; hence the method of considering figures which has been hitherto adopted cannot be applied to such cases, and therefore we must have recourse to some more general method for investigating the properties of figures.

375. We begin by showing how the position of a point in space may be determined.



Let three planes Z A X, Z A Y, and X A Y, be drawn perpendicular to each other, and let the three straight lines A X, A Y, A Z be the intersections of these planes, and A the common point of concourse.

From any point P in space draw the lines PQ, P R, and PS respectively perpendicular to the planes X A Y, Z A X, and Z A Y; then the position of the point P is completely determined when these three perpendicular lines are known.

Complete the rectangular parallelopiped A P, then P Q, P R, and P S are respectively equal to A O, A N, and A M.

These three lines A M, A N, and A O, or more commonly their equals A M, M Q, and Q P, are called the co-ordinates of P, and are denoted by the letters x, y, and z respectively.

The point A is called the origin.

The line A X is called the axis of x, the line A Y is called the axis of y, and the line A Z is called the axis of z.

The plane X A Y is called the plane of $x y$, the plane Z A X is called the plane of $z x$, and the plane Z A Y is called the plane of $z y$ *.

From P we have drawn three perpendicular lines, P Q, P R, and P S, on the three co-ordinate planes. The three points, Q, R, and S are called the projections of the point P on the planes of $x y$, $x z$, and $z y$ respectively.

The method of projections is so useful in the investigation and description of surfaces, that we proceed to give a few of the principal theorems on the subject so far as may be required in this work.

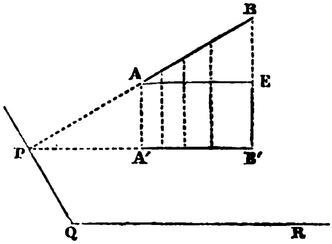
PROJECTIONS.

376. If several points be situated in a straight line, their projections on any one of the co-ordinate planes are also in a straight line.

For they are all comprised in the plane passing through the given straight line, and drawn perpendicular to the co-ordinate plane; and as the intersection of any two planes is a straight line, the projections of the points must be all in one straight line.

This plane, which contains all the perpendiculars drawn from different points of the straight line, is called the projecting plane; and its intersection with the co-ordinate plane is called the projection of the straight line.

377. To find the length of the projection of a straight line upon a plane.



Let A B be the line to be projected on the plane P Q R; produce A B to meet this plane in P; draw A A′ and B B′ perpendicular to the plane, and meeting it in A′ and B′. Join A′ B′; then A′ B′ is the projection of A B.

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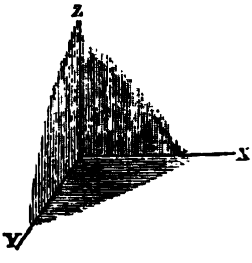
* This system of co-ordinate planes may be represented by the sides and floor of a room, the corner being the origin of the axes, the plane X Y is then represented by the floor of the room, and the two remaining planes by the two adjacent sides of the room.

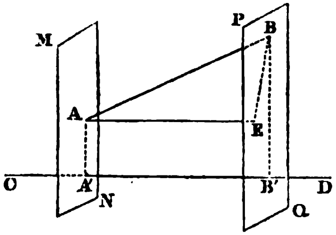


Since A B and A′ B′ are in the same plane, they will meet in P. Let the angle B P B′ or the angle of the inclination of A B to the plane $= \theta$, and in the projecting plane A B′ draw A E parallel to A′ B′, then

$$A'B' = AE = AB \cos. BAE = AB \cos. \theta$$

The same proof will apply to the projection of a straight line upon another straight line, both being in the same plane.

378. To find the length of the projection of a straight line upon another straight line not in the same plane.



Let A B be the line to be projected; C D the line upon which it is to be projected. From A and B draw lines A A′ and B B′ perpendicular to C D, then A′ B′ is the projection of A B.

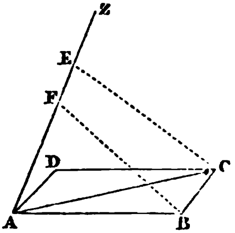
Through A and B draw planes M N and P Q perpendicular to C D. These planes contain the perpendicular lines A A′ and B B′.

From A draw A E perpendicular to the plane P Q, and therefore equal and parallel to A′ B′; join B E; then the triangle A B E having a right angle at E, we have A′ B′ = A E = A B cos. B A E, and angle B A E is equal to the angle θ of inclination between A B and C D; hence

$$A'B' = AB \cos. \theta.$$

Also any line equal and parallel to A B has an equal projection A′ B′ on C D, and the projection of A B on any line parallel to C D is of the same length as A′ B′.

379. The projection of the diagonal of a parallelogram on any straight line is equal to the sum of the projections of the two sides upon the same straight line.



Let A B C D be a parallelogram, A Z any straight line through A inclined to the plane of the parallelogram. From C and B draw perpendiculars C E and B F upon A Z, then A E is the projection of A C upon A Z or A E = A C cos. C A Z; and A F is the projection of A B upon

The line A X is called the axis of x, the line A Y is ca
y, and the line A Z is called the axis of z.

The plane X A Y is called the plane of xy, the plane Z
plane of zx, and the plane Z A Y is called the plane of
From P we have drawn three perpendicular lines, P
on the three co-ordinate planes. The three points, Q, R
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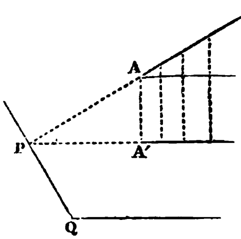
line.

377. To find the length of the projection
plane.



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z, are deter-
rdinate lines :
n A along A Z
f xy: and so
of co-ordinates
space is divided

A Y Z
X A y Z
x A y Z
x A Y Z
X A Y z
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of the point P on two of the co-ordi-
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n, draw S N and R M parallel to A Z,
rallel to A X and A Y, and the position

P of a point from the origin of co-ordi-

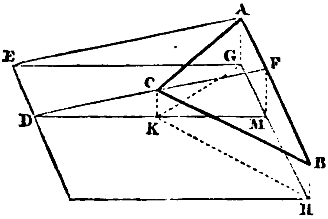
rectangular axes ; A M $= x$. M Q $= y$,
of P.

AZ or $AF = AB \cos. BAZ$. Also FE is the projection of BC or AD upon AZ or $FE = BC \cos. DAZ$

$$\text{and } AE = AF + FE;$$

hence the projection of $AC =$ the sum of projections of AB and BC.

380. To find the projection of the area of any plane figure on a given plane $EDGH$.



Let ABC be a triangle inclined to the given plane $EDGH$ at an angle θ; draw AE, CD, perpendicular to the intersection ED of these planes; then the triangle ABC and its projection GKH have equal bases AB, GH, but unequal altitudes CF, KM;

$$\therefore \text{ area } ABC : GKH :: CF : KM :: DF : DM :: 1 : \cos. \theta$$

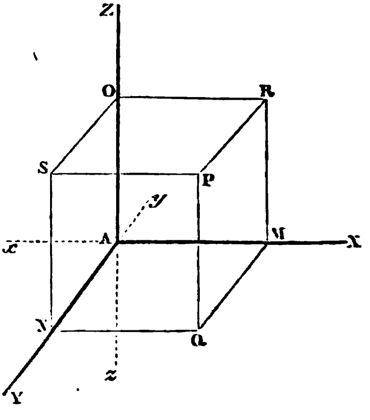
$$\text{or area } GKH = ABC \cos. \theta;$$

and this being true for any triangle, is true for any polygon, and therefore ultimately for any plane area.

CHAPTER II.

THE POINT AND STRAIGHT LINE.

381. We have already explained how the position of a point in space is determined by drawing perpendicular lines from it upon three fixed planes called the co-ordinate planes. If, then, on measuring the lengths of these three perpendicular lines or co-ordinates of P we find $AM=a$, $AN=b$, and $AO=c$, we have the position of a point P completely determined by the three equations $x=a$, $y=b$ and $z=c$; and as these are sufficient for that object, they are called the equations to the point P.



This point may also be defined as in Art. (25) by the equation $(x-a)^2+(y-b)^2+(z-c)^2=0$, since the only values that render this expression real are $x=a$, $y=b$, and $z=c$.

382. The algebraical signs of the co-ordinates x, y, and z, are determined as in Plane Geometry, by the directions of the co-ordinate lines: thus A O is positive or negative according as it is drawn from A along A Z or A z, that is, according as it is above or below the plane of $x\,y$: and so on for the other lines: hence we have the following values of co-ordinates for a point in each of the eight compartments into which space is divided by the co-ordinate planes.

$+x+y+z$	a point P situated in the angle	X A Y Z
$+x-y+z$	. . .	X A y Z
$-x-y+z$	. . .	x A y Z
$-x+y+z$	. . .	x A Y Z
$+x+y-z$	. . .	X A Y z
$+x-y-z$	. . .	X A y z
$-x-y-z$	. . .	x A y z
$-x+y-z$	. . .	x A Y z

383. A point also may be situated in one of the co-ordinate planes, in which case the co-ordinate perpendicular to that plane must $= 0$; thus, if the point be in the plane of $x\,y$, its distance z from this plane must $= 0$: hence the equations to the point in the plane of $x\,y$ are

$$x = a,\ y = b,\ z = 0$$

$$\text{or } (x-a)^2 + (y-b)^2 + z^2 = 0.$$

If the point be in the plane of $x\,z$, the equations are

$$x = a,\ y = 0,\ z = c$$

And if the point be in the plane of $y\,z$

$$x = 0,\ y = b,\ z = c.$$

Also, if the point be on the axis of x, its distance from the planes $x\,y$ and $y\,z = 0$, therefore the equations to such a point are

$$x = a,\ y = 0,\ z = 0;$$

and so on for points situated on the other axes.

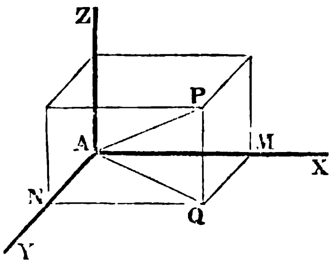
384. The points Q, R, and S, in the last figure, are the projections of the point P on the co-ordinate planes; on referring each of these points to the axes in its own plane, we have

The equations to Q on $x\,y$ are $x = a$, $y = b$
R on $x\,z$ are $x = a$, $z = c$
S on $y\,z$ are $y = b$, $z = c$

Hence we see that the projections of the point P on two of the co-ordinate planes being known, the projection on the third plane is necessarily given: thus, if S and R are given, draw S N and R M parallel to A Z, also N Q and M Q respectively parallel to A X and A Y, and the position of Q is known.

385. To find the distance A P of a point from the origin of co-ordinates A.

Let A X, A Y, and A Z be the rectangular axes; A M $= x$, M Q $= y$, and P Q $= z$, the co-ordinates of P.



The square on A P = the square on A Q + the square on P Q

= the squares on A M, M Q + the square on P Q

or $d^2 = x^2 + y^2 + z^2$.

386. Let α, β, γ, be the angles which A P makes with the axis of $x, y,$ and z, respectively;

$$\text{then } x = AM = AP \cos PAM = d \cos \alpha$$

$$y = MQ = AN = AP \cos. PAN = d \cos. \beta$$

$$z = PQ = AP \sin. PAQ = d \cos. \gamma$$

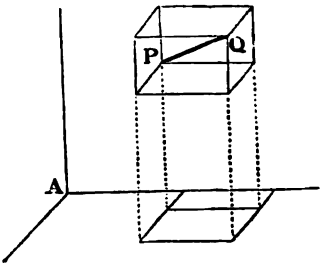
$$\therefore d^2 = x^2 + y^2 + z^2 = d^2 (\cos. \alpha)^2 + d^2 (\cos. \beta)^2 + d^2 (\cos. \gamma)^2$$

$$\therefore (\cos. \alpha)^2 + (\cos. \beta)^2 + (\cos. \gamma)^2 = 1.$$

387. Again $d^2 = x^2 + y^2 + z^2 = x\,d \cos. \alpha + y\,d \cos. \beta + z\,d \cos. \gamma$

$$\therefore d = x \cos. \alpha + y \cos. \beta + z \cos. \gamma.$$

388. To find the distance between two points, let the co-ordinates of the points P and Q be respectively $x\ y\ z$ and $x_1\ y_1\ z_1$; then the distance between these points is the diagonal of a parallelopiped, the three contiguous sides of which are the differences of the parallel co-ordinates; hence, by the last article we have



$$d^2 = (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2$$

If d_1 and d_2 be the distances of the points $x_1\ y_1\ z_1$ and $x_2\ y_2\ z_2$ respectively from the origin, the above expression may be put in the form

$$d^2 = d_1^2 + d_2^2 - 2\,(x_1\,x_2 + y_1\,y_2 + z_1\,z_2).$$

THE STRAIGHT LINE.

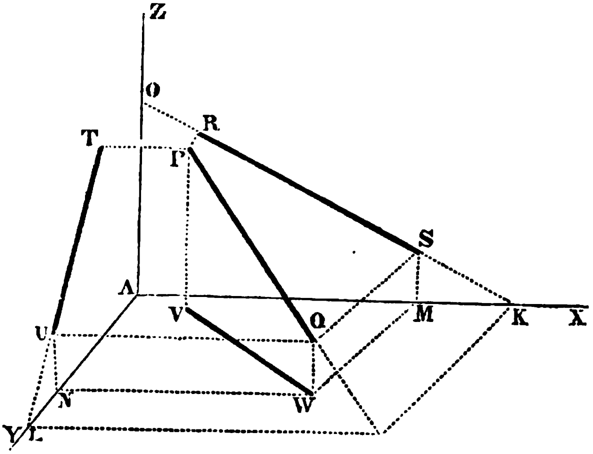
389. A straight line may be considered as the intersection of two planes, and therefore its position will be known if the situation of these planes is known; hence it may be determined by the projecting planes, and the situation of these last is fixed by their intersections with the co-ordinate planes, that is, by the projections of the line upon the co-ordinate planes: hence, the position of a straight line is geometrically fixed by knowing its projections; and it is also algebraically determined by the equations to those projections taken conjointly. Taking the axis of z as the axis of abscissas the equation to the projection on the plane $x\ z$ is of the form

$x = \alpha z + a$ (31), and the equation to the projection on the plane of $y z$ is $y = \beta z + b$.

As these two equations fix the position of the straight line in space, they are, taken together, called the equations to a straight line.

390. To illustrate this subject we shall let P Q be a portion of the straight line, R S its projection on $x z$, T U its projection on $y z$, V W its projection on $x y$;

And let $x = \alpha z + a$, be the equation to R S, and $y = \beta z + b$, be the equation to T U:



then any point Q in the projecting plane P Q R S has the same values of z and x that its projection S has, that is, the co-ordinates A M and M S are the same as N W and W Q; hence there is the same relation between them in each case; and therefore, the equation $x = \alpha z + a$ expresses not only the relation between the x and z of all the points in R S, but also of all the points in the plane P Q R S.

Similarly the equation $y = \beta z + b$ not only relates to T U, but also to all the points in the plane T U Q P.

Therefore, the system of the two equations exists for all the points in the straight line P Q, the intersection of the two projecting planes, and for this line only; hence, the equations to the straight line P Q are

$$\left.\begin{array}{l} x = \alpha z + a \\ y = \beta z + b \end{array}\right\}.$$

The elimination of z between these two equations gives

$$z = \frac{1}{\alpha}(x - a) = \frac{1}{\beta}(y - b)$$

$$\therefore\ y - b = \frac{\beta}{\alpha}(x - a)$$

and this is the relation between the co-ordinates A M and M W of the projection W of any point Q in the line P Q; and therefore, this last equation is that to the projection V W on the plane $x y$.

391. In the equations $x = \alpha z + a$ and $y = \beta z + b$, a is the distance

of the origin from the intersection of R S with A X, or $a =$ A K; similarly $b =$ A L.

Let $x = 0 \therefore z = -\frac{a}{\alpha} = \text{A O} \therefore \text{A K} = -\alpha\,\text{A O}$, but $\text{A K} = \text{A O}$ tan. $\text{A O K} = -\text{A O}$ tan. $\text{Z O R} \therefore \alpha$ is tangent of the angle which R S makes with A Z, and similarly β is tangent of the angle which U T makes with A Z.

392. The straight line will assume various positions according to the algebraical signs of a, b, α and β: however, it would be of very little use to go through all the cases arising from these changes of sign, especially as they offer nothing of consequence, and no one case presents any difficulty. We shall only consider the cases where the absolute value of a, b, α and β is changed.

Let $a = 0$ and $b = 0$, then $x = \alpha z$ and $y = \beta z$, and the two projections pass through the origin, and therefore the line itself passes through the origin; the equation to the third projection is $y = \frac{\beta}{\alpha} x$.

Let $a = 0$ then $x = \alpha z$ and $y = \beta z + b$, the projection on $x z$ passing through the origin, the line itself must pass through the axis A Y perpendicular to $x z$: similarly, if $b = 0$, the equations $x = \alpha z + a$, $y = \beta z$ belong to a line passing through the axis of x, and if the equations are $y = \alpha x$, $y = \beta z + b$, the straight line passes through the axis of z: this last case may be represented by supposing (in the last figure) W V to pass through A, then the equation to V W is of the form $y = \alpha x$, and the equation to O T U is $y = \beta z + b$; now, if two planes be drawn, one through T U perpendicular to $y z$, and the other through V W perpendicular to $x y$, both planes pass through the point O, and therefore the line itself must pass through O.

393. Let $\beta = 0 \therefore x = \alpha z + a$, $y = b$, the line is in a plane parallel to $x z$ and distant from it by the quantity b. If the last figure be adapted to this case we should have U T perpendicular to A Y, and therefore P Q equal and parallel to R S situated in the plane W N U Q perpendicular to $x y$.

Let $\alpha = 0 \therefore x = a$, $y = \beta z + b$, the line is in a plane parallel to to $y z$.

Similarly $z = c$, $y = \alpha' x + a'$ belong to a line in a plane parallel to $x y$.

394. A straight line may also be situated in one of the co-ordinate planes as in the plane of $y z$; for example, the equations to such a line are $y = \beta z + b$, $x = 0$. If the line be in the plane of $x z$ the equations are $x = \alpha z + a$, $y = 0$; and if the line be in the plane of $x y$ the equations become $y = \alpha' x + a'$ $z = 0$.

395. If the straight line be perpendicular to one of the co-ordinate planes, as $x y$ for example; α and β must each equal 0, and therefore the equations to this line are

$$x = a,\ y = b,\ z = \frac{0}{0}.$$

Similarly the equations to a line perpendicular to $x z$ are

$$x = a,\ y = \frac{0}{0},\ z = c$$

and the equations to a line perpendicular to $y\,z$ are

$$x = \frac{0}{0},\ y = b, z = c.$$

396. To find the point where a straight line meets the co-ordinate planes:

Let $x = \alpha\,z + a$ and $y = \beta\,z + b$ be the equations to the line; when it meets the plane of $x\,y$ we have $z = 0$ $\therefore$ $x = a, y = b$ are the equations to the required point.

Similarly $z = -\dfrac{b}{\beta}, x = -\dfrac{\alpha}{\beta}\,b + a$ are the equations to the point where the line meets the plane of $x\,z$, and $z = -\dfrac{a}{\alpha}, y = -\dfrac{\beta}{\alpha}\,a + b$ are the equations to the point where the line pierces the plane of $z\,y$.

397. There are four constant quantities in the general equations to a straight line, and if they are all given, the position of the line is completely determined; for we have only to give to one of the variables as z a value z', and we have

$$x = \alpha\,z + a = \alpha\,z' + a = x' \text{ and } y = \beta\,z + b = \beta\,z' + b = y';$$

or, x' and y' are also necessarily determined; hence, taking A M $= x'$, (see the last figure,) and drawing M W ($= y'$) parallel to A Y, and lastly, drawing from W a perpendicular W Q $= z'$, the point Q thus determined belongs to the line; and similarly, any number of points in the line are determined, or the position of the line is completely ascertained. Again, the straight line may be subject to certain conditions, as passing through a given point, or being parallel to a given line; or, in other words, conditions may be given which will enable us to determine the quantities α, β, a and b, supposing them first to be unknown; in this manner arises a series of Problems on straight lines similar to those already worked for straight lines situated in one plane (40, 50).

PROBLEMS ON STRAIGHT LINES.

398. To find the equations to a straight line passing through a given point:

Let the co-ordinates of the given point be x_1, y_1 and z_1, and let the equations to the straight line be $x = \alpha\,z + a, y = \beta\,z + b$.

Now since this line passes through the given point, the projections of the line must also pass through the projections of the point; hence the projection $x = \alpha\,z + a$ passing through x_1 and z_1, we have $x_1 = \alpha\,z_1 + a$,

$$\therefore\ x - x_1 = \alpha\,(z - z_1)$$

$$\text{and similarly } y - y_1 = \beta\,(z - z_1)$$

hence these are the equations required: α and β being indeterminate, there may be an infinite number of straight lines passing through the given point.

If the given point be in the plane of $x\,y$, we have $z_1 = 0$,

$$\therefore\ \left.\begin{aligned} x - x_1 &= \alpha\,z \\ y - y_1 &= \beta\,z \end{aligned}\right\}$$

If the given point be on the axis of x, we have $z_1 = 0$ and $y_1 = 0$,

$$\left.\begin{array}{l} x - x_1 = \alpha z \\ y = \beta z \end{array}\right\}$$

And the equation would assume various other forms according to the position of the given point.

399. To find the equations to a straight line passing through two given points, $x_1\ y_1\ z_1$ and $x_2\ y_2\ z_2$.

Since the line passes through the point $x_1\ y_1\ z_1$ its equations are

$$x - x_1 = \alpha (z - z_1)$$

$$y - y_1 = \beta (z - z_1).$$

And since the line also passes through $x_2\ y_2\ z_2$ the last equations become

$$x_2 - x_1 = \alpha (z_2 - z_1)$$

$$y_2 - y_1 = \beta (z_2 - z_1)$$

$$\therefore\ \alpha = \frac{x_2 - x_1}{z_2 - z_1} \text{ and } \beta = \frac{y_2 - y_1}{z_2 - z_1};$$

hence the equations to the required line are

$$x - x_1 = \frac{x_2 - x_1}{z_2 - z_1} (z - z_1)$$

$$y - y_1 = \frac{y_2 - y_1}{z_2 - z_1} (z - z_1).$$

These equations will assume many various forms dependent on the position of the given point, for example: If the first point be in the plane of $y\ z$, and the second in the axis of x, we have $x_1 = 0$; $y_2 = 0$, $z_2 = 0$

$$\therefore\ x = \frac{x_2}{-\ z_1} (z - z_1)$$

$$y - y_1 = \frac{y_1}{z_1} (z - z_1).$$

If the second point be the origin, we have $x_2\ y_2\ z_2$ each $= 0$,

$$\therefore\ x - x_1 = \frac{x_1}{z_1} (z - z_1) = \frac{x_1}{z_1} z - x_1$$

$$y - y_1 = \frac{y_1}{z_1} (z - z_1) = \frac{y_1}{z_1} z - y_1;$$

hence the equations to a point passing through the origin are

$$x = \frac{x_1}{z_1} z, \text{ and } y = \frac{y_1}{z_1} z.$$

And these equations may be also obtained by considering that the projections pass through the origin, and therefore their equations are of the form $x = \alpha z$, $y = \beta z$, and the first passing through $x_1\ z_1$ we have $\alpha = \frac{x_1}{z_1}$, and similarly $\beta = \frac{y_1}{z_1}$.

400. To find the equation to a straight line parallel to a given straight line.

Since the lines are parallel their projecting planes on any one of the co-ordinate planes are also parallel, and therefore the projections themselves parallel; hence, if the equations to the given line are

$$x = \alpha z + a, \; y = \beta z + b,$$

the equations to the required line are

$$x = \alpha z + a', \; y = \beta z + b'.$$

If the straight line pass also through a given point $x_1 \, y_1 \, z_1$ its equations are

$$x - x_1 = \alpha (z - z_1), \; y - y_1 = \beta (z - z_1).$$

401. To find the intersection of two given straight lines.

Two straight lines situated in one plane must meet in general, but this is not necessarily the case if the lines be situated anywhere in space; hence there must be a particular relation among the constant quantities in the equation in order that the lines may meet: to find this relation, let the equations to the lines be

$$\left.\begin{aligned} x &= \alpha z + a \\ y &= \beta z + b \end{aligned}\right\} \qquad \left.\begin{aligned} x &= \alpha' z + a' \\ y &= \beta' z + b' \end{aligned}\right\}$$

For the point of intersection the projected values of x, y and z must be the same in all the equations; hence

$$\alpha z + a = \alpha' z + a' \text{ and } z = \frac{a' - a}{\alpha - \alpha'},$$

$$\text{and } \beta z + b = \beta' z + b' \text{ and } z = \frac{b' - b}{\beta - \beta'},$$

$$\therefore \frac{a' - a}{\alpha - \alpha'} = \frac{b' - b}{\beta - \beta'};$$

$$\text{or, } (a' - a)\,(\beta' - \beta) = (b' - b)\,(\alpha' - \alpha).$$

And this is the relation which must exist amongst the constants in order that the two lines may meet.

Having thus determined the necessary relation among the constants, the co-ordinates of intersection are given by the equations

$$z = \frac{a' - a}{\alpha - \alpha'} \text{ or } = \frac{b' - b}{\beta - \beta'}$$

$$y = \beta z + b = \beta \frac{b' - b}{\beta - \beta'} + b = \frac{\beta b' - \beta' b}{\beta - \beta'}$$

$$x = \alpha z + a = \alpha \frac{a' - a}{\alpha - \alpha'} + a = \frac{\alpha a' - \alpha' a}{\alpha - \alpha'}.$$

402. To find the angles which a straight line (l) makes with the co-ordinate axes; and thence with the co-ordinate planes:

Let the equations to the given line be

$$x = \alpha z + a$$
$$y = \beta z + b;$$

the equations to the parallel line through the origin are

$$x = \alpha z, \; y = \beta z;$$

also let r be the distance of any point (x, y, z) in this last line from the origin:

$$\begin{aligned} \therefore\ r^2 &= x^2 + y^2 + z^2 \\ &= (\alpha^2 + \beta^2 + 1)\, z^2 \end{aligned}$$

$$\text{or, } z^2 = \frac{r^2}{1 + \alpha^2 + \beta^2}$$

But $l\,x$, $l\,y$, and $l\,z$ being the angles which either line makes with the axes of x, y, and z respectively, we have from the second line

$$\cos.\ l\,x = \frac{x}{r} = \frac{\alpha z}{r} = \frac{\alpha}{\sqrt{1 + \alpha^2 + \beta^2}}$$

$$\cos.\ l\,y = \frac{y}{r} = \frac{\beta z}{r} = \frac{\beta}{\sqrt{1 + \alpha^2 + \beta^2}}$$

$$\cos.\ l\,z = \frac{z}{r} \quad = \quad \frac{1}{\sqrt{1 + \alpha^2 + \beta^2}}$$

$$\text{Also } (\cos.\ l\,x)^2 + (\cos.\ l\,y)^2 + (\cos.\ l\,z)^2 = 1\,;$$

and this is the equation connecting the three angles which any straight line makes with the rectangular axes.

Since the system is rectangular, the angle which a line makes with any axis is the complement of the angle which it makes with the plane perpendicular to that axis: hence the angles which a line makes with the coordinate planes are given.

403. To find the cosine, sine, and tangent of the angle between two given straight lines.

Let the equations to the two straight lines be

$$\left.\begin{aligned} x &= \alpha\, z + a \\ y &= \beta\, z + b \end{aligned}\right\} \qquad \left.\begin{aligned} x &= \alpha' z + a' \\ y &= \beta' z + b' \end{aligned}\right\}.$$

These two lines may meet, or they may not meet; but in either case their mutual inclination is the same as that of two straight lines parallel to them and passing through the origin; hence the problem is reduced to find the angle between the lines represented by the equations

$$\left.\begin{aligned} x &= \alpha z \\ y &= \beta z \end{aligned}\right\} (1) \qquad \left.\begin{aligned} x &= \alpha' z \\ y &= \beta' z \end{aligned}\right\} (2)$$

Let r = the distance of a point $x\ y\ z$ in (1) from the origin,

r_1 = ... $x_1\, y_1\, z_1$ in (2) . . . . .

d = the distance between these points,

θ = the angle between the given lines,

$$\begin{aligned} \text{then } d^2 &= r^2 + r_1^2 - 2\, r\, r_1 \cos.\ \theta \\ &= (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 \quad (388) \\ &= x^2 + y^2 + z^2 + x_1^2 + y_1^2 + z_1^2 - 2\,(x x_1 + y y_1 + z z_1) \\ &= r^2 + r_1^2 - 2\,(x x_1 + y y_1 + z z_1) \end{aligned}$$

$$\therefore\ r\, r_1 \cos.\ \theta = x x_1 + y y_1 + z z_1$$

$$\text{Now } x x_1 + y y_1 + z z_1 = \alpha z\, \alpha' z_1 + \beta z\, \beta' z_1 + z\, z_1 = (\alpha\alpha' + \beta\beta' + 1)\, z z_1.$$

$$\begin{aligned} \text{And } r\, r_1 &= \sqrt{(x^2 + y^2 + z^2)}\ \sqrt{(x_1^2 + y_1^2 + z_1^2)} \\ &= z\, z' \sqrt{(\alpha^2 + \beta^2 + 1)} \sqrt{(\alpha'^2 + \beta'^2 + 1)}. \end{aligned}$$

$$\therefore\ \cos.\ \theta = \frac{x x_1 + y y_1 + z z_1}{r\, r_1}$$

$$= \frac{\alpha\alpha' + \beta\beta' + 1}{\sqrt{(\alpha^2 + \beta^2 + 1)}\sqrt{(\alpha'^2 + \beta'^2 + 1)}}$$

Hence $\sin.\ \theta = \dfrac{\sqrt{\{(\alpha\beta' - \alpha'\beta)^2 + (\alpha - \alpha')^2 + (\beta - \beta')^2\}}}{\sqrt{(\alpha^2 + \beta^2 + 1)}\sqrt{(\alpha'^2 + \beta'^2 + 1)}}$.

And $\tan.\ \theta = \dfrac{\sin.\ \theta}{\cos.\ \theta} = \dfrac{\sqrt{\{(\alpha\beta' - \alpha'\beta)^2 + (\alpha - \alpha')^2 + (\beta - \beta')^2\}}}{\alpha\alpha' + \beta\beta' + 1.}$

The value of the cosine of the angle between two straight lines may also be expressed in terms of the angles which the two straight lines l and l_1 make with the co-ordinate axes.

For $x = r \cos.\ l\,x$, $y = r \cos.\ l\,y$, $z = r \cos.\ l\,z$,

and $x_1 = r_1 \cos.\ l_1\,x$, $y_1 = r_1 \cos.\ l_1\,y$, $z_1 = r_1 \cos.\ l_1\,z$,

$$\therefore \cos.\ \theta = \frac{x x_1}{r r_1} + \frac{y y_1}{r r_1} + \frac{z z_1}{r r_1}$$

$$= \cos.\ l\,x \cos.\ l\,x + \cos.\ l\,y \cos.\ l\,y + \cos.\ l\,z \cos.\ l_1\,z.$$

404. If the lines are parallel, we must have $\sin.\ \theta = 0$.

$$\therefore (\alpha\beta' - \alpha'\beta)^2 + (\alpha - \alpha')^2 + (\beta - \beta')^2 = 0,$$

an equation which cannot be satisfied unless by supposing $\alpha = \alpha'$, $\beta = \beta'$, and $\alpha\beta' = \alpha'\beta$, the first two of these conditions are the same as those already shown to determine the parallelism of two lines (400), and the third condition is only a necessary consequence of the other two, and therefore implies nothing further.

405. If the lines are perpendicular to each other, we must have $\cos.\ \theta = 0$.

$$\therefore \alpha\alpha' + \beta\beta' + 1 = 0.$$

or, $\cos.\ l\,x \cos.\ l_1\,x + \cos.\ l\,y \cos.\ l_1\,y + \cos.\ l\,z \cos.\ l_1\,z = 0$;

Now, one line in space is considered as perpendicular to a second straight line, whenever it is in a plane perpendicular to this second line; hence an infinite number of lines can be drawn perpendicular to a given line; and this appears from the above equation, for there are four constants involved in the equation to the perpendicular line, and only one equation between them.

406. If the lines also meet, we have then the additional equation,

$$(a' - a)(\beta' - \beta) = (b' - b)(\alpha' - \alpha) \quad (401).$$

However, even yet an infinite number of straight lines can be drawn, meeting the given line at right angles, for an infinite number of planes can be drawn perpendicular to the given line, and in each plane an infinite number of straight lines can be drawn passing through the given line.

407. To find the equation to a straight line passing through a given point $x_1 y_1 z_1$, and meeting a given line (1) at right angles.

Let the equations to the lines be,

$$\left.\begin{aligned} x &= \alpha z + a \\ y &= \beta z + b \end{aligned}\right\} (1) \qquad \left.\begin{aligned} x - x_1 &= \alpha'(z - z_1) \\ y - y_1 &= \beta'(z - z_1) \end{aligned}\right\} (2)$$

hence the two equations of condition are,

$$\alpha\alpha' + \beta\beta' + 1 = 0 \quad (3)$$

$$(a' - a)(\beta - \beta') - (b' - b)(\alpha - \alpha') = 0$$

or since $a' = x_1 - \alpha' z_1$, and $b' = y_1 - \beta' z_1$ -see art 403

$$(x_1 - \alpha' z_1 - a)(\beta - \beta') - (y_1 - \beta' z_1 - b)(\alpha - \alpha') = 0 \quad (4).$$

The elimination of α' and β' from (3) and (4) give the equations

$$\beta' = \frac{\{(x_1 - a)\,\alpha + z_1\}\,\beta - (y_1 - b)\,(1 + \alpha^2)}{(y_1 - b)\,\beta + (x_1 - a)\,\alpha - (\alpha^2 + \beta^2)\,z_1}$$

$$\alpha' = \frac{\{(y_1 - b)\,\beta + z_1\}\,\alpha - (x_1 - a)\,(1 + \beta^2)}{(y_1 - b)\,\beta + (x_1 - a)\,\alpha - (\alpha^2 + \beta^2)\,z_1}.$$

These values of α' and β' substituted in (2) give the final equation to the straight line, passing through a given point, and meeting a given straight line at right angles.

In particular cases other methods may be adopted, for example, to find the equations to a straight line passing through the axis of y at right angles to that axis:
here $x_1 = 0$, and $z_1 = 0$, therefore the equations to the line are

$$x = \alpha z$$
$$y - y_1 = \beta z$$

but because the line is perpendicular to the axis of y we have $\beta = 0$, hence the required equations are $x = \alpha z$, $y = y_1$. By assuming the axes of co-ordinates to be conveniently situated, this and many other problems may be worked in a shorter manner. This will be shown hereafter.

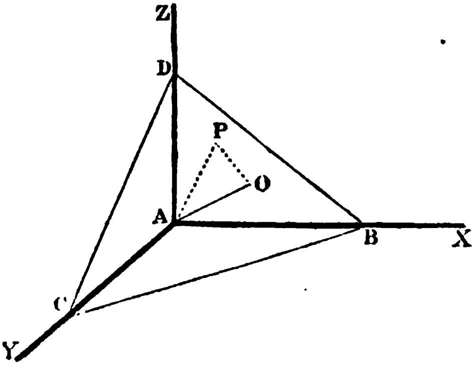
CHAPTER III.

THE PLANE.

408. A PLANE may be supposed to be generated by the motion of a straight line about another straight line perpendicular to it.

Let A be the origin, A X, A Y, A Z the axes, B C D a portion of a plane, A O the perpendicular from the origin upon this plane, P any point in this plane; then, according to the above definition, we suppose the plane to be formed by the revolution of a line like O P round A O, the angle A O P being a right angle.

To find the equation to the plane.
Let x, y, z, be the co-ordinates of P, and x_1, y_1, z_1, those of O, and let the fixed distance AO $= d$.



Then the square on A P = the square on A O + the square on O P;

$$\text{or, } x^2 + y^2 + z^2 = d^2 + (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2$$

$$= d^2 + x^2 + y^2 + z^2 + x_1^2 + y_1^2 + z^2 - 2\,x\,x_1 - 2\,y\,y_1 - 2\,z\,z_1.$$

$$\therefore\ 2\,(x\,x_1 + y\,y_1 + z\,z_1) = d^2 + d^2 = 2\,d^2$$

$$\text{or } x\,x_1 + y\,y_1 + z\,z_1 = d^2.$$

409. Let $\frac{x_1}{d^2} = m$, $\frac{y_1}{d^2} = n$, and $\frac{z_1}{d^2} = p$, then the above equation becomes

$$m\,x + n\,y + p\,z = 1.$$

And it is under this form that we shall generally consider the equation to the plane.

Let $\frac{d^2}{x_1} = a$, $\frac{d^2}{y_1} = b$, and $\frac{d^2}{z_1} = c$, then the equation to the plane is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

And this is perhaps the most intelligible form in which the equation to the plane can be put, the constants a, b and c being equal to A B, A C and A D the respective distances of the origin from the intersection of the plane with the co-ordinate axes; this is found by putting y and z both $= 0$, hence $\frac{x}{a} = 1$, or A B $= a$, and similarly for the other lines.

410. Let the word "plane" be represented by the letter P, and let the angles which A O or d makes with the co-ordinate axes be represented by $d\,x$; $d\,y$; $d\,z$; and let the angles which the plane makes with the same axes be denoted by Px; Py; Pz; then, since A O B is a right angle, and A B O is the angle which the plane makes with A X, we have

$$d = a \cos.\, d\,x = a \sin.\, \mathrm{P}\,x$$

$$d = b \cos.\, d\,y = b \sin.\, \mathrm{P}\,y$$

$$d = c \cos.\, d\,z = c \sin.\, \mathrm{P}\,z;$$

therefore the last equation to the plane may be put in either of the forms

$$x \cos.\, d\,x + y \cos.\, d\,y + z \cos.\, d\,z = d$$

$$\text{or } x \sin.\, \mathrm{P}\,x + y \sin.\, \mathrm{P}\,y + z \sin.\, \mathrm{P}\,z = d.$$

411. Let P, $y\,z$ represent the angle which the plane makes with the co-ordinate plane $y\,z$, then since angle O A B. is equal to the angle of inclination of the plane to $y\,z$, we have $\cos.\, d\,x = \cos.\, \mathrm{P}, y\,z$, hence the equation of the plane becomes

$$x \cos.\, \mathrm{P}, y\,z + y \cos.\, \mathrm{P}, x\,z + z \cos.\, \mathrm{P}, x\,y = d.$$

412. Since by (386) $(\cos.\, d\,x)^2 + (\cos.\, d\,y)^2 + (\cos.\, d\,z)^2 = 1$

we have $(\cos.\, \mathrm{P}, y\,z)^2 + (\cos.\, \mathrm{P}\,x\,z)^2 + (\cos.\, \mathrm{P}\,x\,y)^2 = 1$ *.

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* If A be the area of a plane P, the projections of this area on the co-ordinate planes are represented by A cos. P, $x\,y$; A cos. P, $x\,z$; A cos. P, $y\,z$; hence $(\mathrm{A} \cos.\, \mathrm{P}, x\,y)^2 + (\mathrm{A} \cos.\, \mathrm{P}, x\,z)^2 + (\mathrm{A} \cos.\, \mathrm{P}, y\,z)^2 = \mathrm{A}^2 \left\{(\cos.\, \mathrm{P}, x\,y)^2 + (\cos.\, \mathrm{P}, x\,z)^2 + (\cos.\, \mathrm{P}, y\,z)^2\right\} = \mathrm{A}^2$ by (412). This theorem, referring to the numerical values of the projected areas, is of use in finding the area of a plane between the three co-ordinate planes. Thus, if the equation to a plane be $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, we have by the last figure the area A B C $= \frac{a\,b}{2}$; area A D C $= \frac{b\,c}{2}$, and area A B D $= \frac{a\,c}{2}$; hence the area B C D $= \sqrt{\frac{1}{4}\,(a^2\,b^2 + a^2\,c^2 + b^2\,c^2)}$ by the above theorem. The volume of the pyramid A C D B $= \frac{c}{3}\,\frac{a\,b}{2} = \frac{a\,b\,c}{6}$.

413. To find the angles which a plane makes with the co-ordinate planes in terms of the co-efficients of the equation to the plane.

Let the equation to the plane be

$$mx + ny + pz = 1.$$

Now the equation to a plane expressed in terms of the angles which it makes with the co-ordinate planes is given by (411.)

$$x \cos. P, yz + y \cos. P, xz + z \cos. P, xy = d,$$

hence equating co-efficients, we have

$$m = \frac{\cos. P, yz}{d}, n = \frac{\cos. P, xz}{d}, p = \frac{\cos. P, xy}{d}$$

$$\therefore m^2 + n^2 + p^2 = \frac{1}{d^2} \text{ and } d = \frac{1}{\sqrt{m^2 + n^2 + p^2}};$$

$$\text{and } \cos. P, yz = md = \frac{m}{\sqrt{m^2 + n^2 + p^2}}$$

$$\text{Cos. } P, xz = nd = \frac{n}{\sqrt{m^2 + n^2 + p^2}}$$

$$\text{Cos. } P, xy = pd = \frac{p}{\sqrt{m^2 + n^2 + p^2}}.$$

414. The equation to the plane will assume various forms according to the various positions of the plane.

Let the plane pass through the origin, then $d = 0$; therefore, putting $d = 0$ in the equation, art. (408), we have the equation to the plane passing through the origin; but as the equation to the plane has been obtained on the supposition of d being finite, it becomes necessary to give an independent proof for this particular case.

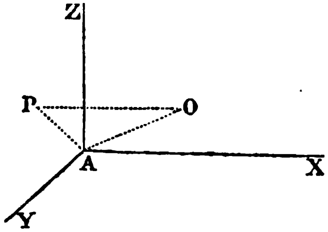
Let A O ($= d$) be the length of a perpendicular from the plane to a given point O; whose co-ordinates are x_1, y_1, z_1; x, y, z, as before, the co-ordinates of any point P in the plane, then

the square on O P = the square on A O + the square on A P;

$$\text{or } (x - x_1)^2 + (y - y_1)^2 + (z - z_1) = d^2 + x^2 + y^2 + z^2.$$

$$\therefore \; - 2\,(x\,x_1 + y\,y_1 + z\,z_1) + d^2 = d^2,$$

$$\text{or } x\,x_1 + y\,y_1 + z\,z_1 = 0.$$



So that the equation to the plane in this case is the same as the original equation without the constant term.

415. Let the plane be parallel to any of the co-ordinate planes, as $x\,y$ for example, then $a = \infty$ and $b = \infty$; therefore the equation $\frac{x}{a} + \frac{y}{b} +$

$\frac{z}{c} = 1$ becomes $0\,x + 0\,y + \frac{z}{c} = 1$; $\therefore\ z = c,\ x = \frac{0}{0}$ and $y = \frac{0}{0}$; of these three equations the first signifies that every point in the plane is equidistant from the plane $x\,y$, and the other two signify that for this single value of z, every possible value of x and y will give points in the plane. The two latter equations are generally omitted; and we say that for a plane parallel to $x\,y$ the equation is $z = c$; similarly for the plane parallel to $x\,z$ it is $x = a$, and for a plane parallel to $y\,z$ the equation is $y = b$.

The equations to a co-ordinate plane, as $x\,y$ for example, are $z = 0$, $x = \frac{0}{0}$; $y = \frac{0}{0}$; or, more simply, $z = 0$.

416. The lines B C, B D, and D C, where the plane intersects the co-ordinate planes, are called the traces of the plane. The equations to these traces are found, from the equation to the plane, by giving to x, y, or z the particular values which they have when the plane intersects the co-ordinate planes.

Let the equation to the plane be $m\,x + n\,y + p\,z = 1$; then for the intersection B C we have the equations

$$z = 0,\ m\,x + n\,y = 1.$$

Similarly the equations to the traces B D and C D are respectively

$$y = 0,\ m\,x + p\,z = 1$$
$$x = 0,\ n\,y + p\,z = 1.$$

PROBLEMS ON THE PLANE.

417. To find the equation to a plane parallel to a given plane.

Let the given plane be $m\,x + n\,y + p\,z = 1$,
and the required plane be $m'\,x + n'\,y + p'\,z = 1$.
Then the planes being parallel, their traces on the co-ordinate planes must be parallel; now their traces on $x\,z$ are

$$m\,x + p\,z = 1,\ m'\,x + p'\,z = 1;$$

$$\therefore\ \frac{m}{p} = \frac{m'}{p'},\ \text{or } m' = \frac{m}{p}\,p';\ \text{similarly } n' = \frac{n}{p}\,p'.$$

Hence the required equation becomes

$$\frac{m}{p}\,p'\,x + \frac{n}{p}\,p'\,y + p'\,z = 1,$$

$$\text{or } m\,x + n\,y + p\,z = \frac{p}{p'}.$$

In this case the resulting equation contains one indeterminate constant p', and therefore shows that an infinite number of planes can be drawn parallel to a given plane, which is also geometrically evident. Three conditions are apparently given, since the three traces of one plane are parallel to the three traces of the other plane; but if the traces on two of the co-ordinate planes be parallel, the traces on the third co-ordinate plane are necessarily parallel; for if $\frac{m}{p} = \frac{m'}{p'}$, and $\frac{n}{p} = \frac{n'}{p'}$, we have $\frac{m}{n} =$

$\frac{m'}{n'}$, or $m x + n y = 1$ parallel to $m' x + n' y = 1$. Thus, in reality, only two conditions are given to determine the three constants.

418. To find the equation to a plane parallel to a given plane, and passing through a given point x_1, y_1, z_1.

Let $m' x + n' y + p' z = 1$ be the required plane, then since the plane passes through $x_1\, y_1\, z_1$ we have

$$m' x_1 + n' y_1 + p' z_1 = 1$$

$$\therefore\ m' (x - x_1) + n' (y - y_1) + p' (z - z_1) = 0.$$

$$\text{Also } \frac{m'}{p'} = \frac{m}{p}, \text{ and } \frac{n'}{p'} = \frac{n}{p};$$

$$\therefore\ \frac{m}{p} p' (x - x_1) + \frac{n}{p} p' (y - y_1) + p' (z - z_1) = 0;$$

$$\text{or } m (x - x_1) + n (y - y_1) + p (z - z_1) = 0.$$

419. To find the intersection of a straight line and plane.

Let $m x + n y + p z = 1$ be the equation to the plane,

$$\left.\begin{array}{l} x = \alpha z + a \\ y = \beta z + b \end{array}\right\} \text{the equations to the line};$$

then, since the co-ordinates of the point of intersection are common, we have

$$m (\alpha z + a) + n (\beta z + b) + p z = 1,$$

$$\therefore\ z = \frac{1 - m a - n b}{m \alpha + n \beta + p},$$

$$\text{and } x = \alpha z + a = \frac{\alpha - n b a + n \beta a + p a}{m \alpha + n \beta + p}$$

$$y = \beta z + b = \frac{\beta - m a \beta + m \alpha b + p b}{m \alpha + n \beta + p}.$$

Thus the required point of intersection is found.

420. To find the conditions that the straight line and plane be parallel or coincide.

If they are parallel, the values of x, y, and z must be infinite

$$\therefore\ m \alpha + n \beta + p = 0.$$

If they coincide, the values of x, y, and z must be indeterminate, or each $= \frac{0}{0}$.

$$\therefore\ m \alpha + n \beta + p = 0, \text{ and } 1 - m a - n b = 0;$$

and these are the two conditions for coincidence, the numerators of x and y being both given $= 0$ by combining the last two equations.

Hence, to find the equation to a plane coinciding with a given straight line, we have the two conditions

$$m a + n b = 1,$$
$$m \alpha + n \beta + p = 0;$$

whence, by elimination, we have

$$m = \frac{\beta + p b}{a \beta - b \alpha} \text{ and } n = - \frac{\alpha + p a}{a \beta - b \alpha};$$

therefore the equation to the plane is

$$(\beta + p\,b)\,x - (\alpha + p\,a)\,y + p\,(a\,\beta - b\,\alpha)\,z = a\,\beta - b\,\alpha,$$

where p remains indeterminate.

421. To find the equation to a plane coinciding with two given lines.

$$m\,x + n\,y + p\,z = 1.$$

$$\left.\begin{aligned} x &= \alpha\,z + a \\ y &= \beta\,z + b \end{aligned}\right\} \qquad \left.\begin{aligned} x &= \alpha'\,z + a' \\ y &= \beta'\,z + b' \end{aligned}\right\}$$

the plane coinciding with the given lines, we have

$$m\,a + n\,b = 1 \quad (1) \qquad m\,\alpha + n\,\beta + p = 0 \quad (3)$$

$$m\,a' + n\,b' = 1 \quad (2) \qquad m\,\alpha' + n\,\beta' + p = 0 \quad (4)$$

From (1) and (2) we have m and n, and these values being substituted in (3) and (4), give two values of p, hence we have the equation of condition

$$(\beta' - \beta)\,(a - a') + (\alpha' - \alpha)\,(b - b') = 0.$$

This equation is verified either if the lines are parallel (in which case $\alpha' = \alpha$ and $\beta' = \beta$), or if they meet; hence in either of these cases a plane may be drawn coinciding with the two lines; the equation to this plane is found, from the values of m, n, and p, to be

$$(b' - b)\,x - (a' - a)\,y + \{\,(a' - a)\,\beta - (b' - b)\,\alpha\,\}\,z = a\,b' - a'\,b.$$

422. If it be required to find the equation to a plane which coincides with one given straight line, and is parallel to another given straight line, we have the three equations

$$\left.\begin{aligned} m\,a + n\,b &= 1 \\ m\,\alpha + n\,\beta + p &= 0 \end{aligned}\right\} \text{ for coincidence with one line,}$$

$$m\,\alpha' + n\,\beta' + p = 0 \text{ for parallelism with the other;}$$

and from these three equations we may determine m, n, and p, and then substitute these values in the general equation to the plane.

423. To find the intersection of two given planes.

Let the equations to the two planes be

$$m\,x + n\,y + p\,z = 1$$

$$m'\,x + n'\,y + p'\,z = 1.$$

By the elimination of z we obtain an equation between x and y, which belongs to the projection of the intersection of the planes on $x\,y$,

hence $(m\,p' - m'\,p)\,x + (n\,p' - n'\,p)\,y = p' - p$

is the projection on $x\,y$ of the required intersection.

Similarly

$$(m\,n' - m'\,n)\,x + (p\,n' - p'\,n)\,z = n' - n$$

is the equation to the projection on $x\,z$.

But the equations to the projections of a line on two co-ordinate planes are called the equations to the line itself; hence the above two equations are the required equations to the intersection.

The third projection is given by the other two, or it may be found separately

$$(n\,m' - n'\,m)\,y + (p\,m' - m\,p')\,z = m' - m.$$

424. To find the intersection of three planes.

Let the intersection of the first and second, found as in the last article, be expressed by the equations

$$x = \alpha z + a$$
$$y = \beta z + b,$$

and let the intersection of the first and third planes be denoted by the equations

$$x = \alpha' z + a'$$
$$y = \beta' z + b'.$$

Then, finding the intersection of these two lines from their four equations, we have the values of x, y, and z, corresponding to the point of intersection of the two lines, and therefore to the point of intersection of the three planes.

In this manner we may find the relation among the co-efficients of any number of planes meeting in one point.

425. To find the relation among the coefficients of the equations to four planes so that they may meet in the same straight line.

Let the equations be

$$m\,x + n\,y + p\,z = 1$$
$$m_1 x + n_1 y + p_1 z = 1$$
$$m_2 x + n_2 y + p_2 z = 1$$
$$m_3 x + n_3 y + p_3 z = 1.$$

Then the first and second plane intersect in a line whose equations are

$$x = \alpha z + a$$
$$y = \beta z + b$$

The first and third intersect in the line

$$x = \alpha_1 z + a_1$$
$$y = \beta_1 z + b_1$$

And the first and fourth in the line

$$x = \alpha_2 z + a_2$$
$$y = \beta_2 z + b_2$$

Now, in order that these intersections all coincide, we must have

$$\alpha = \alpha_1 = \alpha_2 ;\ \beta = \beta_1 = \beta_2 ;\ a = a_1 = a_2 ;\ \text{and } b = b_1 = b_2$$

And the values of α, β, a and b are given in terms of m, n, p, &c., by article (423), hence the relation among the co-efficients is found.

The same relation exists among the co-efficients of any number of planes meeting in one point.

426. To find the relation among the co-efficients of a straight line and plane, so that they may be perpendicular to one another.

Let $(x_1 y_1 z_1)$ be the point in which the plane and line meet, then the equation to the plane is

$$m(x - x_1) + n(y - y_1) + p(z - z_1) = 0 \quad (1)$$

And the equations to the line are

$$\left.\begin{aligned} x &= \alpha z + a \\ y &= \beta z + b \end{aligned}\right\} (2)$$

Also let the equations to a line perpendicular to (2) and passing through the point $(\alpha_1\ y_1\ z_1)$ in (2) be

$$\left.\begin{array}{l} x - x_1 = \alpha'(z - z_1) \\ y - y_1 = \beta'(z - z_1) \end{array}\right\} \quad (3)$$

But since these two lines are perpendicular to one another, we have the cosine of the angle between them = 0,

$$\therefore \ \alpha\alpha' + \beta\beta' + 1 = 0 \quad (402)$$

Now, this equation combined with that to the last line (3), will give the relation among the co-ordinates of x, y, and z, so that the point to which they refer is always in a locus perpendicular to the first given line; hence substituting for α' and β', we have the equation to the plane which is the locus of all the lines perpendicular to (2), this equation is

$$\alpha \frac{x - x_1}{z - z_1} + \beta \frac{y - y_1}{z - z_1} + 1 = 0$$

$$\text{or } \alpha(x - x_1) + \beta(y - y_1) + z - z_1 = 0 \ (4)$$

and as this equation (4) must coincide with (1) we have, by equating the co-efficients,

$$\alpha = \frac{m}{p}, \text{ and } \beta = \frac{n}{p},$$

and these are the conditions required.

427. Hence, if the line be given, the equation to the plane perpendicular to it is

$$\alpha x + \beta y + z = \frac{1}{p}.$$

Or if the plane be given, the equations to the straight line perpendicular to it are

$$x = \frac{m}{p} z + a,$$

$$y = \frac{n}{p} z + b.$$

From the form of these equations to the plane and perpendicular straight line, it appears that the trace of the plane is perpendicular to the projection of the line upon the same co-ordinate plane.

428. If the plane pass through a given point $x_1\, y_1\, z_1$, and be perpendicular to a given straight line, ($x = \alpha z + a$, $y = \beta z + b$) its equation is

$$\alpha(x - x_1) + \beta(y - y_1) + z - z_1 = 0.$$

429. If the straight line pass through a given point, and be perpendicular to a given plane ($mx + ny + pz = 1$) its equations are

$$x - x_1 = \frac{m}{p}(z - z_1),$$

$$y - y_1 = \frac{n}{p}(z - z_1).$$

430. To find the length of a perpendicular from a given point on a given plane.

Let $x_1\, y_1\, z_1$ be the co-ordinates of the given point,

$mx + ny + pz = 1$ the equation to the given plane.

It was shown in Art. 413, that if d be the perpendicular distance of the origin from a plane, whose equation is

$$mx + ny + pz = 1$$

we have $d = \dfrac{1}{\sqrt{m^2 + n^2 + p^2}}.$

Now, the equation to the plane, parallel to the given plane, and passing through the given point, is

$$m(x - x_1) + n(y - y_1) + p(z - z_1) = 0 \text{ (418).}$$

$$\text{or } \frac{mx + ny + pz}{mx_1 + ny_1 + pz_1} = 1.$$

Hence the distance d_1 of the origin from this plane is

$$d_1 = \frac{mx_1 + ny_1 + pz_1}{\sqrt{m^2 + n^2 + p^2}}.$$

But the distance of the given point from the given plane is evidently the distance between the two planes, that is, $= d_1 - d$

$$= \frac{mx_1 + ny_1 + pz_1 - 1}{\sqrt{m^2 + n^2 + p^2}}.$$

431. To find the distance of a point from a straight line.

Let the equations to the given line be $x = \alpha z + a$, $y = \beta z + b$, then the equation to the plane passing through the given point $x_1 y_1 z_1$, and perpendicular to the given line, is

$$\alpha(x - x_1) + \beta(y - y_1) + z - z_1 = 0.$$

Eliminating x, y, and z by means of the above equations to the straight line, we find

$$z = \frac{\alpha(x_1 - a) + \beta(y_1 - b) + z_1}{1 + \alpha^2 + \beta^2};$$

or, if this fraction $= \dfrac{\text{M}}{\text{N}}$, we have

$$z = \frac{\text{M}}{\text{N}}, \quad x = \alpha\frac{\text{M}}{\text{N}} + a, \quad y = \beta\frac{\text{M}}{\text{N}} + b.$$

These are the co-ordinates of the intersection of the given line, with the perpendicular plane passing through the given point; and the required perpendicular line (P) is the distance of the given point from this intersection.

Hence $\text{P}^2 = (x_1 - x)^2 + (y_1 - y)^2 + (z_1 - z)^2$

$$= \left(x_1 - a - \alpha\frac{\text{M}}{\text{N}}\right)^2 + \left(y_1 - b - \beta\frac{\text{M}}{\text{N}}\right)^2 + \left(z_1 - \frac{\text{M}}{\text{N}}\right)^2$$

which, after expansion and reduction, becomes

$$= (x_1 - a)^2 + (y_1 - b)^2 + z_1^2 - \frac{\text{M}^2}{\text{N}}.$$

432. If the given point be the origin, we have $x_1\ y_1\ z_1$, each equal $= 0$

$$\therefore \text{P}^2 = a^2 + b^2 - \frac{(\alpha a + \beta b)^2}{1 + \alpha^2 + \beta^2}.$$

433. To find the angle θ between two given planes.

Let the equations to the planes be

$$mx + ny + pz = 1 \quad (1)$$

$$m_1 x + n_1 y + p_1 z = 1 \quad (2).$$

Then, if from the origin we draw perpendiculars on each of these planes, the angle between these perpendiculars is equal to the angle between the planes: let the equations to the two lines be

$$\left.\begin{matrix} x = \alpha z \\ y = \beta z \end{matrix}\right\} (3) \qquad \left.\begin{matrix} x = \alpha' z \\ y = \beta' z \end{matrix}\right\} (4).$$

In order that (3) may be perpendicular to (1), we must have

$$\alpha = \frac{m}{p}, \beta = \frac{n}{p} \text{ (426), and similarly } \alpha' = \frac{m_1}{p_1}, \beta' = \frac{n_1}{p_1}.$$

Then the angle between the two lines is found from the expression

$$\cos.\,\theta = \frac{\alpha\alpha' + \beta\beta' + 1}{\sqrt{(1 + \alpha^2 + \beta^2)}\sqrt{(1 + \alpha'^2 + \beta'^2)}} \text{ (403).}$$

$$\therefore \cos.\,\theta = \frac{m m_1 + n n_1 + p p_1}{\sqrt{m^2 + n^2 + p^2}\,\sqrt{m_1^2 + n_1^2 + p_1^2}}.$$

434. This value of cos. θ may also be expressed in another form by means of Art. (413.)

$$\cos.\,\theta = \cos.\,P,x \cos.\,P',x + \cos.\,P,y \cos.\,P',y + \cos.\,P,z \cos.\,P',z.$$

$$\text{or } \cos.\,\theta = \cos.\,P,yz \cos.\,P',yz + \cos.\,P,xz \cos.\,P',xz + \cos.\,P,xy \cos.\,P',xy.$$

435. If the planes be perpendicular to each other, we have cos. $\theta = 0$.

$$\therefore m m_1 + n n_1 + p p_1 = 0.$$

Hence, if the equation to any plane be $mx + ny + pz = 1$, the equation to the plane perpendicular to it is

$$m_1 x + n_1 y - \frac{m m_1 + n n_1}{p} z = 1,$$

where two constants remain indeterminate.

436. If the planes be parallel, we have cos. $\theta = 1$; and putting therefore the expression for cos. θ equal to unity, we shall arrive at the results,

$$\frac{m}{n} = \frac{m_1}{n_1} \text{ and } \frac{m}{p} = \frac{m_1}{p_1};$$

the same as already obtained when two planes are parallel.

437. To find the angle between a straight line and a plane.

This angle is the angle which the line makes with its projection on the plane; and therefore, drawing a perpendicular from any point in the line to the plane, is the complement of the angle which this perpendicular makes with the given line.

Let the equations to the plane and the line be

$$mx + ny + pz = 1$$
$$x = \alpha z + a, y = \beta z + b,$$

then the equations to the perpendicular from any point $x_1 y_1 z_1$ in the line to the plane are $x = \frac{m}{p}(z - z_1), y = \frac{n}{p}(z - z_1)$. (429)

$$\therefore \cos.\,(\pi - \theta) = \sin.\,\theta = \frac{\alpha\frac{m}{p} + \beta\frac{n}{p} + 1}{\sqrt{1 + \alpha^2 + \beta^2}\,\sqrt{1 + \frac{m^2}{p^2} + \frac{n^2}{p^2}}}$$

$$= \frac{m\alpha + n\beta + p}{\sqrt{1 + \alpha^2 + \beta^2}\,\sqrt{m^2 + n^2 + p^2}}.$$

CHAPTER IV.

THE POINT, STRAIGHT LINE, AND PLANE REFERRED TO OBLIQUE AXES.

438. If the co-ordinate axes are not rectangular but inclined to each other at any given angles, they are then called oblique axes. The equations to the point, Art. (381.) remain exactly the same as before, but the quantities a, b, and c, are no longer the representatives of lines drawn perpendicular to the co-ordinate planes, but of lines respectively parallel to the oblique axes.

439. To find the distance of a point from the origin referred to oblique axes.

Let A X, A Y, A Z, be the oblique axes; and let x, y, z, be the co-ordinates of P, draw P N perpendicular on A Q produced,

then the sq. on AP = the sqs. on AQ and PQ + twice the rectangle AQ,QN.

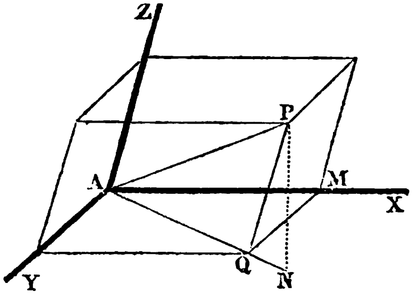
$$\text{Now, } Q\,N = P\,Q \cos. P\,Q\,N = z \cos. Z\,A\,Q$$

$$\text{and } A\,Q \cos. Z\,A\,Q = A\,M \cos. M\,A\,Z + M\,Q \cos. Y\,A\,Z \;(379)$$

$$= x \cos. X\,A\,Z + y \cos. Y\,A\,Z$$

$$\therefore \text{ the rectangle } A\,Q,\ Q\,N = z\,(x \cos. X\,Z + y \cos. Y\,Z)$$

$$\text{also the square on } A\,Q = x^2 + y^2 + 2\,x\,y \cos. Y\,X,$$



$$\therefore d^2 = x^2 + y^2 + z^2 + 2\,x\,y \cos. X\,Y + 2\,x\,z \cos. X\,Z + 2\,y\,z \cos. Y\,Z.$$

440. To find the distance between two points when the axes are oblique

Let $x\ y\ z$ be the co-ordinates of one point,
and $x_1\ y_1\ z_1$ . . the other point,

then the distance between these points is the diagonal of a parallelopiped, of which the sides are the differences of parallel co-ordinates (388); hence,

$$d^2 = (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 + 2\,(x - x_1)\,(y - y_1) \cos. X\,Y$$
$$+ 2\,(x - x_1)\,(z - z_1) \cos. X\,Z + 2\,(y - y_1)\,(z - z_1) \cos. Y\,Z.$$

441. To find the equation to a straight line referred to oblique co-ordinates. The straight line must be considered to be the intersection of two planes formed by drawing straight lines through the several points of the given straight line parallel respectively to the planes of $x\,z$, $y\,z$; the traces of these planes on the co-ordinate planes are of the same *form* as for rect-

angular axes; that is, the equation to the traces, and therefore to the line itself are of the *form*

$$x = \alpha z + a$$
$$y = \beta z + b$$

but the values of α and β are not the tangents of any angles, but the ratio of the sines of the angles which each trace makes with the axes in its plane (51).

The quantities a and b remain the same as when the straight line is referred to rectangular co-ordinates, and since the equations are of the same form as before, those problems which do not affect the inclination of lines will remain the same as before.

442. To find the angle between two straight lines referred to oblique co-ordinates we shall follow the plan adopted in Art. 402.

Let the equations to the parallel lines through the origin be

$$\left.\begin{array}{l} x = \alpha z \\ y = \beta z \end{array}\right\} (1) \qquad \left.\begin{array}{l} x = \alpha' z \\ y = \beta' z \end{array}\right\} (2)$$

And let r be the distance of a point $x\,y\,z$ in (1) from the origin, and r_1 the distance of a point $x_1\,y_1\,z_1$ in (2) from the origin.

Then if d be the distance between these points, we have

$$d^2 = r^2 + r_1^2 - 2\,r\,r' \cos.\theta$$
$$= (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 + 2\,(x - x_1)\,(y - y_1) \cos. XY$$
$$+ 2(x - x_1)\,(z - z_1) \cos. XZ + 2\,(y - y_1)\,(z - z_1) \cos. YZ,$$
$$= r^2 + r_1^2 - 2\,(x\,x_1 + y\,y_1 + z\,z_1)$$
$$- 2\,\{(x_1\,y + x\,y_1) \cos. XY + (x_1\,z + x\,z_1) \cos. XZ + (y_1\,z + y\,z_1) \cos. YZ\}$$
$$\therefore r\,r_1 \cos.\theta = x\,x_1 + y\,y_1 + z\,z_1$$
$$+\{(x_1\,y + x\,y_1) \cos. XY + (x_1\,z + x\,z_1) \cos. XZ + (y_1\,z + y\,z_1) \cos. YZ\}$$
$$\therefore \cos.\theta = \frac{x\,x_1 + y\,y_1 + z\,z_1}{r\,r_1}$$
$$\frac{\alpha\,\alpha^1 + \beta\,\beta^1 + 1 + (\alpha^1\,\beta + \alpha\,\beta^1) \cos. XY + (\alpha^1 + \alpha) \cos. XZ + (\beta^1 + \beta) \cos. YZ}{\sqrt{\{1 + \alpha^2 + \beta^2 + 2\alpha\,\beta \cos. XY + 2\,\alpha \cos. XZ + 2\beta \cos. YZ\}}\;\sqrt{\{1 + \alpha'^2 + \beta'^2 \&c.\}}}$$

443. To find the equation to a plane referred to oblique axes.

We consider a plane as the locus of all the straight lines which can be drawn perpendicular to a given straight line, and passing through a given point in that given straight line.

Let the equations to the given line be

$$x = \alpha z + a$$
$$y = \beta z + b$$

Also the equations to the straight line passing through a point x_1, y_1, z_1, in the above line, are

$$x - x_1 = \alpha'\,(z - z_1)$$
$$y - y_1 = \beta'\,(z - z_1)$$

But these two last lines being perpendicular to each other, we have the angle θ between them $= 90°$, or $\cos.\theta = 0$; hence by the last article:

$$\alpha\,\alpha' + \beta\,\beta' + 1 + (\alpha'\,\beta + \alpha\,\beta') \cos. XY + (\alpha' + \alpha) \cos. XZ + (\beta' + \beta) \cos. YZ = 0$$

and eliminating α' and β'

$$\alpha\,\frac{x-x_1}{z-z_1}+\beta\,\frac{y-y_1}{z-z_1}+1+\left(\frac{x-x_1}{z-z_1}\,\beta+\frac{y-y_1}{z-z_1}\,\alpha\right)\text{cos. X Y}$$

$$+\left(\frac{x-x_1}{z-z_1}+\alpha\right)\text{cos. X Z}+\left(\frac{y-y_1}{z-z_1}+\beta\right)\text{cos. Y Z}=0$$

or, $(\alpha+\beta\text{ cos. X Y}+\text{cos. X Z})\,(x-x_1)+(\beta+\alpha\text{ cos. X Y}+\text{cos. YZ})\,(y-y_1)+(1+\alpha\text{ cos. XZ}+\beta\text{ cos. Y Z})\,(z-z_1)=0$

and this equation, which is the locus of all the straight lines meeting the given straight line at a given point and at right angles, is called the equation to the plane.

444. To find the conditions that a straight line be perpendicular to a given plane;

The method is the same as that in article 426.

The equation to a plane passing through a point $x_1\,y_1\,z_1$ in the given line is

$$m\,(x-x_1)+n\,(y-y_1)+p\,(z-z_1)=0.$$

But the equations to the given line being

$$x=\alpha z+a,\; y=\beta z+b$$

the equation to the plane perpendicular to it is given at the end of the last article; hence, equating co-efficients we have

$$m=\alpha+\beta\text{ cos. X Y}+\text{cos. X Z},$$
$$n=\beta+\alpha\text{ cos. X Y}+\text{cos. Y Z},$$
$$p=1+\alpha\text{ cos. X Z}+\beta\text{ cos. Y Z}.$$

From these equations we have the values of m, n, p; or the values of α and β in terms of m, n, p.

445. To find the angle between a plane and straight line.

Let the given equations be

$$mx+ny+pz=1 \quad (1)$$

$$\left.\begin{aligned}x&=\alpha\,z+a\\ y&=\beta\,z+b\end{aligned}\right\}\;(2)$$

And let the equations to a straight line perpendicular to the given plane be

$$\left.\begin{aligned}x&=\alpha'\,z+a'\\ y&=\beta'\,z+b'\end{aligned}\right\}(3)$$

where α' and β' have the values of α and β in the last article.

Also the angle between the lines (2) and (3) is given in article (442.), and the angle between the plane and the line (1) being the complement of the angle between the two lines (2) and (3) may be obtained.

446. To find the angle between two planes.

The equations to the lines perpendicular to the given planes, and passing through the origin are given by Article (444.); and the angle between these lines, which is the angle between the given planes, is given by Article (442.).

CHAPTER V.

THE TRANSFORMATION OF CO-ORDINATES.

447. To transform an equation referred to an origin A to an equation referred to another origin A′, the axes in the latter case being parallel to those in the former.

The co-ordinates of the new origin being a, b, and c, it is evident that if a point be referred to this new origin and to the new axes, that each original ordinate is equivalent to the new ordinate together with the corresponding ordinate to the new origin; hence if x, y, z be the original coordinate of a point P, and X, Y, Z the new co-ordinates, we have

$$\begin{aligned} x &= a + X, \\ y &= b + Y, \\ z &= c + Z; \end{aligned}$$

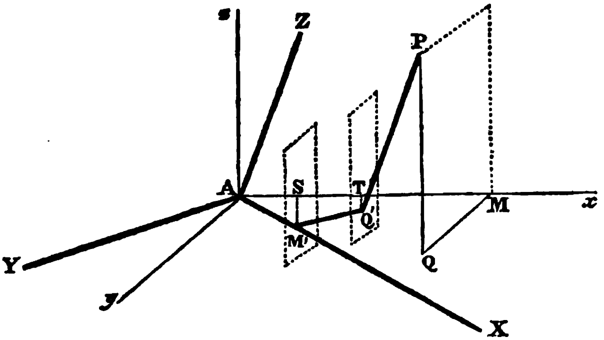
Substituting these values for x, y and z in the equation to the surface, we have the transformed equation between X, Y, and Z referred to the origin A′.

448. To transform the equation referred to rectangular axes to an equation referred to oblique axes having the same origin.

Let A x, A y, A z be the original axes,

A X, A Y, A Z the new axes,

$$\left.\begin{array}{l} AM = x \\ MQ = y \\ QP = z \end{array}\right\} \quad \left.\begin{array}{l} AM' = X \\ M'Q' = Y \\ Q'P = Z \end{array}\right\}$$



Through the points M′, Q′, P draw planes parallel to $y\,z$, or, which is the same thing, perpendicular to A x and meeting A x in S, T and M (these planes are represented by the dotted lines in the figure). Then A S, S T and T M are the respective projections of A M′, M′ Q′ and Q′ P on A x, also

$$\begin{aligned} AM &= AS + ST + TM, \\ &= AM' \cos. XAx + M'Q' \cos. YAx + Q'P \cos. ZAx \quad (378) \end{aligned}$$

$$\therefore \left.\begin{aligned} x &= X\cos. Xx + Y\cos. Yx + Z\cos. Zx \\ y &= X\cos. Xy + Y\cos. Yy + Z\cos. Zy \\ z &= X\cos. Xz + Y\cos. Yz + Z\cos. Zz \end{aligned}\right\}$$

$$\text{Or,}\quad \left.\begin{aligned} x &= mX + m_1 Y + m_2 Z \\ y &= nX + n_1 Y + n_2 Z \\ z &= pX + p_1 Y + p_2 Z \end{aligned}\right\} \; 1.\, ,$$

where m is put for cos. X x, &c.

We have also, by art. 397, the following equation between the angles which one straight line, as A X, makes with the axes of x, y, z.

$$(\cos. Xx)^2 + (\cos. Xy)^2 + (\cos. Xz)^2 = 1,$$

Hence the following system,

$$\left.\begin{aligned} m^2 + n^2 + p^2 &= 1 \\ m_1^2 + n_1^2 + p_1^2 &= 1 \\ m_2^2 + n_2^2 + p_2^2 &= 1 \end{aligned}\right\} \; 2.$$

449. If the new system be rectangular, we have also the equations in art. (405), which signify that the new axes are perpendicular to each other; hence the system

$$\left.\begin{aligned} mm_1 + nn_1 + pp_1 &= 0 \\ mm_2 + nn_2 + pp_2 &= 0 \\ m_1m_2 + n_1n_2 + p_1p_2 &= 0 \end{aligned}\right\} \; 3.$$

Hence we observe that of the nine cosines involved in the system (1) three are determined by the system (2), and other three by the system (3); and therefore that there are only three arbitrary angles remaining.

450. In the place of these three systems the following three may also be used:

$$\left.\begin{aligned} X &= mx + ny + pz \\ Y &= m_1x + n_1y + p_1z \\ Z &= m_2x + n_2y + p_2z \end{aligned}\right\} \; 4.$$

$$\left.\begin{aligned} m^2 + m_1^2 + m_2^2 &= 1 \\ n^2 + n_1^2 + n_2^2 &= 1 \\ p^2 + p_1^2 + p_2^2 &= 1 \end{aligned}\right\} \; 5. \qquad \left.\begin{aligned} mn + mp + np &= 0 \\ m_1n_1 + m_1p_1 + n_1p_1 &= 0 \\ m_2n_2 + m_2p_2 + n_2p_2 &= 0 \end{aligned}\right\} \; 6.$$

For, multiplying the values of x, y and z in (1) by m, n and p respectively; then adding the results together, and reducing by means of (2) and (3), we have $X = mx + ny + pz$; and repeating this operation with the other multipliers $m_1\, n_1\, p_1$ and $m_2\, n_2\, p_2$, we have the system (4). Also, since the distance of P from the origin is the same for both systems, we have $x^2 + y^2 + z^2 = X^2 + Y^2 + Z^2$; putting here, for X, Y and Z, their values in (4), and then equating co-efficients on both sides, we have the two systems (5) and (6).

Whenever we see the systems (2) and (3), we may replace them by (5) and (6); this may be proved independently of any transformation of co-ordinates, by assuming the quantities $m\,n\,p$, &c. to be connected as in (1).

451. The transformation from oblique axes to others oblique, is effected by drawing a perpendicular from M in the last figure upon the

plane of $y\,z$, and by projecting x, X, Y, and Z on this perpendicular, we shall have

$$x \sin. x, y\,z = \mathrm{X} \sin. \mathrm{X}, y\,z + \mathrm{Y} \sin. \mathrm{Y}, y\,z + \mathrm{Z} \sin. \mathrm{Z}, y\,z;$$

and similarly for the other two, x and y,

$$y \sin. y, x\,z = \mathrm{X} \sin. \mathrm{X}, x\,z + \mathrm{Y} \sin. \mathrm{Y}, x\,z + \mathrm{Z} \sin. \mathrm{Z}, x\,z,$$

$$z \sin. z, x\,y = \mathrm{X} \sin. \mathrm{X}, x\,y + \mathrm{Y} \sin. \mathrm{Y}, x\,y + \mathrm{Z} \sin. \mathrm{Z}, x\,y.$$

452. Another useful method of transformation from rectangular axes to others also rectangular, is the following:

Let the equations to the axes of X, Y and Z be respectively

$$\left.\begin{array}{l} x = \alpha z \\ y = \beta z \end{array}\right\} \qquad \left.\begin{array}{l} x = \alpha_1 z \\ y = \beta_1 z \end{array}\right\} \qquad \left.\begin{array}{l} x = \alpha_2 z \\ y = \beta_2 z \end{array}\right\}$$

and let

$$m = \frac{1}{\sqrt{1 + \alpha^2 + \beta^2}}, \quad m_1 = \frac{1}{\sqrt{1 + \alpha_1^2 + \beta_1^2}}, m_2 = \frac{1}{\sqrt{1 + \alpha_2^2 + \beta_2^2}}$$

then by art. (402.) we have

$$\cos. \mathrm{X}\,x = m\,\alpha, \quad \cos. \mathrm{X}\,y = m\,\beta, \ \cos. \mathrm{X}\,z = m; \ \&c.$$

Hence by substitution, the first formulas for transformation in art. (448.) become

$$x = m\,\alpha\,\mathrm{X} + m_1\,\alpha_1\,\mathrm{Y} + m_2\,\alpha_2\,\mathrm{Z}$$

$$y = m\,\beta\,\mathrm{X} + m_1\,\beta_1\,\mathrm{Y} + m_2\,\beta_2\,\mathrm{Z}$$

$$z = m\mathrm{X} + m_1\mathrm{Y} + m_2\,\mathrm{Z}.$$

And the nine angles in (1) are replaced by the six unknown terms $\alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2$.

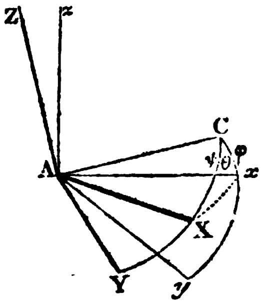
Instead of these systems, we may obtain a system involving only five arbitrary constants by supposing the solid trihedral angle formed by the original co-ordinate planes to turn about the origin into a new position: such a system has been ably discussed by M. Gergonne in the "Annales de Maths.," tome vii. p. 56.

453. It appears throughout these articles that only three arbitrary quantities are absolutely necessary; and therefore it might be supposed that formulas for transformation would be obtained involving only three angles: such formulas have been discovered by Euler, and as they are generally useful in various branches of analysis, we proceed to their investigation.

Let A C be the intersection of the original plane of $x\,y$ with the new plane of X Y, and suppose the plane C X Y A to lie above the plane C $x\,y$ A, which last we may assume to be the plane of the paper.

Let a sphere be described with centre A and radius unity, cutting all the axes in the points indicated by their respective letters.

Let $C x = \phi$, $C X = \psi$, and let the angle $X C x$ between the planes xy and $X Y$ be called θ.



Then the object is to substitute in formula (1) art. (448.) the values of the cosines in terms of the new variables ϕ, ψ, and θ.

This is effected by means of the elementary theorem in spherical trigonometry for finding one side of a triangle in terms of the other two and the included angle. In the triangles $C X x$ and $C Y x$, we have

$$\cos. X x = \cos. \theta \sin \psi \sin. \phi + \cos. \psi \cos. \phi$$

$$\cos. Y x = \cos. \theta \sin. (90^\circ + \psi) \sin. \phi + \cos. (90^\circ + \psi) \cos. \phi$$

$$= \cos. \theta \cos. \psi \sin. \phi - \sin. \psi \cos. \phi.$$

Similarly $\cos. X y$ and $\cos. Y y$ may be found.

Also, supposing $Z x$ and $Z C$ to be joined by arcs of the sphere, we have from the triangle $Z C x$

$$\cos. Z x = \cos. Z C x \sin. Z C \sin. C x + \cos. Z C \cos. C x$$

$$= \cos. (90^\circ + \theta) \sin. 90^\circ \sin. \phi + \cos. 90^\circ \cos \phi$$

$$= - \sin. \theta \sin. \phi.$$

Similarly $\cos. Z y$, $\cos. X z$, and $\cos. Y z$ may be determined.

And $\cos. Z z = \cos. \theta$; hence the system (1) becomes

$$x = X (\cos. \theta \sin. \psi \sin. \phi + \cos. \psi \cos. \phi)$$

$$+ Y (\cos. \theta \cos. \psi \sin. \phi - \sin. \psi \cos. \phi)$$

$$- Z \sin. \theta \sin. \phi$$

$$y = X (\cos. \theta \sin. \psi \cos. \phi - \cos. \psi \sin. \phi)$$

$$+ Y (\cos. \theta \cos. \psi \cos. \phi + \sin. \psi \sin. \phi)$$

$$- Z \sin. \theta \cos. \phi$$

$$z = X \sin. \theta \sin. \psi + Y \sin. \theta \cos. \psi + Z \cos. \theta.$$

These are the formulas investigated, but in a different manner, by Laplace, "Méc. Cél." i. p. 58. They will be found in most works on this subject, but often with some slight alteration in the algebraic signs of the terms, arising from the various positions of A C.

THE INTERSECTION OF A SURFACE BY A PLANE.

454. The last system may be advantageously employed in finding the nature of the intersection of curve surfaces made by planes. If we propose to cut a surface, as a cone for example, by a plane, we should eliminate z from the equations to the surface and plane; but this gives us the equation to the projection of their intersection on $x\,y$, not the equation to the intersection itself; and as the projection will not always suffice to determine the nature of a curve, it is requisite to find the equation to that curve traced on the cutting plane.

This may be done by a transformation of co-ordinates.

Let the cutting plane be that of X Y, and the trace A C the axis of X; the surface will then be referred to new axes X, Y, Z, of which X and Y are in the cutting plane. By putting $Z = 0$ in the equation thus transformed, we shall have the intersection of the surface with the plane X Y, which is the intersection required.

Now, as the present object is only to obtain the curve of intersection, we may at first put $Z = 0$, and then transform the equation.

Let therefore $Z = 0$, and the angle C A X or $\psi = 0$; then the last formulas become

$$\begin{aligned} x &= \text{X} \cos.\phi + \text{Y} \sin.\phi \cos.\theta \\ y &= -\text{X} \sin.\phi + \text{Y} \cos.\phi \cos.\theta \\ z &= \text{Y} \sin.\theta. \end{aligned}$$

These formulas may be separately investigated, with great ease, without deduction from the general case.—See "Francœur," vol. ii. art. 369; or "Puissant, Géométrie," art. 134.

455. In applying these formulas to a particular case, a little consideration will greatly alleviate the labour of transformation: thus, in many cases, we may suppose the cutting plane to be perpendicular to $x\,z$, without at all diminishing the generality of the result, but only adding much to its simplicity; for in this case the trace A C either coincides with A y or y A produced, and therefore $\phi = 90°$; hence the last formulas become

$$\begin{aligned} x &= +\text{Y} \cos.\theta \\ y &= -\text{X} \\ z &= \text{Y} \sin.\theta. \end{aligned}$$

These formulas may be readily investigated by drawing a figure like the last, but letting A C, A X and y A produced coincide, $\phi = 90°$ and $\text{C Y} = 90°$, and then taking the original formulas (1) in art. 448.

456. If in the above cases the origin is also changed, we must introduce the quantities a, b, c into the left side of the above equations.

CHAPTER VI.

THE SPHERE AND SURFACES OF REVOLUTION.

457. A CURVE surface as a sphere being given for discussion, we proceed as in plane geometry to find its equation from some known property of the surface; and generally we arrive at a relation between three unknown quantities x, y, and z, which relation is expressed by the symbol $f(x, y, z) = 0$, or $z = f(x, y)$. This equation is called the equation to the surface, and it corresponds to all points of the surface, and to it alone.

458. Conversely, an equation of the form $f(x, y, z) = 0$, where x, y, and z represent the co-ordinates of a point, refers to some surface. That it cannot belong to all the points in a solid may be thus shown.

Let there be two equations $f(x, y, z) = 0$, and $f'(x, y, z) = 0$; giving to x, y, and z the same values in both these equations, and then eliminating z, we have the equation to the intersection of the above loci projected on the plane of $x y$: this equation is of the form $\phi(x\ y) = 0$, and therefore it belongs to a *line*. Similarly the projections of the intersection on the other co-ordinate planes are *lines*; but if the projections of a locus on three different planes are lines, the locus itself must be a line, that is, it cannot be a surface. Hence the intersection of the two loci of $f(x, y, z) = 0$, and $f'(x, y, z) = 0$ being a line, each of these equations must belong to a surface.

459. Surfaces as well as lines are divided into orders, and for the same object, to avoid the confusion of ideas and to allow us to unite the important properties of generality and simplicity in our investigations as far as possible. Hence a plane which is the locus of a simple equation between three unknown quantities is called a surface of the first order; the locus of an equation of two dimensions between three unknown quantities is called a surface of the second order, and so on. The length, rather than the difficulty of the mathematical operations, renders this part of the subject tedious. Hence we shall omit many of the investigations which merely require manual labour, and rather dwell upon what we consider the important steps.

A much more serious difficulty arises from the state of the figures: we cannot give complete graphical illustrations of this part of geometry, and a mind unaccustomed to the conception of solid figures cannot always comprehend the meaning of the corresponding analytical results. We have endeavoured to obviate this difficulty as much as possible by descriptions of what the figures intend to represent, and to these descriptions we beg the particular attention of our readers, for we are convinced that this part of geometry is by no means difficult, if attention be paid to the form of the body; but without this care it is quite unintelligible.

We commence with the discussion of the Sphere.

THE SPHERE.

460. To find the equation to the surface of a sphere.

Let the surface be referred to rectangular axes, and let x, y, z be the co-ordinates of any point on the surface, and a, b, c the corresponding co-ordinates of the centre. Then since the surface is such that the distance of any point in it from the centre of the sphere is constant or equal to a line r, called the radius, we have by art. (388.)

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2.$$

461. This equation will assume various forms corresponding to the position of the centre.

Let the centre be in the plane of $x\,y$ $\therefore c=0$,

$$\therefore (x-a)^2+(y-b)^2+z^2=r^2.$$

Let the centre be on the axis of z $\therefore a=0$, and $b=0$,

$$\therefore x^2+y^2+(z-c)^2=r^2.$$

462. Let the centre be the origin, $\therefore a=b=c=0$, and the equation is

$$x^2+y^2+z^2=r^2.$$

And this is the equation to the surface of the sphere most generally used.

463. The general equation upon expansion becomes

$$x^2+y^2+z^2-2\,a\,x-2\,b\,y-2\,b\,z+a^2+b^2+c^2-r^2=0.$$

And hence the sphere corresponding to any equation of this form may be described as for the circle, art. 67.

464. The sections of a surface made by the co-ordinate planes are called the principal sections of the surface, and the boundaries of the principal sections are called the traces of the surface on the co-ordinate planes.

The equation to a trace is determined by putting the ordinate perpendicular to the plane of the trace $=0$ in the general equation. Thus, to find the curve in which the sphere cuts the plane of $x\,y$, put $z=0$, and then we have the equation to the points where the plane and sphere meet, which in this case is

$$(x-a)^2+(y-b)^2+c^2=r^2.$$

Hence the section on $x\,y$ is a circle as long as x and y have real values. And, similarly, the other traces are circles.

The theorem that the intersection of any plane with a sphere is a circle, is best proved geometrically, as in Geometry, *b.* v. 19.

465. To find the equation to the tangent plane to a sphere.

Let $x_1\,y_1\,z_1$ be the co-ordinates of the point on the surface through which the tangent plane passes, and let the equation to the spherical surface be

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2;$$

then the equation to the plane passing through the point $x_1\,y_1\,z_1$ is

$$m\,(x-x_1)+n\,(y-y_1)+p\,(z-z_1)=0.$$

Also, the equations to the radius passing through the points $(a\ b\ c)$ $(x_1\,y_1\,z_1)$ are

$$x - x_1 = \frac{x_1 - a}{z_1 - c}(z - z_1),\ y - y_1 = \frac{y_1 - b}{z_1 - c}(\cancel{x} - x_1).$$

$z - z'$ far

And since every line in the tangent plane, and therefore the plane itself, is perpendicular to the radius at the point of tangence, we have from the equations to the plane and line

$$\frac{m}{p} = \frac{x_1 - a}{z_1 - c},\ \frac{n}{p} = \frac{y_1 - b}{z_1 - c}\ (426.)$$

Hence the equation to the tangent plane becomes

$$\frac{x_1 - a}{z_1 - c}(x - x_1) + \frac{y_1 - b}{z_1 - c}(y - y_1) + z - z_1 = 0,$$

or, $(x_1 - a)(x - x_1) + (y_1 - b)(y - y_1) + (z_1 - c)(z - z_1) = 0$.

This equation may be modified by means of the condition

$$(x_1 - a)^2 + (y_1 - b)^2 + (z_1 - c)^2 = r^2,$$

or, $(x_1 - a)(x_1 - a) + (y_1 - b)(y_1 - b) + (z_1 - c)(z_1 - c) = r^2$.

Adding this equation, term by term, to the above one for the tangent plane, we have

$$(x_1 - a)(x - a) + (y_1 - b)(y - b) + (z_1 - c)(z - c) = r^2.$$

466. If the origin is in the centre of the sphere, the equation to the tangent plane is

$$x x' + y y' + z z' = r^2,$$

which equation is at once obtained from that to the sphere $x^2 + y^2 + z^2 = r^2$, or, $x x + y y + z z = r^2$, by putting $x x'$, $y y'$, and $z z'$ for $x x$, $y y$, and $z z$ respectively.

The line in which the tangent plane cuts any co-ordinate plane is found by putting the ordinate perpendicular to that plane $= 0$; and the point in which the tangent plane cuts any axis is found by putting the two variables measured along the other axes each 0.

467. The equation to the spherical surface referred to oblique co-ordinates by (440.) is

$$(x - a)^2 + (y - b)^2 + (z - c)^2 + 2(x - a)(y - b)\cos. XY + 2(x - a)(z - c)\cos. XZ + 2(y - b)(z - c)\cos. YZ = r^2.$$

ON COMMON SURFACES OF REVOLUTION.

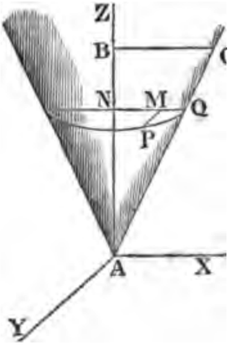
468. A right cone is formed by the revolution of the hypothenuse of a right-angled triangle about one of its sides.

Let A C be the side which revolves about A B as an axis, so that any section Q P perpendicular to the axis is a circle.

Let A X, A Y, A Z be the rectangular axes to which the cone is referred, having the origin at the vertex of the cone, and the axis of Z coincident with the axis of the cone.

$$\left.\begin{aligned} \text{Let } AN &= z \\ NM &= x \\ MP &= y \end{aligned}\right\}\text{be the co-ordinates of any point on the surface.}$$

Then the squares on N M and M P = the square on N P



and N P = N Q = A N tan. C A B,

therefore the equation to the surface is

$$x^2 + y^2 = e^2 z^2,$$

where e = tangent of the semiangle of the cone.

469. Let the line A C be a curve, as a parabola, for example, in which case the surface is called the common paraboloid.

Let the equation to the generating parabola A Q C be N Q = $\sqrt{p z}$. Then the squares on N M, M P = the square on N P = the square on N Q,

$$\therefore x^2 + y^2 = p z.$$

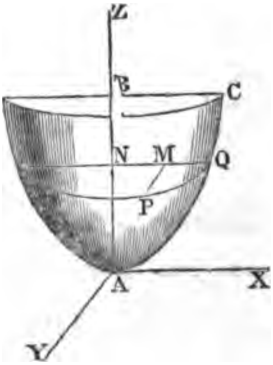
470. Let A C be an ellipse, centre and origin at B.

Let B N = z, N M = x, and M P = y, C B = b and B A = a.

Then the squares on N M and M P = the square on N Q; and N Q being an ordinate to the ellipse A Q C, whose semiaxes are a and b, we have

$$\text{N Q} = \frac{b}{a} \sqrt{a^2 - z^2},$$

and therefore the equation to the surface is



$$x^2 + y^2 = \frac{b^2}{a^2} (a^2 - z^2);$$

$$\text{or, } x^2 + y^2 + \frac{b^2}{a^2} z^2 = b^2.$$

Let a and b change places in the equation, we have then for the surface of revolution round the axis minor the equation

$$x^2 + y^2 + \frac{a^2}{b^2} z^2 = a^2.$$

The former surface is called the prolate spheroid, the latter the oblate spheroid.

471. The equation to the hyperboloid round the transverse axis is

$$x^2 + y^2 - \frac{b^2}{a^2} z^2 = -b^2.$$

And putting a for b and b for a, we have the surface by revolution round the conjugate axis.

472. In general the equation to all these surfaces may be comprehended under the form $x^2 + y^2 = f(z)$ if A Z be the axis of revolution; or, $z^2 + y^2 = f(x)$ if A X be the axis of revolution.

To find the curve of intersection of a plane and a surface of revolution.

473. Let the section be made by a plane perpendicular to $x\,z$, and as the nature of the curve is the same in whatever part of the cutting plane we place the origin, we shall let the origin be in the plane $x\,z$.

Then the formulas for transformation are

$$\begin{aligned} x &= a + y \cos.\theta \\ y &= -x \\ z &= c + y \sin.\theta. \end{aligned}$$

Hence by substitution in the equation to a surface, we shall have the required curve of intersection.

474. Let the surface be a paraboloid

$$x^2 + y^2 = p\,z$$

$$\therefore\ (a + y \cos.\theta)^2 + x^2 = p\,(c + y \sin.\theta)$$

or, $y^2 (\cos.\theta)^2 + x^2 + (2\,a \cos.\theta - p \sin.\theta)\,y = 0$, since $a^2 = p\,c$;

hence the curve of intersection is a line of the second order.

It is an ellipse generally (76); a circle if $\theta = 0$; and a parabola similar to the generating one, if $\theta = 90°$.

475. Let the surface be the spheroid formed by the revolution of an ellipse round its axis major

$$x^2 + y^2 + \frac{b^2}{a^2} z^2 = b^2;$$

by substitution this equation becomes

$$y^2 \left\{ (\cos.\theta)^2 + \frac{b^2}{a^2} (\sin.\theta)^2 \right\} + x^2 + 2\,y \left\{ c \frac{b^2}{a^2} \sin.\theta - a_1 \cos.\theta \right\} = 0.$$

This is the equation to an ellipse generally, and to a circle when $\theta = 0$.

476. Let the surface be the hyperboloid, whose equation is

$$x^2 + y^2 - \frac{b^2}{a^2} z^2 = -b^2,$$

the sections will be found to depend on the value of tan. θ: if tan. θ is less than $\frac{b}{a}$, the curve is an ellipse; if it is equal to $\frac{b}{a}$, the curve is a parabola; and if tan. θ is greater than $\frac{b}{a}$, it is an hyperbola; and lastly, a circle if $\theta = 0$.

CHAPTER VII.

SURFACES OF THE SECOND ORDER.

477. THE general equation to surfaces of the second order is

$$a x^2 + b y^2 + c z^2 + 2 d x y + 2 e x z + 2 f y z + 2 g x + 2 h y + 2 i z + k = 0$$

the number 2 being prefixed to some of the terms merely for convenience. In order to discuss this equation, that is, to examine the nature and position of the surfaces which it represents, we shall render it more simple by means of the transformation of co-ordinates.

Let the origin be transferred by putting

$$x = x' + m, \; y = y' + n, \; z = z' + p,$$

substituting these values in the general equation, and then putting the Sum of
the terms containing the first powers of the variables each $= 0$, we have the equation

$$a x'^2 + b y'^2 + c z'^2 + 2 d x' y' + 2 e x' z' + 2 f y' z' + k' = 0.$$

This equation remains the same if we change x', y', z', into $-x', -y', -z'$ respectively; thence we conclude that any straight line drawn through the origin, and intercepted by the surface, will be divided into two equal parts at the origin; this new origin therefore will be the centre of the surface, attributing to this expression the same signification as we did in treating of curves of the second order (81.)

478. The values of m, n, and p, are to be determined from the three equations

$$\begin{aligned} a m + d n + e p + g &= 0, \text{ co-efficient of } x', \\ b n + d m + f p + h &= 0, \quad \ldots \quad y', \\ c p + e m + f n + i &= 0, \quad \ldots \quad z'. \end{aligned}$$

Eliminate p from the first and second of these equations, and also from the first and third, then from the two resulting equations eliminate n, and we shall arrive at an equation of the first order involving m, whence we have the value of m, and therefore of n and p.

The denominator of the values of m, n and p is

$$a b c + 2 d e f - a f^2 - b e^2 - c d^2$$

hence, if this quantity $= 0$, the values of m and p are infinite, or the surface has no centre when there is this relation among the co-efficients of the original equation. This circumstance corresponds to the case of the parabola in lines of the second order (81.)

479. To destroy the co-efficients of the terms involving $x' y'$, $x' z'$, and $y' z'$, we must have recourse to another transformation of co-ordinates.

Taking the formulas in (452) we have

$$x' = m\,\alpha\,x'' + m_1\,\alpha_1\,y'' + m_2\,\alpha_2\,z''$$
$$y' = m\,\beta\,x'' + m_1\,\beta_1\,y'' + m_2\,\beta_2\,z''$$
$$z' = m\,x'' \quad + m_1\,y'' \quad + m_2\,z''$$

Substituting in the general equation, and then putting the co-efficients of $x''\,y''$, $x''\,z''$, and $y''\,z''$, each $= 0$, we have the three equations

$$(a\,\alpha + d\,\beta + e)\,\alpha_1 + (d\,\alpha + b\,\beta + f)\,\beta_1 + e\,\alpha + f\,\beta + c = 0 \ldots\ldots x''\,y''$$
$$(a\,\alpha + d\,\beta + e)\,\alpha_2 + (d\,\alpha + b\,\beta + f)\,\beta_2 + e\,\alpha + f\,\beta + c = 0 \ldots\ldots x''\,z''$$
$$(a\,\alpha_2 + d\,\beta_2 + e)\,\alpha_1 + (d\,\alpha_2 + b\,\beta_2 + f)\,\beta_1 + e\,\alpha_2 + f\,\beta_2 + c = 0 \ldots y''\,z''$$

Our object is now to ascertain if this transformation can always be effected, that is, to determine the possibility of the values of the six unknown quantities in the last three equations.

480. The equations to the new axis of y'' are $x = \alpha_1\,z$, $y = \beta_1\,z$ (452.); hence, by substitution, the first of the above three equations becomes

$$(a\,\alpha + d\,\beta + e)\,x + (d\,\alpha + b\,\beta + f)\,y + (e\,\alpha + f\,\beta + c)\,z = 0,$$

which is the equation to a plane passing through the origin.

Now the co-ordinates of every point in this plane satisfy the condition that the co-efficient of $x''\,y'' = 0$, that is, give the necessary relation between α_1 and β_1; hence, if the new axis of y'' be drawn in this plane the condition is still satisfied. Thus, the direction of the axis of x'' being quite arbitrary, that of y'' is determined to be in the particular plane given above; and the term $x''\,y''$ is gone.

Again, by a similar elimination of α_2 and β_2 from the co-efficient of $x''\,z''$, and from the equations of z'' ($x = \alpha_2\ z$, $y = \beta_2\ z$), we have, from the similarity of the equations, the same plane as before; hence, if the axis of z'' be also drawn in this plane, the term $x''\,z''$ will disappear.

Also, α_2 and β_2 being thus obtained, the relation between α_1 and β_1 may be found from the co-efficient of $y''\,z'' = 0$.

Thus, fixing upon any position of the axis of x'', that is, giving any values to α and β, we have determined a plane passing through the origin, in which plane any two straight lines whatever drawn from the origin may be the axes of y'' and z'', and one of them as z'' being so drawn, α_2 and β_2 are given, and then the relation between α_1 and β_1 is determined from the co-efficient of $x''\,y'' = 0$.

But since the relation between these quantities α_1 and β_1, and not the quantities themselves, is given by the last equation, it appears that there are an infinite number of systems to which, if the axes be transferred, the products of the variables may be destroyed.

481. Let the new axes be rectangular.

In this case the axis of x'' must be perpendicular to the plane of $y''\,z''$, or the line whose equations are $x = \alpha\,z$, $y = \beta\,z$ is perpendicular to the plane

$$(a\,\alpha + d\,\beta + e)\,x + (d\,\alpha + b\,\beta + f)\,y + (e\,\alpha + f\,\beta + c)\,z = 0$$

$$\therefore\ a\,\alpha + d\,\beta + e = (e\,\alpha + f\,\beta + c)\,\alpha \text{ (426)}$$
$$d\,\alpha + b\,\beta + f = (e\,\alpha + f\,\beta + c)\,\beta$$

Substituting in the first of these equations the value of α obtained from the second, we have the following equation for β:

$$\begin{aligned}&\{(a-b)fe+(f^2-e^2)d\}\beta^3\\ &+\{(a-b)(c-b)e+(2d^2-f^2-e^2)e+(2c-a-b)fd\}\beta^2\\ &+\{(c-a)(c-b)d+(2c^2-f^2-d^2)d+(2b-a-c)fe\}\beta\\ &+\{(a-c)fd+(f^2-d^2)e\}=0.\end{aligned}$$

This equation of the third degree has at least one real value for β, and hence a real value of α; thus the position of the axis of x'' is found, and also the position of the perpendicular plane in which y'' and z'' are situated.

Again, we might find a plane $x''\,z''$ perpendicular to y'', and such that the terms in $x''\,y''$, $y''\,z''$ should disappear, and the necessary conditions will, as appears from the similarity of the equations, lead to the same equation of the third degree in β_1, and the same is true for the axis of z''.

Hence the three roots of the above equation of the third degree are the three real values of β, β_1 and β_2.

These three quantities give the three corresponding values of α, α_1 and α_2, and since there are only one value of each quantity, it appears that there is only one system of rectangular axes to which the curve surface can be referred so as not to contain the products of the variables. For further information on this subject, see "Annales Math." ii. p. 144.

482. By the last transformation, the equation when the locus has a centre is reduced to the form

$$a_1\,x''^2+b_1\,y''^2+c_1\,z''^2+k_1=0$$

$$\text{or, } \mathrm{L}\,x^2+\mathrm{M}\,y^2+\mathrm{N}\,z^2=1$$

by substitution and the suppression of accents, which are no longer necessary.

The order of transformation might have been inverted, by first destroying the products of the variables exactly in every respect as in the last article, and then the resulting equation must be deprived of three terms by a simple change of the origin; the result, after both transformations, is

$$\mathrm{L}\,x^2+\mathrm{M}\,y^2+\mathrm{N}\,z^2+\mathrm{P}\,x=0.$$

483. The central equation involves three distinct cases, which depend on the signs of the quantities L, M, and N.

(1) They may be all positive.

(2) Two may be positive, and the third negative.

(3) One may be positive, and the other two negative.

They cannot be all negative.

Substituting for L, M and N, the constants $\frac{1}{a^2}\;\frac{1}{b^2}\;\frac{1}{c^2}$ respectively, where a is $>b$ and $b>c$, the three cases are

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

The readiest way of obtaining the form of these surfaces is by sections either in planes parallel to the co-ordinate planes, or on the co-ordinate planes. We remark again, that in the latter case they are called the principal sections or traces.

THE ELLIPSOID.

484. $$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

For the trace on $x\,y$, $z=0$, $\therefore \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

.... $x\,z$, $y=0$, $\therefore \frac{x^2}{a^2}+\frac{z^2}{c^2}=1$

.... $y\,z$, $x=0$, $\therefore \frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Therefore the principal sections are ellipses.

Let $z=m$ $\therefore$ the section parallel to $x\,y$ is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1-\frac{m^2}{c^2}$ *

$y=n$ .... $x\,z$ is $\frac{x^2}{a^2}+\frac{z^2}{c^2}=1-\frac{n^2}{c^2}$

$x=p$ .... $y\,z$ is $\frac{y^2}{b^2}+\frac{z^2}{c^2}=1-\frac{p^2}{c^2}$.

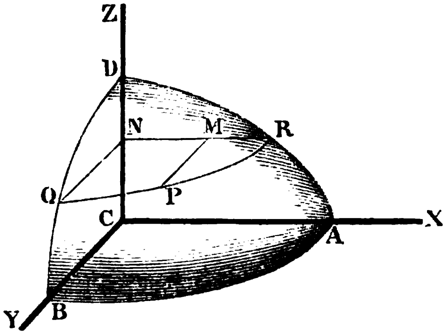
The first of these equations is an ellipse from m, or $z=0$ to $z=c$; when $z=c$ the curve becomes a point, and when z is greater than c the ellipse is imaginary, therefore the surface is limited in the direction of z. Similarly it may be proved, that the other sections are ellipses, and the surface is limited in the directions of x and y. From the circumstance of this surface being thus limited in every direction, and also from the above sections being all ellipses, this surface is called the ellipsoid.

The diameters $2\,a$, $2\,b$, $2\,c$ of the principal sections are called the diameters of the ellipsoid, and their extremities are the vertices of the surface.

If $b=a$, the equation becomes $\frac{x^2}{a^2}+\frac{y^2}{a^2}+\frac{z^2}{c^2}=1$, which is the equation to a spheroid by revolution round the axis of z.

If any other two co-efficients are equal, we have spheroids round the other axes; and if $a=b=c$, the surface becomes a sphere.

485. To render the conception of this surface clear we subjoin a figure representing the eighth part of an ellipsoid.



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* This equation belongs to the projection on $x\,y$, but since the plane of $x\,y$ is parallel to that of $z=m$, the projection is exactly the same in form as the curve of section itself.

A B is part of the ellipse on $x\,y$
A D . . . . $x\,z$
B D . . . . $y\,z$,

and the section Q P R parallel to $x\,y$ is also an ellipse.

The surface may be conceived to be generated by a variable ellipse C A B moving upwards parallel to itself with its centre in C Z. Let N Q R be one position of this variable ellipse; and let

$$\begin{array}{lll} \text{C N} = z, & \text{C A} = a, & \text{N R} = x_1; \\ \text{N M} = x, & \text{C B} = b, & \text{N Q} = y_1; \\ \text{M P} = y, & \text{C D} = c, & \end{array}$$

Then from the ellipse Q P R we have

$$\frac{x^2}{x_1^2} + \frac{y^2}{y_1^2} = 1$$

Also from the ellipses D R A and D Q B we have

$$\frac{x_1^2}{a^2} + \frac{z^2}{c^2} = 1, \text{ and } \frac{y_1^2}{b^2} + \frac{z^2}{c^2} = 1$$

Therefore $\frac{x_1^2}{a^2} = \frac{y_1^2}{b^2}$; and multiplying the first equation by $\frac{x_1^2}{a^2}$ or its equal $\frac{y_1^2}{b^2}$, we have $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{x_1^2}{a^2} = 1 - \frac{z^2}{c^2}$.

$$\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

THE HYPERBOLOID.

486. Case 2. $$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

The principal sections are

$$\text{on } x\,y,\ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (1)$$

$$\text{on } x\,z,\ \frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 \quad (2)$$

$$\text{on } y\,z,\ \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \quad (3).$$

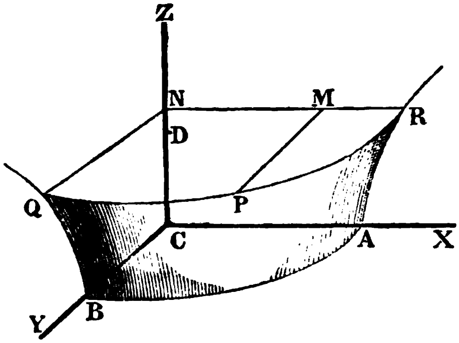
(1) is the equation to an ellipse whose axes are $2\,a$ and $2\,b$; (2) and (3) are hyperbolas with the same imaginary conjugate axis $2c\sqrt{-1}$; if x is less than a, or y less than b, z is imaginary.

Giving to z, y, and x the values m, n, and p, respectively, we have the section parallel to $x\,y$ an ellipse, to $y\,z$ and $x\,z$ hyperbolas.

487. The accompanying figure represents a portion of the eighth part of this surface. A B is the ellipse on $x\,y$, A R the hyperbola on $x\,z$, and B Q is the other hyperbola on $y\,z$. This surface may also be conceived

to be generated by a variable ellipse C A B moving parallel to itself with its centre in C Z. Let N Q R be one position of this variable ellipse; and let

$$\begin{array}{lll} CN = z, & CA = a, & NR = x_1; \\ NM = x, & CB = b, & NQ = y_1; \\ MP = y, & CD = c, & \end{array}$$



Then from the ellipse P Q R, we have

$$\frac{x^2}{x_1^2} + \frac{y^2}{y_1^2} = 1$$

Also from the hyperbolas A R and B Q we have

$$\frac{x_1^2}{a^2} - \frac{z^2}{c^2} = 1, \text{ and } \frac{y_1^2}{b^2} - \frac{z^2}{c^2} = 1,$$

therefore $\frac{x_1^2}{a^2} = \frac{y_1^2}{b^2}$; and multiplying the first equation by $\frac{x_1^2}{a^2}$ or its equal $\frac{y_1^2}{b^2}$, we have

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{x_1^2}{a^2} = 1 + \frac{z^2}{c^2}$$

$$\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

This surface is called the hyperboloid of one sheet because it forms one continuous surface or sheet.

If $a = b$ the surface becomes the common hyperboloid of revolution round the *conjugate* axis.

488. Through the origin draw a line, whose equations are $x = \alpha z$, $y = \beta z$, and substituting in the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$, we have

$$\left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} - \frac{1}{c^2}\right) z^2 = 1$$

$$\therefore z = \pm \frac{abc}{\sqrt{b^2c^2\alpha^2 + a^2c^2\beta^2 - a^2b^2}};$$

hence this line meets the surface as long as the denominator of the fraction is real and finite; let $b^2c^2\alpha^2 + a^2c^2\beta^2 = a^2b^2$, then the line only meets the surface at an infinite distance, or is an asymptote to the surface. The last equation gives the relation between α and β, when the corresponding line is an asymptote; and if for α and β we substitute their general values $\frac{x}{z}$ and $\frac{y}{z}$, we obtain an equation between x, y, z, whose locus will consist of all the asymptotes to the surface, because the co-ordinates of any point in it have the required relation above.

The equation to this surface is

$$b^2 c^2 x^2 + a^2 c^2 y^2 = a^2 b^2 z^2$$

$$\text{or, } \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}:$$

We shall hereafter show (art. 514.) that this is the equation to a cone whose vertex is the origin, and whose base, or section parallel to the axis, is an ellipse.

489. Case 3.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

The principal sections are

$$\text{on } xy, \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \qquad (1)$$

$$\text{on } xz, \frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 \qquad (2)$$

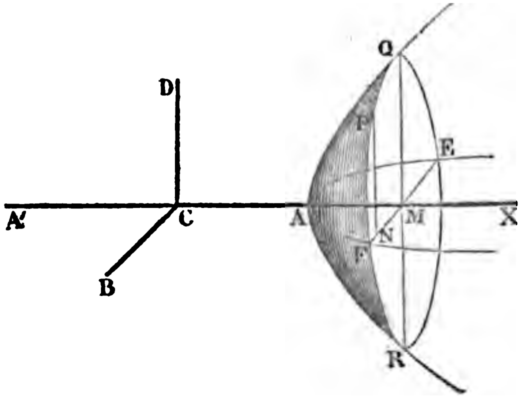
$$\text{on } yz, \frac{y^2}{b^2} + \frac{z^2}{c^2} = -1 \qquad (3).$$

(1) is an hyperbola whose axes are $2a$ and $2b\sqrt{-1}$; (2) is an hyperbola whose axes are $2a$ and $2c\sqrt{-1}$; (3) is imaginary, therefore the plane of yz never meets the surface.

Of the sections parallel to the co-ordinate planes, those parallel to xy and xz are hyperbolas, and that parallel to yz is an ellipse, whose equation is

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{p^2}{a^2} - 1;$$

hence this ellipse is imaginary, if p or x is less than $\pm a$; therefore, if two planes are drawn parallel to $y z$, and at distances $\pm a$ from the centre, no part of the surface can be between these planes.



In the figure E A F represents the hyperbolic section on $x y$, and Q A R that on $x z$; E Q F R is an elliptic section parallel to $y z$. There is an equal and opposite sheet with its vertex at A′; hence the surface is called the hyperboloid of two sheets.

490. The equation to the surface is deduced from the figure; let $AM = x$, $MN = y$, $NP = z$; $QM = z_1$, $MF = y_1$;

Then from the elliptic section Q P F R we have

$$\frac{z^2}{z_1^2} + \frac{y^2}{y_1^2} = 1.$$

Also from the hyperbolas E A F and Q A R we have

$$\frac{y_1^2}{b^2} - \frac{x^2}{a^2} = -1, \text{ and } \frac{z_1^2}{c^2} - \frac{x^2}{a^2} = -1$$

therefore $\frac{y_1^2}{b^2} = \frac{z_1^2}{c^2}$; and multiplying the first equation by $\frac{z_1^2}{c^2}$ or its equal $\frac{y_1^2}{b^2}$, we have

$$\therefore \frac{z^2}{c^2} + \frac{y^2}{b^2} = \frac{z_1^2}{c^2} = \frac{x^2}{a^2} - 1$$

$$\therefore \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

491. $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$, is the equation to the conical asymptote; hence both in case (2) and (3) we have the conical asymptote by omitting the constant term in the equations.

ON SURFACES WHICH HAVE NO CENTRE.

492. In this case the general equation can be deprived of the products of the variables, as in (479); it will then be of the form

$$a x^2 + b y^2 + c z^2 + 2 g x + 2 h y + 2 i z + k = 0.$$

In order to deprive this equation of three more terms, let

$$x = m + x', \; y = n + y', \; z = p + z',$$

$$\therefore a x'^2 + b y'^2 + c z'^2 + 2\,(a m + g)\,x' + 2\,(b n + h)\,y' + 2\,(c p + i)\,z' + k' = 0:$$

Let the co-efficients of x', y' and $z' = 0$;

$$\therefore m = -\frac{g}{a}, \; n = -\frac{h}{b}, \; p = -\frac{i}{c}:$$

But since this class has no centre, the values of some, or all the quantities m, n, p, must be infinite; therefore, either one, two, or three of the co-efficients a, b, c, must $= 0$. Thus the original transformation which deprived the equation of the terms $x y$, $x z$, and $y z$, has of itself destroyed one or two of the co-efficients of x^2, y^2, or z^2; this corresponds to the case in art. 92. Now, all three co-efficients cannot $= 0$, for then we fall upon the equation to a plane: hence we have only two cases left, when a vanishes, or when a and b both vanish.

493. Let $a = 0$, then, as we have three quantities, m, n and p to determine, we may let $k' = 0$ as well as the co-efficients of y' and z'; hence the equation is reduced to the form

$$by'^2 + cz'^2 + 2gx' = 0.$$

$$\text{or} \left(-\frac{b}{2g}\right) y'^2 + \left(-\frac{c}{2g}\right) z'^2 = x'.$$

This equation has two varieties depending upon the signs of the quantities $-\frac{b}{2g}$ and $-\frac{c}{2g}$.

494. Case 1. Let the signs of y'^2 and z'^2 be both alike and positive, (if they were negative we should change the sign of x' to reduce the equation to the same form) substituting $\frac{1}{l}$ for $-\frac{b}{2g}$, and $\frac{1}{l'}$ for $-\frac{c}{2g}$, and suppressing the accents on x, y and z as no longer necessary, the equation is of the form

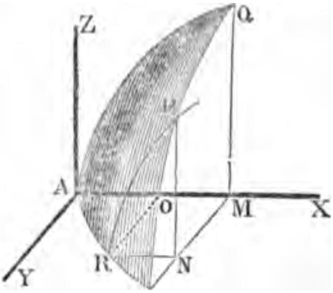
$$\frac{y^2}{l} + \frac{z^2}{l'} = x.$$

For the principal sections we have

on xy, $y^2 = lx$. (1)
on xz, $z^2 = lx'$. (2)
on yz, $l'y^2 + lz^2 = 0$ (3)

(1) and (2) are parabolas extending on the side of x positive; (3) is a point, which is the origin itself.

For the sections parallel to



$$xy, \text{ put } z = p, \therefore \frac{y^2}{l} = x - \frac{p^2}{l'} \quad (1)$$

$$xz, \text{ put } y = n, \therefore \frac{z^2}{l'} = x - \frac{n^2}{l} \quad (2)$$

$$yz, \text{ put } x = m, \therefore \frac{y^2}{l} + \frac{z^2}{l'} = m \quad (3)$$

(1) and (2) are parabolas, equal to those of the principal sections respectively, (the equation differing by a constant term, implies that the origin is differently situated with regard to the curve): (3) is an ellipse.

495. In the figure A Q and A R are parts of the parabolas on xz and xy, and the surface is described by the motion of the parabola A Q, parallel to itself, its vertex moving along the parabola A R. Let P R N be one position of the generating parabola, and let A M $= x$, M N $= y$, N P $= z$, and draw R O parallel to A Y or M N; then from the parabola R P we have

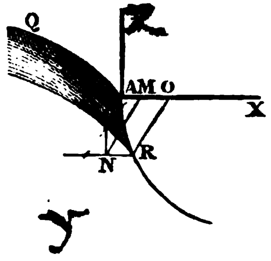
$$z^2 = l' \text{RN} = l'(\text{AM} - \text{AO}) = l'\left(x - \frac{y^2}{l}\right)$$

$$\therefore \frac{z^2}{l'} + \frac{y^2}{l} = x.$$

This surface is called the elliptic paraboloid, and is composed of one entire sheet, like the paraboloid of revolution.

496. Case 2. Let the signs of y'^2 and z'^2 be different.

$$\therefore \frac{y^2}{l} - \frac{z^2}{l'} = x.$$



For the principal sections we have
on xy, $y^2 = lx$ (1)
on xz, $z^2 = -l'x$ (2)
on yz, $l'y^2 - lz^2 = 0$ (3)
(1) and (2) are parabolas, the first corresponding to x positive, and the second to x negative; (3) belongs to two straight lines through the origin.

The sections in planes parallel to xy and xz are parabolas, and those parallel to yz are hyperbolas.

497. AQ is the parabola on xz, and AR is that on xy; and the surface is described by the motion of the parabola AQ parallel to itself, its vertex moving along the parabola AR. Let RPN be one position of the generating parabola, and let $AM = x$, $MN = y$, and $NP = z$, and draw RO parallel to MN; then from the parabola RQ we have;

$$z^2 = l'\,RN = l'(AO - AM) = l'\left(\frac{y^2}{l} - x,\right)$$

$$\therefore \frac{y^2}{l} - \frac{z^2}{l'} = x.$$

This surface is called the hyperbolic paraboloid.

498. The equations to the elliptic and hyperbolic paraboloids may be deduced from those of the ellipsoid and hyperboloid of one sheet, as the equation to the parabola was deduced from that to the ellipse (228) by supposing the centre to be infinitely distant.

Let the origin be transferred to a vertex of the surface, by putting $x - a$ for x, then the equation to the ellipsoid and hyperboloid is

$$\frac{(x-a)^2}{a^2} + \frac{y^2}{b^2} \pm \frac{z^2}{c^2} = 1.$$

Let m and m' be the distances of the vertex from the foci of the sections on xy and xz;

$$\therefore b^2 = a^2 - (a - m)^2 = 2am - m^2$$
$$\text{and } c^2 = 2am' - m'^2;$$

therefore, by substitution, the equation

$$\frac{x^2}{a^2} - \frac{2x}{a} + 1 + \frac{y^2}{b^2} \pm \frac{z^2}{c^2} = 1$$

$$\text{becomes } \frac{x^2}{a^2} - \frac{2x}{a} + \frac{y^2}{2am - m^2} \pm \frac{z^2}{2am' - m'^2} = 0$$

$$\text{or } \frac{x^2}{a} - 2x + \frac{y^2}{2m - \dfrac{m^2}{a}} \pm \frac{z^2}{2m' - \dfrac{m'^2}{a}} = 0,$$

$$\text{or } \frac{y^2}{2m} \pm \frac{z^2}{2m'} - 2x = 0, \text{ when } a \text{ is infinite.}$$

And hence results obtained for the ellipsoid and hyperboloid will be true for the paraboloids, after making the above substitutions.

499. We stated in article 492, that both a and b might vanish; in this case the equation will be

$$cz^2 + 2gx + 2hy + 2iz + k = 0.$$

And by the transformation in art. 492, we cannot destroy the co-efficients of x and y, but we may destroy that of z, and also the constant term k; hence the transformed equation is reduced to the form

$$cz^2 + 2gx + 2hy = 0;$$

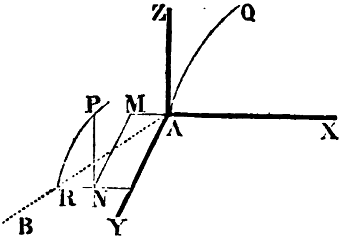
$$\text{or } z^2 = lx + l'y \text{ if } -\frac{2g}{c} = l, \text{and } -\frac{2h}{c} = l'.$$

500. There are two cases depending on the signs of l and l', which may be both positive, or one positive and the other negative.

Case 1. l and l' both positive.

The section on xy is $lx + l'y = o$ (1)
on xz is $z^2 = lx$ (2)
on yz is $z^2 = l'y$ (3)

(1) is a straight line A B; (2) is a parabola A Q; (3) is also a parabola, not in the figure; the sections on the planes parallel to the above are similar in each case. The surface is formed by the motion of the parabola A Q parallel to itself, its vertex describing the straight line A R; let R P N be one position of the generating parabola; let A M $= x$, M N $= y$, N P $= z$,



$$\text{then } z^2 = l.\,\text{RN} = l\left(\frac{l'}{l}y + x\right) = l'y + lx.$$

Since this surface is a cylinder with a parabolic base, it is not usually classed among the surfaces of the second order.

Case 2. If the signs of l and l' be different, the surface will be the same, but situated in a different manner.

CHAPTER VIII.

CYLINDRICAL AND CONICAL SURFACES.

501. Our notion of surfaces will be very much enlarged, if we take into consideration the general character of classes of surfaces, defining them by their peculiar method of generation, and then expressing that definition in a general algebraical form. For example, we have been accustomed, in common geometry, to consider a cylinder as a surface generated by a straight line, which is carried round the circumference of a given circle, and always parallel to a given straight line. (Geom. b. v.

def. 1.) But it is evident that if the base be not a circle, but any other curve, as a parabola, for instance, we shall have a surface partaking of the essential cylindrical character, and which, with others of the same kind, come under a more extended definition; and similarly for conical and many other surfaces.

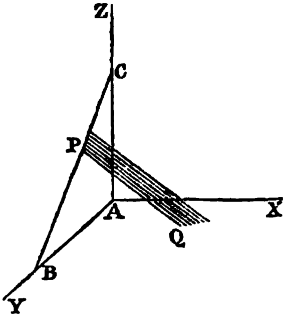
Having seized upon this general character, method of generation, or law by which the lines are compelled to move, the next step is to express this fact in algebraical language; that is, to obtain an equation between co-ordinates x, y, and z, of any point on the surface, which equation shall belong to the class of surfaces in the first instance, and then can be adapted to any particular surface in that class.

THE PLANE.

502. In order to prepare the reader for this subject, we shall take a simple case: to find the surface generated by the motion of a straight line, parallel to itself, and constrained to pass through a given straight line.

Let A X, A Y, A Z be rectangular axes, and let the equations to the given straight line B C (supposed for the sake of simplicity to be in the plane of $y\,z$) be

$$\left.\begin{array}{r} n\,Y + p\,Z = 1 \\ X \quad\quad = 0 \end{array}\right\} 1.$$



Also, let the equations to the generating line P Q, in any one of its positions, be

$$\left.\begin{array}{l} x = \alpha\,z + a \\ y = \beta\,z + b \end{array}\right\} 2.$$

Now, α and β are the tangents of the angles which the projections of P Q make with the axes A X and A Y respectively; and in the motion of P Q, parallel to itself, the projections also remain parallel to themselves respectively; and hence α and β are always constant, and therefore are known or given quantities. But a and b being the co-ordinates of the point where the line P Q meets the plane of $x\,y$, they change with every change of position of P Q; and therefore, being variable, must not appear in the final equation to the surface. Now, these variable quantities, a and b, can be expressed in terms of the other variable quantities x, y, z; and hence we can thus estimate them from the two given systems above.

At the point P, where P Q meets B C, we have, by comparison of (1) and (2),

$$X = x = 0$$
$$Z = z = -\frac{a}{\alpha}$$
$$Y = y = -\frac{a\beta}{\alpha} + b$$

But the system (1) is true for any values of X, Y, Z; therefore, by substitution in (1), we have

$$n\left(-\frac{a\beta}{\alpha} + b\right) + p\left(-\frac{a}{\alpha}\right) = 1;$$

and this is the equation connecting a and b together, or expressing the relation which the variable quantities a and b have to each other, or the relation which any quantities equal to a and b have to each other; that is, substituting for a and b the quantities $x - \alpha z$, and $y - \beta z$ from (2), we shall have the relation between the quantities x, y, and z, which is called the equation to a surface.

$$\therefore -n\frac{\beta}{\alpha}\left(x - \alpha z\right) + n(y - \beta z) - \frac{p}{\alpha}\left(x - \alpha z\right) = 1;$$

$$\text{or,} \; -\frac{n\beta + p}{\alpha}x + ny + pz = 1,$$

which is the equation to a plane; and this is the most general method of determining the equation to a plane; for it can be thus found for any system of co-ordinate axes, and it is determined from the most obvious character of the plane.

We now proceed to the discussion of surfaces formed by the motion of a straight line constrained to move after some given law or condition.

ON CYLINDRICAL SURFACES.

503. Definition. A cylindrical surface is generated by a straight line, which moves parallel to itself in space, and describes, with its extremity, a given curve.

The straight line which moves is called the Generatrix; and the given curve is called the Directrix.

To find the equation to the surface,

Let the equation to the generatrix, in any one of its positions, be

$$x = \alpha z + a$$
$$y = \beta z + b$$

Now, the generatrix, in its movement, always moving parallel to itself; the quantities α and β remain the same for every position of the generatrix; but the quantities a and b, which are the co-ordinates of the point where the generatrix meets the plane of $x\,y$, are constant for the same position of the generatrix, but vary when the generatrix passes from one position to another. Thus, when any point on the surface changes its position without quitting the generatrix, a and b are both constant; and when the point moves from one position of the generatrix to another, a and b are both variable; hence these two quantities, being constant together, and variable together, must be dependent on each other in some way or another; which general dependence is expressed by saying that one of them is a function of the other,

$$\therefore b = \phi(a);$$

or, putting for b and a their values as above, we have

$$y - \beta z = \phi(x - \alpha z),$$

which is the general equation to cylindrical surfaces.

504. The form of the function ϕ will depend upon the nature of the directrix in any particular case.

Let the equations to the directrix be

$$\left.\begin{array}{l} F(X, Y, Z,) = 0 \\ f(X, Y, Z,) = 0 \end{array}\right\}$$

Then as the generatrix must in all its positions meet the directrix, the equations to this curve and to the generatrix must exist simultaneously for the points of intersection; thus having four equations we may eliminate x, y, z, and arrive at an equation between a, b, and constant quantities, which will determine the form of the function ϕ.

Substituting in this equation for a and b their values $x - \alpha z$, $y - \beta z$, we have the actual equation to the particular cylinder required.

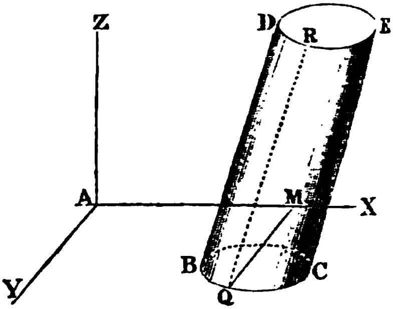
505. Ex. 1. Let the directrix be the circle B Q C, in the plane of $x y$, and let x_1 and y_1 be the co-ordinates of its centre; then the equations to the directrix are

$$\left.\begin{array}{r} (X - x_1)^2 + (Y - y_1)^2 = r^2 \\ Z = 0 \end{array}\right\} (1)$$

Let B D, Q R, C E, be various positions of the generatrix whose general equation is

$$\left.\begin{array}{l} x = \alpha z + a \\ y = \beta z + b \end{array}\right\} (2)$$

to express that the generatrix meets the circle as at Q, the equations (1) and (2) must exist together,



$$\therefore Z = z = 0$$
$$X = x = a$$
$$Y = y = b$$

substituting these values in (1), we have

$$(a - x_1)^2 + (b - y_1)^2 = r^2 \qquad (3)$$

hence the form of the function ϕ is determined.

Substituting in (3) the values of a and b from (2), we have

$$(x - \alpha z - x_1)^2 + (y - \beta z - y_1)^2 = r^2$$

This is the equation to an oblique cylinder, with circular base, situated in the plane of $x y$.

506. Let the centre of the circle be at the origin,

$$\therefore x_1 = 0 \text{ and } y_1 = 0$$
$$\therefore (x - \alpha z)^2 + (y - \beta z)^2 = r^2$$

And if the origin be at the extremity of a diameter parallel to the axis of x,

$$(x - \alpha z)^2 + (y - \beta z)^2 = 2 r (x - \alpha z)$$

507. Let the axis of the cylinder be parallel to the axis of z; then α and β

each $= 0$, since they are the tangents of the angles which the projection of the generatrix on $x\,z$ and $y\,z$ make with A Z;

And if the axis coincide with A Z $\therefore\ (x - x_1)^2 + (y - y_1)^2 = r^2$;

$$x^2 + y^2 = r^2;\ z = 0;$$

in these cases the cylinder is called a right cylinder, and its equation is the same as that of the directrix.

If the directrix be a circle on $x\,z$, the equation to the right cylinder will be

$$x^2 + z^2 = r^2.$$

508. Let the directrix be a parabola on $x\,y$, vertex at the origin, and axis coincident with the axis of x.

Then the equations to the directrix and generatrix are

$$\left.\begin{array}{l} Y^2 = p\,X \\ Z = 0 \end{array}\right\} 1 \qquad \left.\begin{array}{l} x = \alpha\,z + a \\ y = \beta\,z + b \end{array}\right\} 2;$$

therefore at the points of junction we have

$$\begin{array}{l} Z = z = 0 \\ X = x = a \\ Y = y = b; \end{array}$$

then by substituting in (1) we have

$$b^2 = p\,a$$

$$\therefore\ (y - \beta\,z)^2 = p\,(x - \alpha\,z)$$

which is the equation to an oblique parabolic cylinder, whose base is on $x\,y$.

509. Let the directrix be a parabola on $x\,z$, axis A X, and vertex at A; and let the generatrix be parallel to the plane $x\,y$.

The equations are

$$\left.\begin{array}{l} Z^2 = p\,X \\ Y = 0 \end{array}\right\} 1 \qquad \left.\begin{array}{l} y + \alpha\,x = a \\ z = b \end{array}\right\} 2$$

Then the equation to the surface is

$$z^2 = \frac{p}{\alpha}\,y + p\,x. \qquad \text{See article (499).}$$

ON CONICAL SURFACES.

510. Definition. A conical surface is generated by the movement of a straight line, which passes constantly through a given point, and also describes a given curve.

The given point is called the centre of the surface, the straight line which moves is called the generatrix, and the given curve is called the directrix.

Let a, b, c, be the co-ordinates of the centre; then the equations to the generatrix are

$$\begin{array}{l} x - a = \alpha\,(z - c) \\ y - b = \beta\,(z - c). \end{array}$$

Now when a point on the surface changes its position without quitting the generatrix, the quantities α, β are constant, but when the point passes from one generatrix to another, they are both variable; hence being constant together, and variable together, they are functions of one another;

$\therefore\ \beta = \phi\,(\alpha)$, or substituting their equals,

$\dfrac{y - b}{z - c} = \phi\left(\dfrac{x - a}{z - c}\right)$ which is the general equation to conical surfaces.

511. The form of the function ϕ will depend upon the nature of the directrix in any particular case.

By combining the equations to the generatrix and directrix we may, as for cylindrical surfaces, eliminate x, y, z, in a particular case, and thus arrive at an equation between α and β, which will determine the form of the function ϕ.

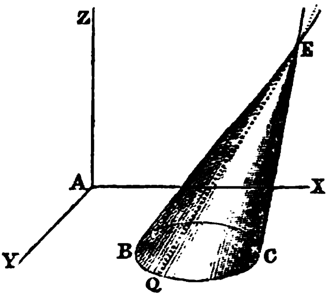
Substituting in this equation for α and β their values $\frac{x - a}{z - c}$ and $\frac{y - b}{z - c}$ we obtain the actual equation to the particular conical surface.

512. Ex. Let the directrix be a circle B Q C in the plane of $x\, y$.

The equations to this directrix are

$$\left.\begin{array}{c}(X - x_1)^2 + (Y - y_1)^2 = r^2\\ Z = 0\end{array}\right\} \quad (1)$$

And the equations to the generatrix B E, or Q E passing through the point E (a, b, c), are



$$\left.\begin{array}{l}x - a = \alpha\, (z - c\\ y - b = \beta\, (z - e\end{array}\right\} \quad (2)$$

To express that the generatrix meets the circle, the equations (1) and (2) must coexist.

$$\begin{array}{l}\therefore Z = z = 0\\ X = x = a - \alpha\, c\\ Y = y = b - \beta\, c\end{array}$$

hence by substitution in (1) we have

$$(a - \alpha\, c - x_1)^2 + (b - \beta\, c - y_1)^2 = r^2 \; (3)$$

Putting for α and β their values from (2) and reducing

$$\left(\frac{a\, z - c\, x}{z - c} - x_1\right)^2 + \left(\frac{b\, z - c\, y}{z - c} - y_1\right)^2 = r^2$$

This is the equation to an oblique cone with a circular base situated in the plane of $x\, y$.

Let the centre of the circle be at the origin $\therefore x_1 = o$ and $y_1 = o$;

$$\therefore (a\, z - c\, x)^2 + (b\, z - c\, y)^2 = r^2\, (z - c)^2$$

513. Let the axis of the cone be parallel to the axis of z $\therefore a = x_1$ and $b = y_1$, and the general equation becomes

$$\left(\frac{x - a}{z - c}\right)^2 + \left(\frac{y - b}{z - c}\right)^2 = \frac{r^2}{c^2}$$

In this case the cone is called a right cone.

Also, if in this case the origin be at the centre of the circle, we have $a = o$ and $b = o$,

$$\therefore x^2 + y^2 = \frac{r^2}{c^2}\, (z - c)^2$$

514. Directrix an ellipse on $x\, y$, whose centre is the origin, and the centre of the cone in the axis of z; then the equation to the cone is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \left(\frac{z - c}{c}\right)^2$$

or putting $r - c$ for c, that is, measuring from the centre of the cone

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}.$$

In this simple case, the equation to the surface is easily found by the method in article (468). Taking the figure in that article, and supposing every section, like P Q, to be an ellipse, whose axes x_1 and y_1 are always proportional to the axes a and b of an ellipse whose centre is in A Z, and at a distance c from A, we have the equation to P Q

$$\frac{x^2}{x_1^2} + \frac{y^2}{y_1^2} = 1,$$

$$\text{but } y_1 = \frac{b}{a} x_1, \text{ and } x_1 = \frac{a}{c} z$$

$$\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}.$$

515. Let the directrix be a parabola parallel to $x\,y$, and vertex in the axis of z. The equations to the directrix and generatrix are

$$\left.\begin{array}{l} Y^2 = p\,X \\ Z \;= d \end{array}\right\} 1 \qquad \left.\begin{array}{l} x - a = \alpha\,(z - c) \\ y - b = \beta\,(z - c) \end{array}\right\};$$

at the points of junction we have

$$\begin{array}{l} Z = z = d \\ X = x = a + \alpha\,(d - c) \\ Y = y = b + \beta\,(d - c) \end{array}$$

hence the final equation is

$$\left\{b + \frac{y - b}{z - c}(d - c)\right\}^2 = p\left\{a + \frac{x - a}{z - c}(d - c)\right\}.$$

516. Let the vertex or centre of the cone be at the origin $\therefore a = b = c = o$, and the equation to a cone whose directrix is $\{ y^2 = p\,x,\ z = d \}$ and whose vertex is at the origin, is

$$d\,y^2 = p\,x\,z.$$

517. The following method of finding the equation to a right cone whose vertex is at the origin, is sometimes useful.

Let the length of the axis of the cone be k, and suppose this axis to pass through the origin, and be perpendicular to a given plane or base whose equation therefore will be of the form

$$\alpha\,x + \beta\,y + \gamma\,z = k$$

where α, β, γ are the co-sines of the angles which k makes with the axis of x, y, and z (410).

Also suppose x, y, and z to be the co-ordinates of a point on the circumference of this base, and let θ be the angle which the generatrix of the cone makes with its axis, then by the property of the right-angle triangle we have the equation

$$k = \sqrt{(x^2 + y^2 + z^2)}\cos.\theta$$

Hence by equating the values of k we have the equation,

$$(\alpha\,x + \beta\,y + \gamma\,z)^2 = (x^2 + y^2 + z^2)\,(\cos.\theta)^2.$$

And this is the equation to any point in the surface, since α, β, γ remain the same for a plane parallel to the base and passing through any point $(x\ y\ z)$ of the surface.

If the axis of the cone coincides with the axis of z, we have $\alpha = \beta = 0$ and $\gamma = 1$; $z^2 = (x^2 + y^2 + z^2)\,(\cos.\ \theta)^2$

518. To find the curve of intersection of a plane and an oblique cone, we may suppose the cutting plane to pass through the origin of co-ordinates without detracting from the generality of the result. Substituting for x, y, z, in the equation, their values in 455, we readily find that the sections are lines of the second order and their varieties.

ON CONOIDAL SURFACES.

519. Definition. A conoidal surface is generated by the movement of a straight line constantly parallel to a plane, one extremity of the line moving along a given straight line, the other describing a given curve.

We shall commence with a simple case. Let the axis of z be one directrix, and let the generatrix be parallel to the plane of $x\ y$: then the equations to the generatrix in any one position are

$$\left.\begin{array}{l} y = \alpha\, x \\ z = b \end{array}\right\}$$

Now it is evident that when a point moves on the surface without quitting the generatrix, α and b are both constant, but when it passes from one position of the generatrix to another α and b are both variable; hence these quantities, being constant together and variable together, are functions of one another.

$\therefore\ b = \phi\,(\alpha)$ or substituting their values.

$$z = \phi\left(\frac{y}{x}\right)$$

which is the general equation to all conoidal surfaces.

520. The form of the function ϕ will depend upon the nature of the second directrix.

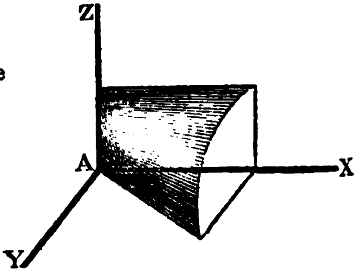
By combining the equations to the generatrix and this directrix, we may, as before, eliminate x, y, z, and arrive at an equation between b and α, we must then substitute the values of b and α, their general values z and $\frac{y}{x}$, and we shall obtain the equation to the particular conoidal surface.

521. Let the second directrix be a circle parallel to $y\ z$, and the centre in the axis of x, therefore the equations to this directrix are

$$\left.\begin{array}{r} Z^2 + Y^2 = r^2 \\ X = a \end{array}\right\}\ (1)$$

Then where this directrix meets the generatrix we have

$$Z = z = b$$
$$X = x = a$$
$$Y = y = \alpha\, a$$
$$\therefore\ b^2 + \alpha^2 a^2 = r^2$$



Hence the required equation is

$$z^2 + a^2\,\frac{y^2}{x^2} = r^2.$$

This surface partaking of the form and generation of both the cone and the wedge, was called the cono-cuneus by Wallis, who investigated many of its properties.

If the axis of x be one directrix, and the other be a circle parallel to $x\,z$, and the generatrix be parallel to $y\,z$, the equation is

$$x^2 + \frac{a^2\,z^2}{y^2} = r^2.$$

522. Let the axis of z be one directrix, any straight line the other, and let the generatrix move parallel to $x\,y$.

Then the equations to the second directrix are

$$\begin{aligned} X &= \mu\,Z + m \\ Y &= \nu\,Z + n \end{aligned}$$

Also the equations to the generatrix being $y = \alpha\,x$, $z = b$, we have at the points of junction

$$\begin{aligned} Z &= z = b \\ Y &= y = \nu\,b + n \\ X &= x = \frac{\nu\,b + n}{\alpha} \end{aligned}$$

$$\therefore\ \frac{\nu\,b + n}{\alpha} = \mu\,b + m$$

$$\therefore\ (\nu\,z + n)\,\frac{x}{y} = \mu\,z + m$$

$$\text{or} \qquad \nu\,z\,x - \mu\,z\,y + n\,x - m\,y = 0.$$

523. Let the axis of z be one directrix, and let the second directrix be the thread of a screw whose axis is coincident with the axis of z.

The thread of a screw, or the curve called the helix, is formed by a thread wrapped round the surface of a right cylinder, so as always to make the same angle with the axis; or if the base of a right-angled triangle coincide with the base of the cylinder, and the triangle be wrapped round the cylinder, the hypothenuse will form the helix A P.

To find the equations to the helix,

Let the centre of the cylindrical base be the origin of rectangular axes. C M $= x$, M Q $= y$, P Q $= z$ and the radius of the cylinder $= a$.

Then P Q bears a constant ratio to A Q; namely, that of the altitude to the base of the describing triangle

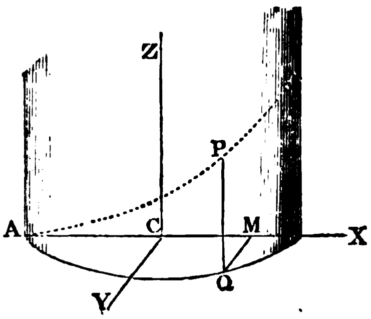
$$\therefore\ \mathrm{P\,Q} = e\,\mathrm{A\,Q}$$

and A Q is a circular arc whose sine is y and radius a:

$$\therefore\ z = e\,a\,\sin.^{-1}\frac{y}{a},$$

$$\text{or } z = e\,a\,\cos.^{-1}\frac{x}{a};$$

$$\text{also } x^2 + y^2 = a^2$$



And these are the equations to the projections of the helix.

To return to the problem, which is to find the surface described by a line subject to the conditions that it be parallel to the base of the cylinder, that it passes through the axis, and that it follows the course of the helix.

The equations to the directrix (if c be the interval between two threads) are

$$z = e\,a \sin.^{-1}\frac{y}{a} + c$$

$$x^2 + y^2 = a^2$$

And the equations to the generatrix being $y = \alpha\,x$, $z = b$; we have

$$z = b;\ x = \frac{y}{\alpha} = \sqrt{a^2 - y^2}\ \therefore\ y = \frac{\alpha\,a}{\sqrt{1+\alpha^2}}$$

$$\therefore b = e\,a \sin.^{-1}\frac{\alpha}{\sqrt{1+\alpha^2}} + c$$

hence the equation to the surface is

$$z = e\,a \sin.^{-1}\frac{y}{\sqrt{y^2 + x^2}} + c$$

This surface is the under side of many spiral staircases.

524. A straight line passes through two straight lines whose equations are $x = a$, $y = b$; and $x = a_1$, $z = b_1$; and also through a given curve $z = f(y)$ in the plane of $z\,y$; to find the equation to the surface traced out by the straight line.

The three directrices are

$$\left.\begin{array}{l} X = a \\ Y = b \end{array}\right\} 1 \qquad \left.\begin{array}{l} X = a_1 \\ Z = b_1 \end{array}\right\} 2 \qquad \left.\begin{array}{l} Z = f(Y) \\ X = 0 \end{array}\right\} (3)$$

And let the equations to the generatrix be

$$x = \alpha\,z + m$$

$$y = \beta\,z + n$$

and consequently $y = \dfrac{\beta}{\alpha}x + p$, if $p = n - \dfrac{\beta}{\alpha}m$;

Then since this line meets the three given lines, we have the following equations

$$b = \frac{\beta}{\alpha}a + p,\quad a_1 = \alpha\,b_1 + m,\quad -\frac{m}{\alpha} = f\left(-\frac{m}{\alpha}\beta + n\right)$$

We must now eliminate α, β, m, n from these equations, and that to the generatrix.

By subtraction we have

$$y - b = \frac{\beta}{\alpha}(x - a);\ x - a_1 = \alpha(z - b_1).\ \therefore \alpha = \frac{x - a_1}{z - b_1},$$

$$\therefore \frac{m}{\alpha} = \frac{x}{\alpha} - z = \frac{a_1 z - b_1 x}{x - a_1}, \text{ and}$$

$$\frac{n\alpha - m\beta}{\alpha} = \frac{\alpha(y - \beta z) - \beta(x - \alpha z)}{\alpha} = \frac{\alpha y - \beta x}{\alpha} = y - \frac{\beta x}{\alpha} = \frac{b x - a y}{x - a}.$$

Hence the final equation is

$$\frac{b_1 x - a_1 z}{x - a_1} = f\left(\frac{b\,x - a\,y}{x - a}\right)$$

525. The following problem is easily solved in the same manner. To find the equation to a surface formed by a straight line moving parallel to the plane of $x\,z$, and having its extremities in two given curves $z = f(y)$ on $z\,y$, and $x = \phi(y)$ on $x\,y$.

The equation is $$\frac{z}{f(y)} + \frac{x}{\phi(y)} = 1.$$

526. In questions of this kind some care is requisite in selecting the position of the axes and co-ordinate planes, so that the equations, both those given and those to be found, may present themselves in the simplest form. For example, —to find the surface formed by the motion of a straight line constantly passing through three other given straight lines;

Take three lines parallel to the given lines for the axes of co-ordinates; then the equations to the three directrices are

$$\left.\begin{array}{l} X = a_1 \\ Y = b_1 \end{array}\right\} \qquad \left.\begin{array}{l} X = a_2 \\ Z = c_2 \end{array}\right\} \qquad \left.\begin{array}{l} Y = b_3 \\ Z = c_3 \end{array}\right\}$$

and the equations to the generating line in any position are

$$x = \alpha z + a,\ y = \beta z + b,$$

and consequently $y = \dfrac{\beta}{\alpha} x + c$, where $c = b - \dfrac{\beta}{\alpha} a$;

Then since this line meets each of three given lines, we have the following equations:

$$b_1 = \frac{\beta}{\alpha} a_1 + c\,;\ a_2 = \alpha\, c_2 + a\,;\ b_3 = \beta\, c_3 + b.$$

We must now eliminate a, b, α, β from these three equations and that to the generatrix; by subtraction we have

$$y - b_1 = \frac{\beta}{\alpha}(x - a_1)\,;\ x - a_2 = \alpha(z - c_2)\,;\ y - b_3 = \beta(z - c_3)$$

hence, eliminating α and β, we have the required equation

$$(x - a_1)(y - b_3)(z - c_2) = (x - a_2)(y - b)(z - c_3)$$

which is of the second order, since the term $x\,y\,z$ disappears. See Hymers's Anal. Geom. p. 23, Cambridge, 1830.

CHAPTER IX.

ON CURVES OF DOUBLE CURVATURE.

527. Definition. A curve of double curvature is one whose generating point is perpetually changing not only the direction of its motion, as in plane curves, but also the plane in which it moves.

If a circle be described on a flat sheet of paper, it is a plane curve; let the sheet of paper be rolled into a cylindrical form, then the circle has two curvatures, that which it originally had, and that which it has acquired by the flexion of the paper, hence in this situation it is called a curve of double curvature.

528. Curves of double curvature arise from the intersection of two surfaces; for example, place one foot of a pair of compasses on a cylindrical surface, let the other in revolving constantly touch the surface, it will describe a curve of double curvature, which, though not a circle, has yet all

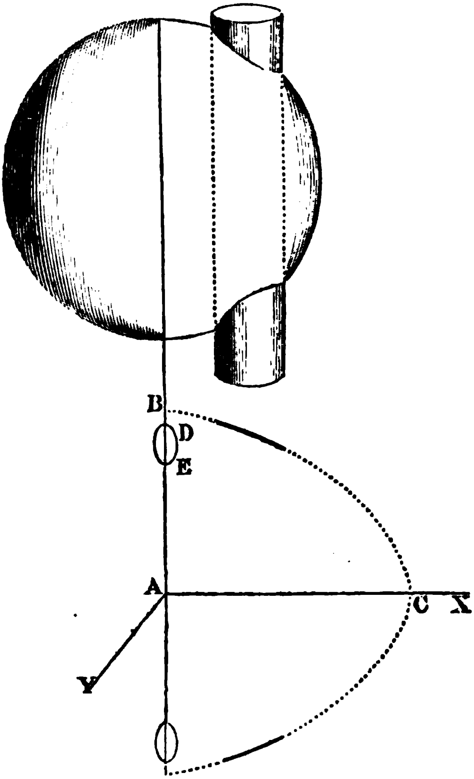
its points at equal distances from the fixed foot of the compasses. The curve is then part of a spherical surface, whose radius is equal to the distance between the feet of the compasses, and consequently is the intersection of this sphere with the cylinder.

529. The equations to the two surfaces taken together are the equations to their intersection, and consequently are the equations to the curve of double curvature.

By the separate elimination of the variables in the two equations, we obtain the respective projections of the curve upon the co-ordinate planes. Two of these are sufficient to define the curve of double curvature; for we may pass two cylinders through two projections of the curve, at right angles to each other, and to the co-ordinate planes, the intersection of these cylinders is the required curve. This is analogous to the consideration of a straight line, being the intersection of two planes.

We proceed to examine curves of double curvature arising from the intersections of surfaces.

530. Let the curve arise from the intersection of a sphere and right cylinder; the origin of co-ordinates being at the centre of the sphere, the axis of the cylinder in the plane xz and parallel to the axis of z.



Let the distance between the centres of the sphere and cylinder $= c$, then the equation to the sphere is $x^2 + y^2 + z^2 = a^2$, and the equation to the cylinder is $(x - c)^2 + y^2 = b^2$, (507.)

$$\text{eliminating } y,\ z^2 = a^2 + c^2 - b^2 - 2\,c\,x \ (1),$$

$$\text{eliminating } x,\ z^2 = a^2 - b^2 - c^2 \mp 2\,c\,\sqrt{b^2 - y^2} \ (2).$$

From (1) the projection of the curve on $x\,z$ is a portion of a parabola B C whose vertex is C, where $\mathrm{A\,C} = \dfrac{a^2 + c^2 - b^2}{2\,c}$. and $\mathrm{A\,B} = \sqrt{a^2 + c^2 - b^2}$.

From (2) the projection on $y\,z$ consists of two ovals, whose positions are determined by the two extreme values of z,

$$\mathrm{A\,D} = \pm\sqrt{a^2 - (b - c)^2}$$

$$\mathrm{A\,E} = \pm\sqrt{a^2 - (b + c)^2}.$$

As c increases, that is, as the cylinder moves further from A, A E decreases, and the ovals approach nearer to each other, as in fig. (1); when $c = a - b$, that is, when the sphere but just encloses the cylinder A E = 0, and the ovals meet, fig. (2). As c increases, we obtain fig. (3), which gradually approaches fig. (4); and lastly, when $c = a$ vanishes entirely.

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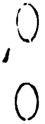

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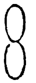

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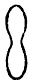
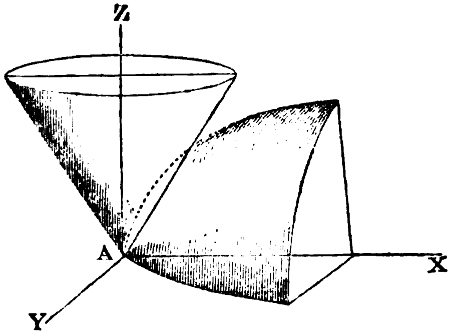


4

Different values, as $c, \dfrac{a}{2}$, &c., may be given to b, and we may then trace the projections: they offer no difficulty, but we recommend their investigation, as the complete examination of one example greatly facilitates the comprehension of all others.

531. Ex. 2. A right cone and a paraboloid of revolution have their vertices coincident, the axis of the cone being perpendicular to the axis of the paraboloid.



The equation to the cone is $x^2 + y^2 = e^2\,z^2$, (468) and that to the paraboloid, $y^2 + z^2 = p\,x$ (469); hence the projection on $x\,z$ is $x^2 + p\,x = (1 + e^2)\,z^2$, which is an hyperbola, whose axes are p and $\dfrac{p}{\sqrt{e^2 + 1}}$. (157).

[illegible]

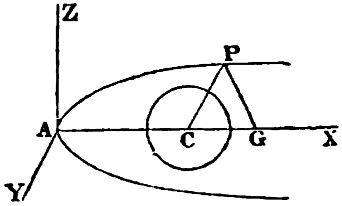
Again, $x^2 + y^2 = e^2 (p x - y^2)$ $\therefore$ $(1 + e^2) y^2 = e^2 p x - x^2$: hence the projection on $x y$ is an ellipse, whose vertex is A and axes $e^2 p$ and $\frac{e^2 p}{\sqrt{1 + e^2}}$ (103).

The equation to the projection on $y z$ is $(y^2 + z^2)^2 + p^2 y^2 = e^2 p^2 z^2$; this is the equation to a Lemniscata, and becomes the Lemniscata of Bernouilli, when $e = 1$, that is, when the cone is right-angled (314).

532. To find the curve of intersection of two surfaces, we have eliminated the variables separately, and thus obtained the equations to the projections on the co-ordinate planes; conversely, by combining these last equations either by addition or multiplication, &c., so as to have an equation between the three variables, we may obtain the surface on which the curve of double curvature may be described. This surface does not at all define the curve of double curvature; since an infinite number of curves may be traced on this individual surface, to all of which the general equation to the surface belongs.

The results of the above combination are often interesting. For example: Let the curve be the intersection of a parabolic cylinder on $x y$, with a circular cylinder on $x z$, the origin being the vertex of the parabola, and the centre of the circle being in the axis of the parabola, which is also the axis of x.

Let $y^2 = 2 p x$ be the equation to the parabola A P on $x y$,
$(x - a)^2 + z^2 = r^2$.... circle on $x z$,
Combining these equations by addition,
$(x - a)^2 - 2 p x + y^2 + z^2 = r^2$,
or $(x - a - p)^2 + y^2 + z^2 = r^2 + p^2 + 2 a p$.



Which is the equation to a sphere whose centre is at a distance A G $= a + p$, measured from A along A X. Now, p is the subnormal C G to the point P of the parabola, P C being the ordinate at C (242); hence, all the points of the curve of double curvature are on the surface of a sphere whose centre is at the extremity of the subnormal of a point in the parabola, the ordinate of which point passes through the centre of the given circle.

533. The intersections of surfaces are not always curves of double curvature, but often they are plane curves. We proceed, then, to show how plane curves may be detected, and their equations determined.

Whenever we obtain a straight line for a projection, the curve cannot be one of double curvature.

Ex. Let the curve be the intersection of two parabolic cylinders, whose equations are

$$x^2 = a z$$
$$b y = x^2.$$

Eliminating x, we have $b y = a z$, hence the projection on $y z$ is a straight line; and as no projection of a curve of double curvature can be a straight line, it follows that the curve of intersection is a plane curve.

534. Again, If we can so combine the equations to the projections as to produce the general equation to a plane, the curve, which is necessarily traced on that plane, is itself a plane curve. For example: let the curve

arise from the intersection of two parabolic cylinders, whose equations are

$$x^2 = a\,z$$
$$b\,y = x^2 + c\,x.$$

In the second equation, substituting $a\,z$ for x^2, we obtain

$$b\,y = a\,z + c\,x;$$

which equation belonging to a plane, the curve is a plane curve.

535. There is another and more general method of detecting plane curves.

From the two equations to the surfaces eliminate one of the variables, as z, for example, we obtain an equation $F\,(x, y) = 0$.

Now, if the curve be plane, it may arise from the intersection of either of the surfaces with a plane whose equation is $z = m\,x + n\,y + p$; eliminate z between this equation to the plane and that to one of the surfaces, the result is $f\,(x, y) = 0$, which must be identical with $F\,(x, y) = 0$; therefore, comparing $F\,(x,\ y) = 0$, with $f\,(x, y) = 0$, we may obtain various equations to determine m, n, and p; which values of m, n, and p must satisfy all the equations in which these quantities appear; if not, the curve is one of double curvature.

For example; take the intersection of a sphere and cylinder, art. 530.

The equation to the Sphere is $x^2 + y^2 + z^2 = a^2$ (1)
.... Cylinder $(x - c)^2 + y^2 = b^2$ (2)
.... Plane $z = m\,x + n\,y + p$ (3)

Eliminating z between (1) and (3), we have $f\,(x, y) = 0$

$$(m^2+1)\,x^2 + (n^2+1)\,y^2 + 2\,m\,n\,x\,y + 2\,m\,p\,x + 2\,n\,p\,y + p^2 - a^2 = 0 \quad (4)$$

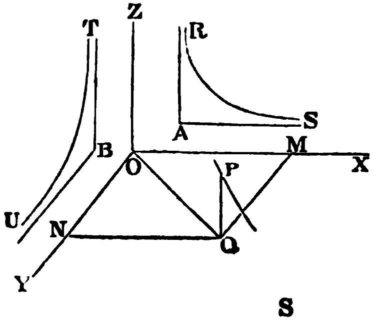
Comparing (2) and (4), we have $m = 0$, $n = 0$ from the co-efficients of x^2 and y^2; but the condition of $m = 0$ destroys the coefficient of x in (4); and thereby shows that (4) cannot be made identical with (2). The curve is therefore a curve of double curvature.

But let the equation to the cylinder be $x^2 + y^2 = b^2$, then $m = 0$ and $n = 0$ render (4) and (2) identical; therefore the curve is a plane curve, situated in a plane, whose equation is $z = \sqrt{a^2 - b^2}$; this is clear, also, from geometrical considerations.

536. To find the curve represented by the equations

$$\frac{a}{x} + \frac{c}{z} = 1,\ \frac{b}{y} + \frac{c}{z} = 1.$$

These equations, taken separately, belong to two right hyperbolic cylinders; one with the base in $x\,z$, and the other in $y\,z$. (209, Ex. 3.)



R S is the hyperbola on $x\,z$, its centre being at A; T U is the hyperbola on $y\,z$, its centre being at B.

Also, $\dfrac{a}{x} = \dfrac{b}{y}$, or $y = \dfrac{b}{a}\,x$.

Hence the projection of the intersection of the above cylinders on $x\,y$ is a straight line O Q, and therefore the curve is a plane curve, situated in the plane Z O Q, perpendicular to $x\,y$.

537. As we cannot have a very clear notion of the curve itself, merely from the idea of the two hyperbolic cylinders, we shall find the equation to the curve in the plane Z O Q; that is, in its own plane.

Let P be any point in the curve; O M $= x$, M Q $= y$, P Q $= z$. Then, in order to find the relation between O Q ($= u$) and Q P ($= z$), we shall express O M and O N in terms of O Q, and substitute in the given equations.

The equation to O Q is $y = \frac{b}{a} x = x \tan.\theta$ (if $\frac{b}{a} = \tan.\theta$),

$$\therefore \text{O M} = \text{O Q} \cos.\theta, \text{ and O N} = \text{O Q} \sin.\theta.$$

Hence the equation $\frac{a}{x} + \frac{c}{z} = 1$ becomes $\frac{a}{u\cos.\theta} + \frac{c}{z} = 1$,

and the equation $\frac{b}{y} + \frac{c}{z} = 1$ becomes $\frac{b}{u\sin.\theta} + \frac{c}{z} = 1$.

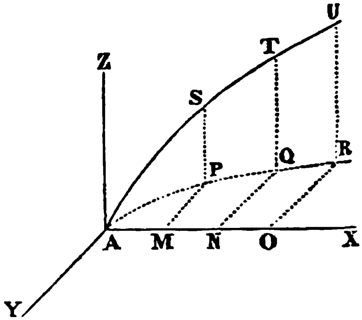
Since $b = a\tan.\theta$, or $b\cos.\theta = a\sin.\theta$, these two equations are the same, and either of them belongs to the required curve; hence the curve is an hyperbola, whose equation referred to its centre is

$$u z = \frac{ac}{\cos.\theta} = \frac{bc}{\sin.\theta}. \quad (209)$$

538. To describe a curve of double curvature by points.

Let $f(x, y) = 0$, and $\phi(x, z) = 0$, be two of its projections.

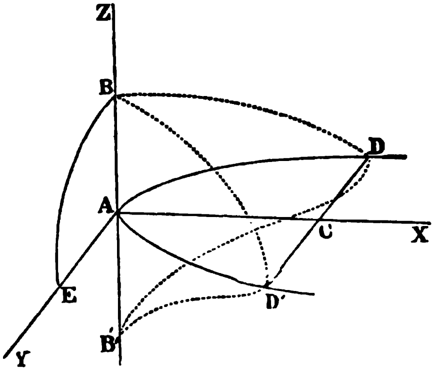
Upon $x y$ trace the curve A P Q R, whose equation is $f(x, y) = 0$.



For any value of x, as A M, we obtain a corresponding value M P of y: from $\phi(x, z) = 0$, we can also obtain a corresponding value of z. From P draw P S perpendicular to x y, and equal to this value of z; then S is a point in the curve. By repeating this process we may obtain any number of points S T U, &c., in the curve.

It is evident, that if any value given to x or y renders z imaginary, no part of the curve can be constructed corresponding to such values of x or y. Also, that if z be negative, P S must be drawn below the plane x y.

539. Ex. 1. Let the curve arise from the intersection of a parabolic cylinder on x y, and a circular cylinder on y z, the axes perpendicular to each other; and the vertex of the parabola together with the centre of the circle at the origin of co-ordinates.



Let $y^2 = a\,x$ be the equation to the parabola D A D′,
$y^2 + z^2 = a^2$ .... circle E B,
$\therefore\ z^2 + a\,x = a^2$ is a parabola on $x\,z$.
Let A B $= a$, A C $= a$, and let the ordinate C D $= a$.

To trace the curve, we have the three equations on the co-ordinate planes,

$$z = \pm\sqrt{a\,(a - x)}$$
$$z = \pm\sqrt{a^2 - y^2},$$
$$y = \pm\sqrt{a\,x}.$$

If $x = 0$, $y = 0$, and $z = a$, $\therefore$ the curve passes through B ; as x increases, y increases, and z diminishes ;

When $x = a$, $y = a$, and $z = 0$, therefore the curve decreases in altitude from B down to meet the parabola in D. This gives the dotted branch BD.

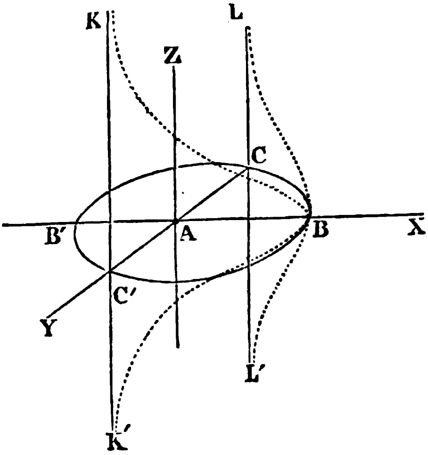
If x is greater than a, z is imaginary; therefore the curve does not extend beyond D.

But since $z = \pm\sqrt{a\,(a - x)}$ there is another ordinate corresponding to every value of x between o and a; hence there is another branch, equal and opposite to B D, but below the plane $x\,y$. This is represented by D B′.

Again, since when y is negative, the values of z do not change, there is another arc, B D′ B′, represented by the double dotted line, which is exactly similar to B D B′.

Therefore, the curve is composed of four parts, B D, D B′, B D′, and D′ B′, equal to one another, and described upon the surface of the parabolic cylinder, whose base is D A D′. These branches form altogether a figure something like that of an ellipse, of which the plane is bent to coincide with the cylinder.

540. Ex. 2. Let the circle, whose equation is $x^2 + y^2 = a^2$, be the projection of the curve of double curvature on $x\,y$; and the curve, of which the equation is $a^2\,y^2 = a^2\,x^2 - y^2\,z^2$, be the projection on $y\,z$, to trace the curve.



Let B C B′ C′ be the circle on $x\,y$ whose equation is $x^2 + y^2 = a^2$; then the equation on $y\,z$ being $a^2 y^2 = a^2 z^2 - y^2 z^2$, the equation on $x\,z$ is $x^2 z^2 = a^4 - a^2 x^2$.

$$\therefore z = \pm \frac{a}{x}\sqrt{a^2 - x^2},$$

$$\text{or } z = \pm \frac{a\,y}{\sqrt{a^2 - y^2}},$$

$$\text{and } y = \pm \sqrt{a^2 - x^2}.$$

If $x = 0$, $y = a$, $z =$ infinity, therefore the vertical line C L through C is an asymptote to the curve. As x increases, y decreases, and z decreases, therefore the curve approaches the plane of $x\,y$. If $x = a$, $y = o$, $z = o$, therefore the curve passes through B. If x is greater than a, y and z are each impossible, therefore no part of the curve is beyond B: for any value of y there are two of z, therefore for the values of y in the quadrant A C B, there are two equal and opposite branches, L B, B L′.

Similarly there are two other equal branches, K B, B K′, for the quadrant B A C′; and as the same values of y and z recur for x negative, there are four other branches equal and opposite to those already drawn, which correspond to the semicircle C B′ C′, and which proceed from B′.

These two examples are taken from Clairaut's Treatise on Curves of Double Curvature; a work containing numerous examples and many excellent remarks on this subject.

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USEFUL BOOKS ON THIS BRANCH OF SCIENCE.

Agnesi.—Analytical Institutions. Trans. London, 1801.
Biot.—Essai de Géométrie Analytique. Paris, 1813.
Bourcharlat.—Théorie des Courbes et des Surfaces. Paris, 1810.
Bourdon.—Application de l'Algèbre à la Géométrie. Paris, 1825.
Clairaut.—Recherches sur les Courbes à double Courbure. Paris, 1731.
Cramer.—Introduction à l'Analyse des Lignes Courbes. Genève, 1750.
Francœur.—Course of Pure Mathematics. Trans. Cambridge, 1830.
Garnier.—Géométrie Analytique. Paris, 1813.
Gua.—Usage de l'Analyse de Descartes. Paris, 1740.
Hamilton.—Principles of Analytical Geometry. Cambridge, 1826.
Hamilton.—Analytical System of Conic Sections. Cambridge, 1830.
Hymers.—Treatise on Analytical Geometry of three Dimensions. Cambridge, 1830.
Lacroix.—Traité Elémentaire de Trigonométrie, &c. Paris, 1810.
Lardner.—System of Algebraical Geometry. London, 1823.
Lefebvre de Fourcy.—Leçons de Géométrie Analytique. Paris, 1827.
Monge.—Application de l'Analyse à la Géométrie. Paris, 1809.
Newton.—Arithmetica Universalis. Enumeratio linearum tertii ordinis. Vol. i. Horsley, ed. London, 1779.
Peacock.—Examples on the Differential and Integral Calculus. Cambridge, 1820.
Puissant.—Recueil de Propositions de Géométrie. Paris, 1824.
Reynaud.—Traité d'Application de l'Algèbre à la Géométrie. Paris, 1819.
Stirling.—Lineæ tertii ordinis Newtonianæ. London, 1717.
The Elements of Curves. Oxford, 1828.
Young.—Elements of Analytical Geometry. London, 1830.

Cambridge Philosophical Transactions.
Encyclopædia Metropolitana; arts. Conic Sections; Analytical Geometry.
Annales des Mathématiques, par J. D. Gergonne. Nismes.
Journal de l'Ecole Polytéchnique.
Correspondance sur l'Ecole Polytéchnique.
Journal für die reine und angewandte Mathematic, von A. L. Crelle. Berlin.

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ERRATA.

Page 7, line 1, read Let $x = \sqrt{3} = \sqrt{2 + 1}$. In the last figure let A B, B C, and C D each be equal to the linear unit, then A D $= \sqrt{3}$.

40.....2, read $\therefore y' = \pm \frac{r}{\sqrt{1 + a^2}}$.

40.....3, for $\frac{r^2}{y}$ read $\frac{r^2}{y'}$.

40....17, for 24 read 25.

48....12, for y^3 read y^2.

110....24, for e read c.

111....17, for tan. θ . tan. $= \frac{c}{a}$, read tan. θ . tan. $\theta' = \frac{c}{a}$.

112....20, for $\frac{y^2}{b^3}$ read $\frac{y^2}{b^2}$.

114....18, for conjugate read semi-conjugate.

123....30, for $x' + m\,S = P$, read $x' + m = S\,P$.

153, in the table, column 7, insert c.

190, line 5 from bottom, for $3\,a$, read $3\,a^2$.

209, line 10, read cos. $l\,x$ cos. $l_1\,x$ + cos. $l\,y$ cos. $l_1\,y$.

217, line 13, for (2) read (1).

221, line 27, read cos. $\theta = \frac{x\,x_1 + y\,y_1 + z\,z_1 + \&c.}{r\,r_1} =$

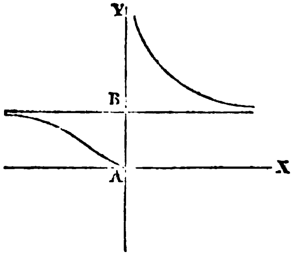
224, line 10, for 397 read 402.

247, line 3, $\therefore (x - x_1)^2 + (y - y_1)^2 = r^2$;
and if the axis coincide with A Z, $x^2 + y^2 = r^2, z = 0$;

249, line 1, for $r - c$ read z, and for c read $z - c$.

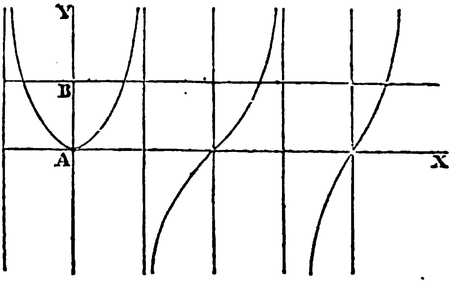
ERRATA IN THE FIGURES.

Art. 352. $y = a^{\frac{1}{x}}$



Art. 353. $y = x^{x}$. See the figure in the same page just above the Art. 353; th
letter B should be at the point where the upper curve meets A Y.

Art. 355 $y = x \tan. x$



Art. 363. The involute of the circle.